# Discovering Theorems in Game Theory: Two-Person Games with Unique Nash Equilibria

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### Abstract

In this paper we provide a logical framework for using computers to discover theorems in (two-person finite) games in strategic form, and apply it to discover classes of games that have unique (pure-strategy) Nash equilibria. We consider all possible classes of games that can be expressed by a conjunction of two binary clauses, and our program re-discovered Kats and Thisse's class of weakly unilaterally competitive two-person games, and came up with several other classes of games that have unique Nash equilibria. It also came up with new classes of strict games that have unique Nash equilibria, where a game is strict if for each player different profiles have different payoffs. Partly motivated by these findings, we also (manually) prove a result that says that a strict game has a unique Nash equilibrium iff it is best-response equivalent to a strictly competitive game.

## Introduction

In game theory, a key notion is that of Nash equilibria. A game in strategic form can have one, more than one, or zero Nash equilibria (see, e.g. [Osborne and Rubinstein, 1994]), and there has been extensive work on classes of games that always have Nash equilibria, such as potential games [Monderer and Shapley, 1996] and supermodular games [Topkis, 1998], as well as classes of games with unique Nash equilibria, such as strictly competitive games [Moulin, 1976; Friedman, 1983]

In this paper, as part of our project on using computers to discover theorems in game theory, we consider the possibility of using computers to discover new classes of two-person games that have unique (pure) Nash equilibria. Our starting point is the class of two-person strictly competitive games. We first formulate the notions of games, strictly competitive games and Nash equilibria in first-order logic. Under our formulation, a class of games corresponds to a first-order sentence. In particular, the sentence that corresponds to the class of strictly competitive games is a conjunction of two binary clauses with all variables universally quantified. So we implemented a program that examine all these universally quantified conjunctions of binary clauses to see if there is another such condition that also captures a class of games with unique Nash equilibria. We did not expect much as these conditions are rather simple, but to our surprise, our program returned a condition that is more general than the strict competitiveness condition. As it turned out, it exactly corresponds to Kats and Thisse's [1992] class of *weakly unilaterally competitive* two-person games. Our program also returned some other conditions. Two of them capture a class of "unfair" games where one player has advantage over the other. The remaining ones capture games where everyone gets what he wants - each receives his maximum payoff in every equilibrium state, thus there is no real competition among the players. Thus one conclusion that we can draw from this experiment is that among all classes of games that can be expressed by a conjunction of two binary clauses, the class of weakly unilaterally competitive games is the most general class of "competitive" and "fair" games that have unique Nash equilibria. Of course, this does not mean that the other conditions are not worth investigating. For instance, sometimes one may be forced to play an unfair game.

For the same set of conditions, we also consider strict two-person games where different profiles have different payoffs for each player. Among the results returned by our program, two of them are exactly the two conjuncts in Kats and Thisse's weakly unilaterally competitive condition, but the others all turn out to be special cases of games with dominant strategies. Motivated by these results, we consider certain equivalent classes of games, and show that a strict game has a unique Nash equilibrium iff it is best-response equivalent [Rosenthal, 1974] to a strictly competitive game.

The rest of the paper is organized as follows. We first review some basic concepts in two-person games in strategic form, and then reformulate them in first-order logic. We then show that for a class of conditions, whether any of them entails the uniqueness of Nash equilibria needs only to be checked on games up to certain size. We then describe a computer program based on this result, and report our experimental results.

## Two-person games

A (two-person) game (in strategic form) is a tuple $(A, B, \leq_1, \leq_2)$, where $A$ and $B$ are sets of strategies of players 1 and 2, respectively, and $\leq_1$ and $\leq_2$ are total orders on $A \times B$ called *preference relations* for players 1 and 2, respectively.

Instead of two preference relations, a two-person game can also be specified by two payoff functions, one for each player, which map profiles to numbers. The relationship between these two formulations are as follows: for any profiles $s$ and $s'$, $s \leq_i s'$ iff $u_i(s) \leq u_i(s')$, where $u_i$ is the payoff function for player $i$. In the following, we shall use these two formulations interchangeably.

For each $b \in B$, we define $B_1(b)$ to be the set of best responses by player 1 to the strategy $b$ by player 2:

$$B_1(b) = \{a \mid a \in A, \text{ and for all } a' \in A, (a', b) \leq_1 (a, b)\}.$$

Similarly, for each $a \in A$, the set of best responses by player 2 is:

$$B_2(a) = \{b \mid b \in B, \text{ and for all } b' \in B, (a, b') \leq_2 (a, b)\}.$$

A profile $(a, b) \in A \times B$ is a *Nash equilibrium* if both $a \in B_1(b)$ and $b \in B_2(a)$. Notice that we consider only pure-strategy Nash equilibria. A game can have exactly one, more than one, or no Nash equilibria. We say that a game has *unique* Nash equilibria if for each player, all Nash equilibria of the game have the same payoffs, that is, whenever $(a_1, b_1)$ and $(a_2, b_2)$ are two Nash equilibria of the game, then

$$(x_1, y_1) \leq_i (x_2, y_2) \wedge (x_2, y_2) \leq_i (x_1, y_1),$$

for $i = 1, 2$. Notice that according to this definition, if a game has no Nash equilibria, then it is also a game with unique Nash equilibria. Thus games with unique Nash equilibria are really games that have at most one equivalent class of Nash equilibria.

One interesting class of two-person games is that of *strictly competitive* games. A game is strictly competitive [Moulin, 1976; Friedman, 1983] if for every pair of profiles $s_1$ and $s_2$ in $A \times B$, we have that $s_1 \leq_1 s_2$ iff $s_2 \leq_2 s_1$. Thus in strictly competitive games, the two players' preferences are exactly opposite.

Strictly competitive games have many nice properties. If $(a, b)$ and $(a', b')$ are both Nash equilibria of a strictly competitive game, then (1) they are *equivalent* in the sense that $(a, b) \leq_i (a', b')$ and $(a', b') \leq_i (a, b)$ for both $i = 1, 2$; (2) they are interchangeable in the sense that $(a', b)$ and $(a, b')$ are also Nash equilibria. Thus if a strictly competitive game has Nash equilibria, then they are unique. Furthermore they can be computed using the minmax procedure.

Another class of games that we shall consider in this paper is that of *strict* games. A game is strict if for both players, different profiles have different payoffs, that is, $(a, b) = (a', b')$ whenever $(a, b) \leq_i (a', b')$ and $(a', b') \leq_i (a, b)$, where $i = 1, 2$. As we shall see, strict games have some nice properties that general games do not have.

## Formulating two-person games in first-order logic

We consider a first-order language with two sorts $\alpha$ and $\beta$, equality, and two predicates $\leq_1$ and $\leq_2$. We use "$\wedge$" for conjunction, "$\vee$" for disjunction, "$\neg$" for negation, "$\supset$" for implication, and "$\equiv$" for equivalence. Negation has the highest precedence, followed by conjunction and disjunction, implication, and then equivalence. The rule of precedence can be overridden by a new line. For instance, the following expression

$$\begin{aligned} &p \supset q \wedge \\ &\quad q \supset p \end{aligned}$$

stands for the sentence $(p \supset q) \wedge (q \supset p)$.

In our language, sort $\alpha$ is for player 1's strategies, and $\beta$ for player 2's strategies. In the following, we use variables $x, x_1, x_2, ..$ to range over $\alpha$, and $y, y_1, y_2, ...$ to range over $\beta$. The two predicates represent the two players' preference relations. In the following, as we have already done above, we write $\leq_i (x_1, y_1, x_2, y_2)$ in infix notation as $(x_1, y_1) \leq_i (x_2, y_2)$, $i = 1, 2$, and $(x_1, y_1) \simeq_i (x_2, y_2)$ as a shorthand for

$$(x_1, y_1) \leq_i (x_2, y_2) \wedge (x_2, y_2) \leq_i (x_1, y_1),$$

where $i = 1, 2$. We also write $(x_1, y_1) <_i (x_2, y_2)$ as a shorthand for

$$(x_1, y_1) \leq_i (x_2, y_2) \wedge \neg(x_2, y_2) \leq_i (x_1, y_1).$$

The two relations need to be total orders (in the rest of the paper, unless otherwise stated, all free variables in a displayed formula are assumed to be universally quantified from outside):

$$(x, y) \leq_i (x, y), \tag{1}$$

$$(x_1, y_1) \leq_i (x_2, y_2) \vee (x_2, y_2) \leq_i (x_1, y_1), \tag{2}$$

$$\begin{aligned} &(x_1, y_1) \leq_i (x_2, y_2) \wedge (x_2, y_2) \leq_i (x_3, y_3) \supset \\ &\qquad (x_1, y_1) \leq_i (x_3, y_3), \end{aligned} \tag{3}$$

where $i = 1, 2$. In the following, we denote by $\Sigma$ the set of the above sentences. Thus two-person games correspond to first-order models of $\Sigma$, and two-person finite games correspond to first-order finite models of $\Sigma$. This correspondence extends to other type of games as well. For instance, let $\Sigma_s$ be the union of $\Sigma$ with the following two axioms:

$$(x_1, y_1) \simeq_1 (x_2, y_2) \supset (x_1 = x_2 \wedge y_1 = y_2),$$
$$(x_1, y_1) \simeq_2 (x_2, y_2) \supset (x_1 = x_2 \wedge y_1 = y_2).$$

Then strict games and models of $\Sigma_s$ are isomorphic.

We now show how some other notions in game theory can be formulated in first-order logic. The condition for a profile $(\xi, \zeta)$ to be a Nash equilibrium is captured by the following formula:

$$\forall x.(x, \zeta) \leq_1 (\xi, \zeta) \wedge \forall y.(\xi, y) \leq_2 (\xi, \zeta) \tag{4}$$

In the following, we shall denote the above formula by $NE(\xi, \zeta)$.

The following sentence expresses the uniqueness of Nash equilibria in a game:

$$\begin{aligned} &NE(x_1, y_1) \wedge NE(x_2, y_2) \supset \\ &\qquad (x_1, y_1) \simeq_1 (x_2, y_2) \wedge (x_1, y_1) \simeq_2 (x_2, y_2) \end{aligned} \tag{5}$$

A game is strictly competitive if it satisfies the following property:

$$(x_1, y_1) \leq_1 (x_2, y_2) \equiv (x_2, y_2) \leq_2 (x_1, y_1). \tag{6}$$

Thus it should follow that

$$\Sigma \models (6) \supset (5). \qquad (7)$$

Notice that we have assumed that all free variables in a displayed formula are universally quantified from outside. Thus (6) is a sentence of the form $\forall x_1, x_2, y_1, y_2 \varphi$. Similarly for (5).

Theorems like (7) can actually be generated automatically using the following theorem.

**Theorem 1** *Suppose $Q$ is a formula without quantifiers, $\vec{x}_1$ and $\vec{x}_2$ tuples of variables of sort $\alpha$, and $\vec{y}_1$ and $\vec{y}_2$ tuples of variables of sort $\beta$. We have that*

1. $\Sigma \models \exists\vec{x}_1\exists\vec{y}_1\forall\vec{x}_2\forall\vec{y}_2 Q \supset (5)$
   *iff for all model $G$ of $\Sigma$ such that $|A| \leq |\vec{x}_1| + 2$ and $|B| \leq |\vec{y}_1| + 2$, we have that*
   $G \models \exists\vec{x}_1\exists\vec{y}_1\forall\vec{x}_2\forall\vec{y}_2 Q \supset (5)$,
   *where $A$ is the domain of $G$ for sort $\alpha$, and $B$ the domain of $G$ for sort $\beta$.*
2. $\Sigma \models \exists\vec{x}_1\exists\vec{y}_1\forall\vec{x}_2\forall\vec{y}_2 Q \supset \neg\exists x, y.NE(x, y)$
   *iff for all model $G$ of $\Sigma$ such that $|A| \leq |\vec{x}_1| + 1$ and $|B| \leq |\vec{y}_1| + 1$ we have that*
   $G \models \exists\vec{x}_1\exists\vec{y}_1\forall\vec{x}_2\forall\vec{y}_2 Q \supset \neg\exists x, y.NE(x, y)$,
   *where $A$ is the domain of $G$ for sort $\alpha$, and $B$ the domain of $G$ for sort $\beta$.*

**Proof:** "Only if" is trivial. To show "if", suppose there is a game $G$ (model of $\Sigma$) such that it satisfies the condition $\exists\vec{x}_1\exists\vec{y}_1\forall\vec{x}_2\forall\vec{y}_2 Q$ but has two non-equivalent Nash equilibria, $(a, b)$ and $(a', b')$: either $(a, b) \not\simeq_1 (a', b')$ or $(a, b) \not\simeq_2 (a', b')$. Thus there is a tuple $\vec{a}_1$ of elements from $A$ and a tuple $\vec{b}_1$ of elements from $B$ such that $|\vec{a}_1| = |\vec{x}_1|$, $|\vec{b}_1| = |\vec{y}_1|$, and $G$ satisfies

$$(\forall\vec{x}_2\forall\vec{y}_2 Q)|_{\vec{x}_1/\vec{a}_1, \vec{y}_1/\vec{b}_1},$$

which is obtained from $(\forall\vec{x}_2\forall\vec{y}_2 Q)$ by replacing in it every free occurrence of each variable in $\vec{x}_1$ by its corresponding element in $\vec{a}_1$, and every free occurrence of each variable in $\vec{y}_1$ by its corresponding element in $\vec{b}_1$. Now construct a new game $G' = (A', B', \leq_1', \leq_2')$ as follows:

- $A' = \{a, a'\} \cup \vec{a}_1$ and $B' = \{b, b'\} \cup \vec{b}_1$.
- $\leq_1'$ is the restriction of $\leq_1$ on $A'$, and $\leq_2'$ is the restriction of $\leq_2$ on $B'$.

Notice that this game is well-defined as $\leq_1'$ and $\leq_2'$ are both total orders, i.e. $G' \models \Sigma$. Clearly, the size of $G'$ is smaller or equal to $(|\vec{x}_1|+2) \times (|\vec{y}_1|+2)$, both $(a, b)$ and $(a', b')$ are still non-equivalent Nash equilibria of $G'$, and the formula $\exists\vec{x}_1\exists\vec{y}_1\forall\vec{x}_2\forall\vec{y}_2 Q$ is still true in $G'$. ■

In other words, to prove that a sentence of the form $\exists\vec{x}_1\exists\vec{y}_1\forall\vec{x}_2\forall\vec{y}_2 Q$ is a sufficient condition for the uniqueness of Nash equilibria, it suffices to verify that this is the case for all games of sizes up to $(|\vec{x}_1| + 2) \times (|\vec{y}_1| + 2)$, and to prove that it is a sufficient condition for the non-existence of Nash equilibria, it suffices to verify this for games of sizes up to $(|\vec{x}_1| + 1) \times (|\vec{y}_1| + 1)$.

Theorem 1 holds for many specialized games as well. For instance, it holds for strict games as well.

**Theorem 2** *Theorem 1 holds when $\Sigma$ is replaced by $\Sigma_s$.*

In fact, Theorem 1 holds when $\Sigma$ is replaced by any set of universally quantified sentences.

## Theorem discovering

Since $p \equiv q$ is logically equivalent to $(\neg p \vee q) \wedge (p \vee \neg q)$, the condition (6) for strictly competitive games can be written as a conjunction of two binary clauses:

$$(l_1 \vee l_2) \wedge (l_3 \vee l_4), \qquad (8)$$

where each $l_i$, $1 \leq i \leq 4$, is a literal, i.e. either an atom or the negation of an atom. As we mentioned, we want to know if there are other sentences of the form (8) that also capture classes of games with unique Nash equilibria. In the following, we say that a condition $\varphi$ is a *uniqueness condition* if whenever a game satisfies this condition, it has unique Nash equilibria, that is, if $\Sigma \models \varphi \supset (5)$.

Based on Theorem 1, a straightforward way of discovering uniqueness conditions of the form (8) is as follows: For each condition of the form (8), check that if a $2 \times 2$ game does not have unique Nash equilibria, then it does not satisfy this condition. There are 810,000 such conditions, 1950 non-isomorphic $2 \times 2$ two-person games, and among them 709 games that do not have unique Nash equilibria. Thus this strategy can be implemented on a modern computer even by brute-force search.

The search space can also be pruned by noticing that the conditions of the form (8) are not independent. For instance, condition

$$(x_1, y_1) \leq_1 (x_2, y_2)$$

entails (is stronger than) condition

$$(x_1, y_1) \leq_1 (x_1, y_2).$$

Once we know that a condition $C$ is a uniqueness condition, those that entail $C$ are no longer interesting as they become special cases of $C$, thus can be pruned.

However, checking logical entailment is in general not decidable for first-order logic. But as a strategy for pruning search space, we can use a weaker notion called *subsumption* on conditions of the form (8): $C$ subsumes $C'$ if there is a substitution $\sigma$ such that $C\sigma = C'$. For our language, subsumption can be checked efficiently, and the search tree can be designed in such a way that the condition associated with a node always subsumes the conditions associated with the ancestors of the node. Thus once a condition is found to be a uniqueness condition, the entire sub-tree under this condition can be pruned.

However, we still need a way to check for complete logical entailment under $\Sigma$ for conditions of the form (8). This is because we want every condition returned by our program to be a most general, "weakest" uniqueness condition in the sense that it does not entail any other uniqueness condition of the form (8). Fortunately, this can be done using the following proposition.

**Proposition 1** *To check whether condition $\forall\vec{x}_1\vec{y}_1 Q_1$ entails condition $\forall\vec{x}_2\vec{y}_2 Q_2$ for all two-person games, it suffices to check this for all games up to $max\{|\vec{x}_2|, 1\} \times max\{|\vec{y}_2|, 1\}$, where $Q_1$ and $Q_2$ are formulas without quantifiers. This result holds for strict games as well.*

Notice that what we have described applies to the task of discovering uniqueness conditions of the form (8) for strict two-person games as well.

We now report our experimental results, first for general two-person games, and then for strict two-person games.

## General games

For two-person general games, our program returns the following seven uniqueness conditions for 2x2 games.

$$(x_1, y) \leq_1 (x_2, y) \supset (x_2, y) \leq_2 (x_1, y) \wedge (x, y_1) \leq_2 (x, y_2) \supset (x, y_2) \leq_1 (x, y_1) \quad (9)$$

$$(x_1, y) \leq_1 (x_2, y) \supset (x_1, y) \leq_2 (x_2, y) \wedge (x, y_1) \leq_2 (x, y_2) \supset (x, y_2) \leq_1 (x, y_1) \quad (10)$$

$$(x_1, y) \leq_1 (x_2, y) \supset (x_2, y) \leq_2 (x_1, y) \wedge (x, y_1) \leq_2 (x, y_2) \supset (x, y_1) \leq_1 (x, y_2) \quad (11)$$

$$(x_1, y_1) \leq_1 (x_2, y_1) \supset (x_1, y_2) \leq_2 (x_2, y_2) \wedge (x, y_1) \leq_2 (x, y_2) \supset (x, y_1) \leq_1 (x, y_2) \quad (12)$$

$$(x_1, y) \leq_1 (x_2, y) \supset (x_1, y) \leq_2 (x_2, y) \wedge (x_1, y_1) \leq_2 (x_1, y_2) \supset (x_2, y_1) \leq_1 (x_2, y_2) \quad (13)$$

$$(x_1, y_1) \leq_1 (x_2, y_2) \supset (x_1, y_1) \leq_2 (x_2, y_1) \wedge (x_1, y_1) \leq_2 (x_2, y_2) \supset (x_2, y_1) \leq_1 (x_2, y_2) \quad (14)$$

$$(x_1, y_1) \leq_1 (x_2, y_2) \supset (x_1, y_2) \leq_2 (x_2, y_2) \wedge (x_1, y_1) \leq_2 (x_2, y_2) \supset (x_1, y_1) \leq_1 (x_1, y_2). \quad (15)$$

By Theorem 1, these are also uniqueness conditions for all two-person games. Furthermore, since these are the only conditions returned by our program, for any sentence $C$ of the form (8), if it is a uniqueness condition, then it must entail one of the above conditions under $\Sigma$. In other words, the above seven conditions are the weakest (most general) uniqueness conditions of the form (8).

Notice that condition (10) and condition (11) are symmetric in the sense that one can be obtained from the other by swapping the roles of the two players. So are (12) and (13), and (14) and (15). On the other hand, (9) is symmetric to itself. It is easy to see that if two conditions are symmetric, then one is a uniqueness condition iff the other is.

Condition (9) looks like condition (6) for strictly competitive games, except that the strategy of one of the players is fixed in each implication. As it turned out, it captures exactly the class of two-person games that are *weakly unilaterally competitive* [Kats and Thisse, 1992]:

> "a game belongs to this class if a unilateral move by one player which results in an increase in that player's payoff also causes a (weak) decline in the payoffs of all other players. Furthermore, if that move causes no change in the mover's payoff then all other players' payoffs remain unchanged."

Clearly, if a game is strictly competitive, then it is also weakly unilaterally competitive, but the converse is not true in general. Kats and Thisse [1992] showed that if a game is weakly unilaterally competitive, then it has at most one Nash equilibrium. For us, for two-person games, this follows directly from our computer output and Theorem 1.

Condition (10) can be given a similar interpretation:

> A two-person game satisfies this condition if a unilateral move by player 1 which results in a (weak) increase in his payoff also causes a (weak) increase in the payoff of player 2, but a unilateral move by player 2 which results in a (weak) increase in his payoff will causes a (weak) decline in the payoff of player 1.

Thus in this class of games, the two players are not equal, and it clearly favors player 2. The game may be competitive for player 1, but not for player 2.

**Proposition 2** *Given a game that satisfies (10), if player 2's payoff is maximal at $(a, b)$, i.e. $(a', b') \leq_2 (a, b)$ for all $a', b'$, then there is a strategy $a^*$ such that $(a^*, b)$ is a Nash equilibrium and $(a^*, b) \simeq_2 (a, b)$.*

Thus for the class of games that satisfy condition (10), the optimal strategy for player 2 is to do the strategy for which there is a strategy by the other player that will give him the maximum payoff. The following is an example of such games (as usual, player 1 is the row player, and player 2 the column player; the first number in a cell is the payoff of the row player, the second the column player):

| | | |
|---|---|---|
| 3, 6 | 4, 5 | 5, 1 |
| 2, 3 | 1, 4 | 6, 2 |

It has a unique equilibrium $(3, 6)$.

As we mentioned, condition (11) is symmetric to condition (10), with the roles of the two players swapped. For the classes of games corresponding to the other conditions, (12) - (15), both players can obtain their maximal payoffs.

**Proposition 3** *Given a game that satisfies one of the conditions (12) - (15), if player 1's (player 2's) payoff at $(a, b)$ maximal, then there is a strategy $b^*$ ($a^*$) such that $(a, b^*)$ ($(a^*, b)$) is a Nash equilibrium where both players receive the maximum payoffs.*

Thus, from these two propositions, we see that the classes of games represented by the conditions (10) - (15) are not really "competitive" games. We can then conclude that among the classes of games that can be represented by a conjunction (8) of two binary clauses, the class of weakly unilaterally competitive games is the most general class of "competitive" and "fair" games that have unique Nash equilibria. As we mentioned above, by this we do not mean that other types of games are not interesting. In real life, unfair games like those described by (10) may well arise.

## Strict games

We now describe our experimental results for strict games. Recall that these are games where for each player, different profiles have different payoffs.

### Games with dominant strategies

We first consider conditions that mention only $\leq_1$:

$$s_1 \leq_1 s_2 \vee s_3 \leq_1 s_4.$$

For this class of conditions, our program outputs the following six uniqueness conditions on 2x2 strict games:

$$(x_1, y_1) \leq_1 (x_2, y_1) \vee (x_2, y_1) \leq_1 (x_1, y_2),$$
$$(x_1, y_1) \leq_1 (x_2, y_1) \vee (x_2, y_2) \leq_1 (x_1, y_1),$$
$$(x_1, y_1) \leq_1 (x_2, y_1) \vee (x_2, y_2) \leq_1 (x_1, y_2),$$
$$(x_1, y_1) \leq_1 (x_2, y_2) \vee (x_2, y_1) \leq_1 (x_1, y_1),$$
$$(x_1, y_1) \leq_1 (x_2, y_2) \vee (x_2, y_2) \leq_1 (x_1, y_2).$$

By Theorem 2, these are also uniqueness conditions for all strict two-person games. Notice that these conditions do not mention $\leq_2$. This means that if player 1's preference relation satisfies any of the above conditions, then the game has a unique Nash equilibrium, no matter what the other player's preference relation is.

For instance, the first condition can be written as

$$\neg(x_1, y_1) \leq_1 (x_2, y_1) \supset (x_2, y_1) \leq_1 (x_1, y_2).$$

For strict games, this is equivalent to

$$(x_2, y_1) <_1 (x_1, y_1) \supset (x_2, y_1) \leq_1 (x_1, y_2)$$

as $\neg(x_1, y_1) \leq_1 (x_2, y_1)$ iff $(x_2, y_1) <_1 (x_1, y_1)$. It is not hard to see that the above condition implies the following condition:

$$\exists x \forall x', y.(x', y) \leq_1 (x, y),$$

meaning that no matter what player 2 does, the best response for player 1 is always the same. For strict games, this means that player 1 has a *strictly dominant strategy* [Shor, web accessed January 2007]: a strategy $x$ is a strictly dominant strategy if for all other strategy $x'$ of player 1, and any strategy $y$ of player 2, $(x', y) <_1 (x, y)$. As it turned out, this is also the case for the other five conditions above, as the following proposition shows.

**Proposition 4** *A strict game $G = (A, B, \leq_1, \leq_2)$ has a strictly dominant strategy for player 1 if and only if for any preference relation $\leq'_2$ for player 2, the game $G' = (A, B, \leq_1, \leq'_2)$ has exactly one Nash equilibrium.*

Given this result, there is no need to consider any condition of the form (8) that mentions only one player's preference relation.

It is interesting to note that for the prisoner's dilemma

| | |
|---|---|
| 4, 4 | 0, 5 |
| 5, 0 | 1, 1 |

each player has a strictly dominant strategy, thus should play this strategy. The dilemma is that each player can get a higher payoff by a unilateral move away from his dominant strategy.

## Weakly unilaterally competitive games for individual players

For other conditions of the form (8), our program returns 16 uniqueness conditions for strict games. However, each of them has a symmetric one when the roles of the two players are swapped. Thus there are really only eight such conditions, given below:

$$(x_1, y) \leq_1 (x_2, y) \vee (x_1, y) \leq_2 (x_2, y), \quad (16)$$
$$(x_1, y_1) \leq_1 (x_1, y_2) \vee (x_1, y_2) \leq_2 (x_2, y_1), \quad (17)$$
$$(x_1, y_1) \leq_1 (x_1, y_2) \vee (x_2, y_2) \leq_2 (x_1, y_1), \quad (18)$$
$$(x_1, y_1) \leq_1 (x_1, y_2) \vee (x_2, y_2) \leq_2 (x_2, y_1), \quad (19)$$
$$(x_1, y_1) \leq_1 (x_2, y_2) \vee (x_1, y_2) \leq_2 (x_1, y_1), \quad (20)$$
$$(x_1, y_1) \leq_1 (x_2, y_2) \vee (x_2, y_2) \leq_2 (x_1, y_2), \quad (21)$$
$$(x_1, y_1) \leq_1 (x_1, y_2) \vee (x_1, y_1) \leq_2 (x_2, y_1), \quad (22)$$
$$(x_1, y_1) \leq_1 (x_2, y_1) \vee (x_2, y_2) \leq_2 (x_2, y_1). \quad (23)$$

In particular, we found that for strict games, a conjunction $C_1 \wedge C_2$ of two binary clauses is a uniqueness condition iff either $C_1$ or $C_2$ is a uniqueness condition.

The first condition is equivalent to

$$(x_2, y) \leq_1 (x_1, y) \supset (x_1, y) \leq_2 (x_2, y) \quad (24)$$

as in strict games, $s_1 \leq_1 s_2$ iff $s_1 <_1 s_2 \vee s_1 = s_2$. This is exactly one of the two conjuncts in the condition (9) for weakly unilaterally competitive games.

Now swap the roles of the two players in (24), we get the following condition

$$(x, y_1) \leq_2 (x, y_2) \supset (x, y_2) \leq_1 (x, y_1), \quad (25)$$

which is exactly the other conjunct in the condition (9).

In the following, we call a game that satisfies (24) a *weakly unilaterally competitive for player 1*, and a game that satisfies (25) a *weakly unilaterally competitive for player 2*. Thus a game is weakly unilaterally competitive if it is weakly unilaterally competitive for both players. The following example shows that a game can be weakly unilaterally competitive for player 1 but not for player 2.

| | |
|---|---|
| 2, 1 | 3, 4 |
| 1, 2 | 4, 3 |

This example also shows that a weakly unilaterally competitive game for player 1 may not be *almost strictly competitive* [Aumann, 1962]: a game is almost strictly competitive if

1. the set of payoff vectors of the Nash equilibria is the same as the set of payoff vectors of the *twisted equilibria*; and
2. there is a Nash equilibrium that is also a twisted equilibrium,

where $(a, b)$ is a twisted equilibrium if no player can decrease the payoff of the other player by a unilateral change of his own strategy: for every $a' \in A$ ($b' \in B$), $(a, b) \leq_2 (a', b)$ ($(a, b) \leq_1 (a, b')$). For this example, it is easy to see that the only equilibrium of the game, $(4, 3)$, is not a twisted equilibrium.

As it turns out, (24) and (25) are the only non-trivial conditions. The last two conditions (22) and (23) can never be satisfied by games larger or equal to 3x3. The remaining five conditions (17) - (21) are games with strictly dominant strategies.

**Proposition 5** *If $G$ is a strict game and satisfies one of the conditions (17) - (21), then one of the players has a strictly dominant strategy in $G$.*

## Strictly competitive game classes

To summarize, for strict games, the only interesting uniqueness conditions that can be expressed by a conjunction of two binary clauses and include games that do not have dominant strategies are weakly unilaterally competitive conditions for individual players, (24) and (25). This led us to wonder if these two conditions are also necessary conditions for a strict game to have a unique Nash equilibrium. However, it is easy to see that this is not the case. In fact, a universal condition like (8) can never be both a necessary and a sufficient condition for a game to have unique Nash equilibria. This is because for any given game, no matter how many Nash equilibria it has, we can always extend it by one more strategy for each player, and make it into a game with a unique Nash equilibrium by assigning payoffs large enough to a profile made of the two new strategies. However, if a universal condition is satisfied by a game, it is also satisfied by any of its sub-games.

This led us to consider not individual games, but classes of games under certain equivalence relation.

Two games $G_1 = (A, B, \leq_1, \leq_2)$ and $G_2 = (A', B', \leq'_1, \leq'_2)$ are *unilaterally order equivalent*[1] if

- $A = A'$, and $B = B'$.
- For every $a \in A$, $b, b' \in B$, $(a, b) \leq_2 (a, b')$ iff $(a, b) \leq'_2 (a, b')$.
- For every $b \in B$, $a, a' \in A$, $(a, b) \leq_1 (a', b)$ iff $(a, b) \leq'_1 (a', b)$.

They are *best-response equivalent* [Rosenthal, 1974] if for all $a \in A$, $B_2(a)$ in $G_1$ and $G_2$ are the same, and for all $b \in B$, $B_1(b)$ in $G_1$ and $G_2$ are the same. Clearly, if $G_1$ and $G_2$ are unilaterally order equivalent, then they are also best-response equivalent, but the converse is not true in general. Both notions of equivalence preserve Nash equilibria.

We have the following result.

**Theorem 3** *A strict game has at most one Nash equilibrium iff it is best-response equivalent to a strictly competitive game.*

To prove this theorem, for any given game $G = (A, B, \leq_1, \leq_2)$, we associate the following relation with it:

$$\begin{aligned} R &= \{((a,b),(a',b)) \mid a' \in B_1(b), a \notin B_1(b)\} \cup \\ &\quad\ \{((a,b),(a,b')) \mid b \in B_2(a), b' \notin B_2(a)\}. \end{aligned}$$

The theorem then follows from the following two lemmas about $R$. Notice that relation $R$ can also be viewed as a directed graph: if $(s, s') \in R$, then there is an arc from $s$ to $s'$.

**Lemma 1** *$G$ is best-response equivalent to a strictly competitive game iff $R$ has no cycle.*

**Lemma 2** *If $G$ is a strict game and $R$ has a cycle, then $G$ has more than one Nash equilibria.*

[1] We call it unilaterally order equivalence to distinguish it from *order equivalence* [Rosenthal, 1974] that requires both the row and column orders in the two games to be the same for both players.

Theorem 3 does not hold if we replace best-response equivalence by unilaterally order equivalence. The following strict game

| | | |
|---|---|---|
| 1, 1 | 6, 2 | 8, 3 |
| 2, 6 | 5, 5 | 9, 4 |
| 3, 9 | 4, 8 | 7, 7 |

has a unique equilibrium $(3, 9)$ but is not unilaterally order equivalent to any strictly competitive games.

Theorem 3 does not hold either for general two-person games. For instance, the following game

| | |
|---|---|
| 1, 1 | 2, 2 |
| 2, 2 | 1, 1 |

has a unique equilibrium $(2, 2)$ but is not best-response equivalent to any strictly competitive games.

# Concluding remarks

We have provided a logical framework for doing computer-aided theorem discovery in two-person game theory, and applied it successfully to the task of discovering classes of two-person games with unique Nash equilibria. The general methodology used here is similar to the one used by Lin [2004] for discovering state invariants in planning.

There are many directions for future work. An obvious one is to see how interesting theorems can be discovered using Theorem 1 on classes of games that do not have any Nash equilibrium.

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