The Knuth-Yao Quadrangle Inequality Speedup is a Consequence of Total Monotonicity

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Motivation

- Nothing new: material here goes back 20-30 years.
- There are two classic Dynamic Programming Speedups in the literature
  - Knuth-Yao Quadrangle Inequality Speedup
  - SMAWK Algorithm for Totally Monotone Matrices
- They “feel” similar. Are they related?
- Both techniques have been used quite often in improving DP algorithms for various type of constrained source coding.

Outline

- Background
  - Kunth-Yao (KY) Quadrangle Inequality (QI) Speedup
  - SMAWK Algorithm for finding
    Row Minima of Totally Monotone (TM) Matrices
- The D^d Decomposition
  A transformation from QI to TM such that
  SMAWK solves KY problem as quickly as KY.
- The L^m and R^m Decompositions
  Another transformation from QI to TM that
  (1) implies KY speedup and (2) enables online solution.

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  - D. E. Knuth (1971) and F. F. Yao (1980,1982)
  - $\Theta(n)$ speedup: $O(n^3)$ down to $O(n^2)$

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  - $\Theta(n)$ speedup: $O(n^2)$ down to $O(n)$
- How are the two techniques related?

Quadrangle Inequality

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  - Construct a search tree for n keys
  - n internal nodes corresponds to successful search
  - $n+1$ external nodes corresponds to unsuccessful search
  - Minimize the expected number of comparisons

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- Optimal BST
  - Construct a search tree for n keys
  - n internal nodes corresponds to successful search
  - $n+1$ external nodes corresponds to unsuccessful search
  - Minimize the expected number of comparisons

- Solution: Dynamic Programming

$$B_{i,j} = \begin{cases} w(i,j) + \min_{i<t\leq j}\{B_{i,t-1} + B_{t,j}\} & (i<j) \\ 0 & (i=j) \end{cases}$$

for some $w(i,j)$ that can be computed in $O(1)$ time.

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- Standard Calculation

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  $n = 6$

	0	1	2	3	4	5	6
0	0						
1		0					
2			0				
3				0			
4					0		
5						0	
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	0	1	2	3	4	5	6
0	0	230					
1		0	146				
2			0	75			
3				0	43		
4					0	44	
5						0	52
6							0

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1		0	146	260			
2			0	75	141		
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	0	1	2	3	4	5	6
0	0	230	433	586			
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2			0	75	141	250	
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	0	1	2	3	4	5	6
0	0	230	433	586	698		
1		0	146	260	349	491	
2			0	75	141	250	357
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  - An example:

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	0	1	2	3	4	5	6
0	0	230	433	586	698	862	
1		0	146	260	349	491	624
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  - Diagonal by diagonal
  - An example: $n = 6$
  - Running time: $O(n^3)$

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$$K_B(i,j-1) \leq K_B(i,j) \leq K_B(i+1,j)$$

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	$j-1$	j
i	$K_B(i,j-1)$	$K_B(i,j)$
$i+1$		$K_B(i+1,j)$

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0		0					
1			1				
2				2			
3					3		
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2				2	2	2	
3					3	4	4
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2				2	2	2	4
3					3	4	4
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6							

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  - Function $f(i,j), (0 \le i \le j \le n)$
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$$f(i,j) + f(i',j') \le f(i',j) + f(i,j')$$

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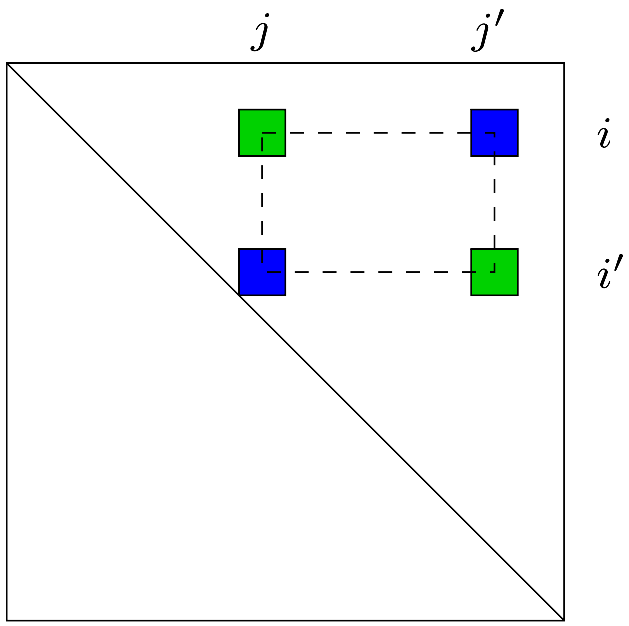

Speedup using Quadrangle Inequality

$$B_{i,j} = w(i,j) + \min_{i<t\leq j}\{B_{i,t-1} + B_{t,j}\}$$

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  - The specific $w(i,j)$ satisfies QI (and the additional constraints).

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7	2	4	3	8	9	$\text{RM}_M(1) = 2$
5	1	5	1	6	5	$\text{RM}_M(2) = 4$
7	1	2	0	3	1	$\text{RM}_M(3) = 4$
9	4	5	1	3	2	$\text{RM}_M(4) = 4$
8	4	5	3	4	3	$\text{RM}_M(5) = 6$
9	6	7	5	6	5	$\text{RM}_M(6) = 6$

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8	4	5	3	4	3	RM$_M(5) = 6$
9	6	7	5	6	5	RM$_M(6) = 6$

- An $m \times n$ matrix M is Totally Monotone (TM) if every 2×2 submatrix is Monotone.

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  - Usually $\Theta(n)$ speedup: $O(n^2)$ down to $O(n)$.

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  TM property is often established via Monge property.

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  TM property is often established via Monge property.

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  $$M_{i,j} + M_{i',j'} \leq M_{i',j} + M_{i,j'}$$

- Theorems

  M is Monge $\Rightarrow$ M is Totally Monotone

  M is Monge $\not\Leftarrow$ M is Totally Monotone

The Monge Property

Quadrangle Inequality

Function $f(i,j)$

$\forall i \leq i' \leq j \leq j'$

$$f(i,j) + f(i',j') \leq f(i',j) + f(i,j')$$

Monge

Matrix M

$\forall i \leq i'$ and $\forall j \leq j'$

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- QI vs. Monge

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$\forall i \leq i' \leq j \leq j'$

$$f(i, j) + f(i', j') \leq f(i', j) + f(i, j')$$

Monge

Matrix M

$\forall i \leq i'$ and $\forall j \leq j'$

$$M_{i,j} + M_{i',j'} \leq M_{i',j} + M_{i,j'}$$

- QI vs. Monge
  - Different names for same type of inequality.

The Monge Property

Quadrangle Inequality	Monge
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      Only need the row minima, but not other entries.

Relationship?

Quadrangle Inequality Totally Monotone (Monge)

Relationship?

Quadrangle Inequality	Totally Monotone (Monge)
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Relationship?

Quadrangle Inequality

A matrix to be calculated

Need all $O(n^2)$ entries

Totally Monotone (Monge)

A matrix given implicitly

Need only $O(n)$ row minima

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Quadrangle Inequality	Totally Monotone (Monge)
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- This talk

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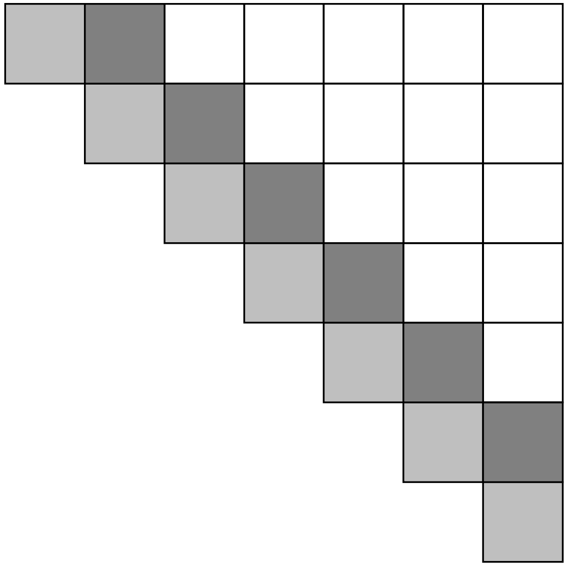

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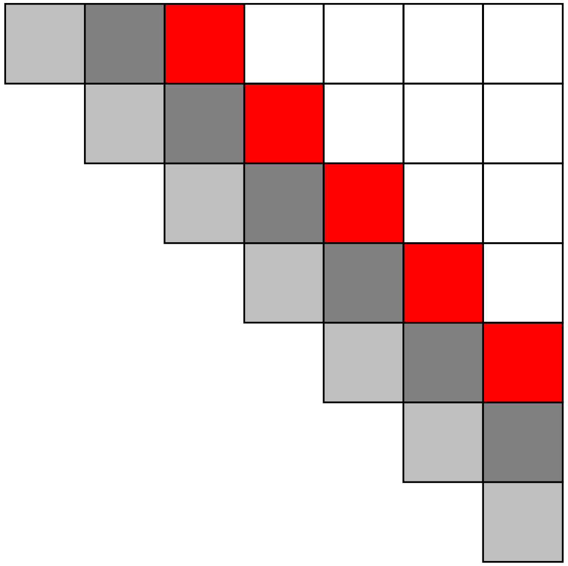

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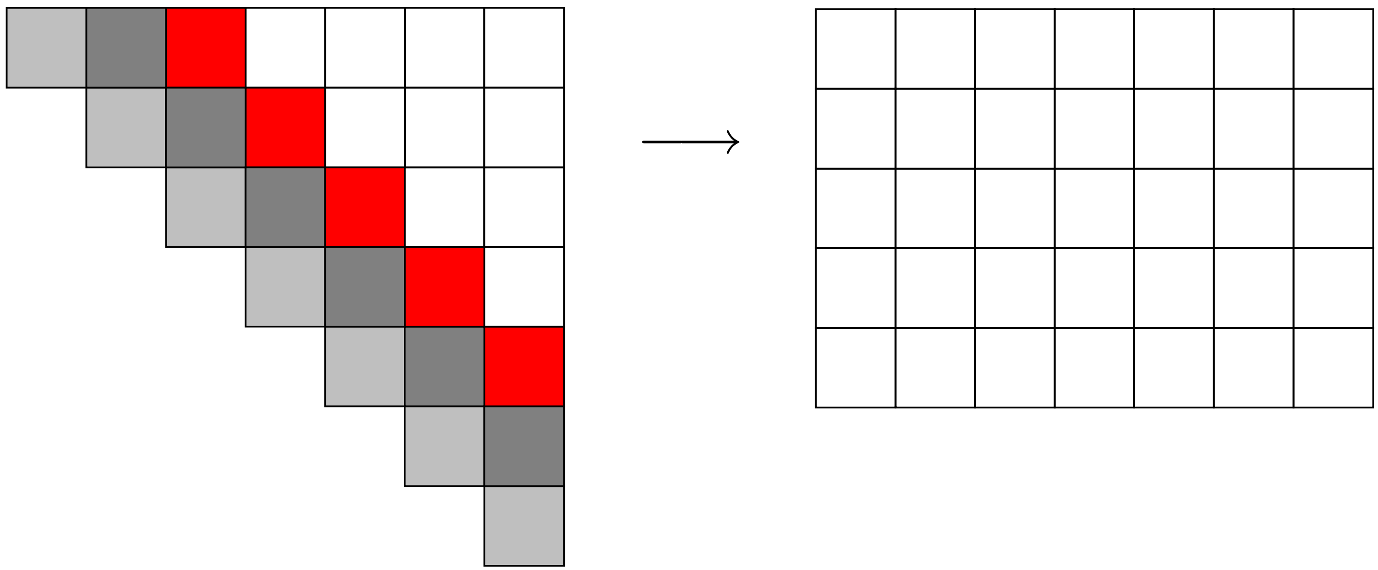

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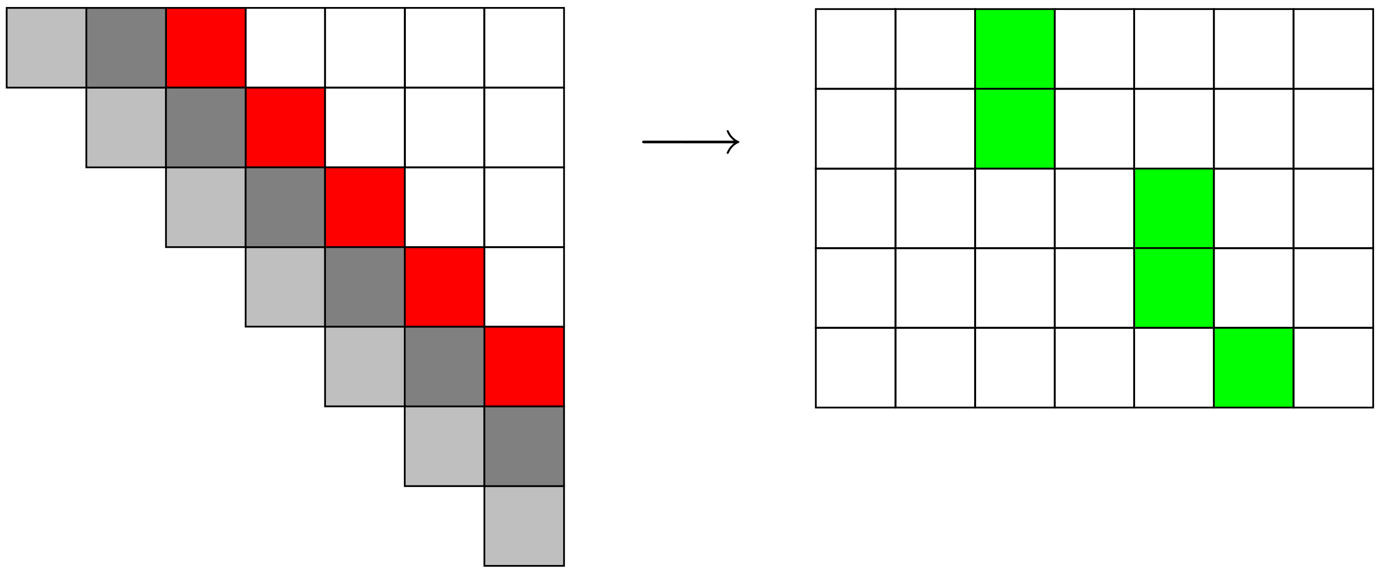

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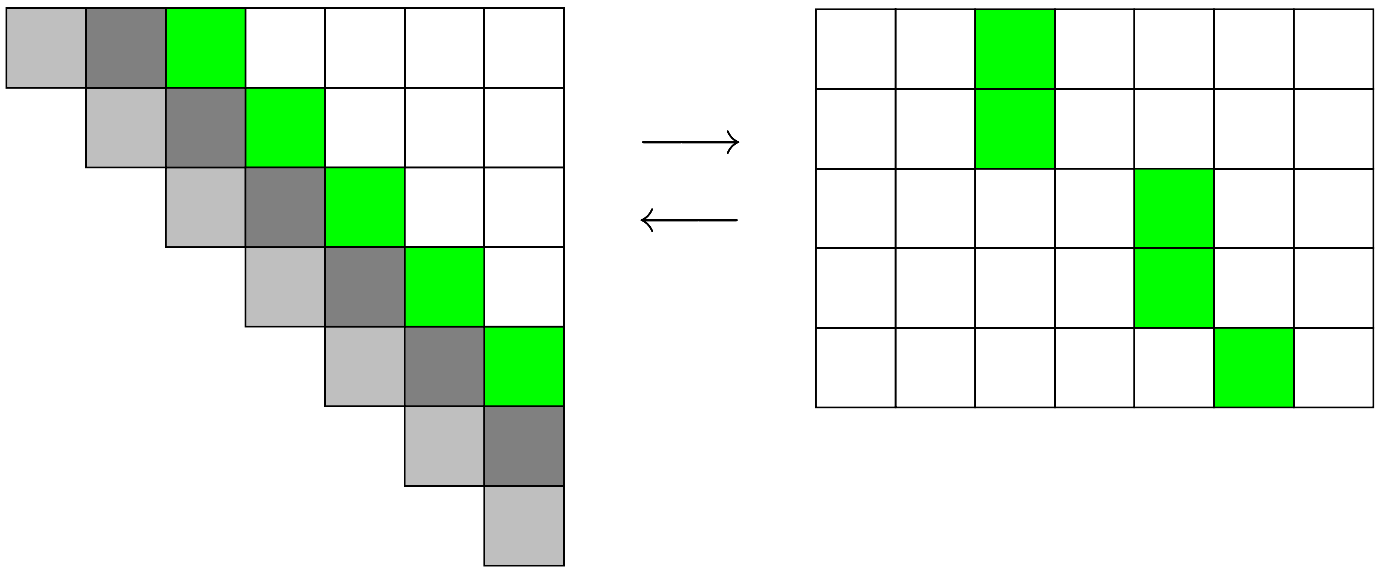

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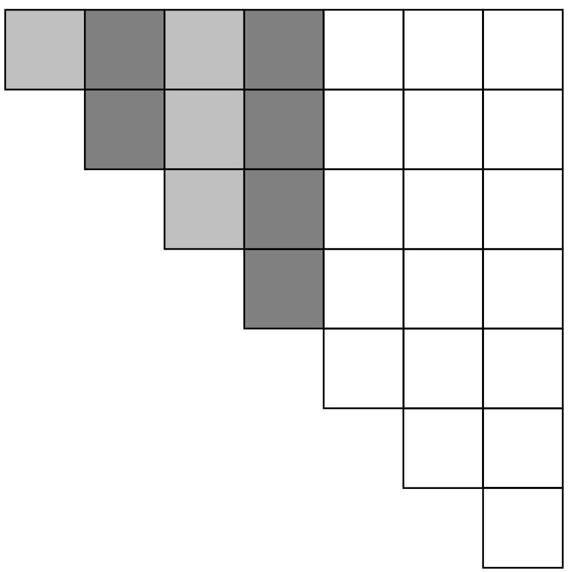

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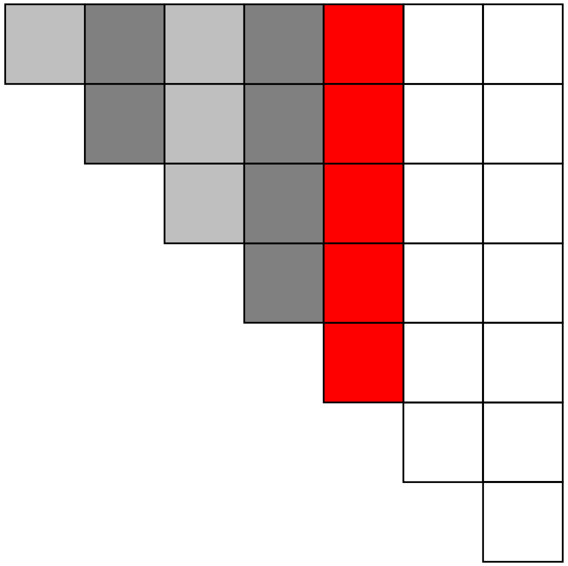

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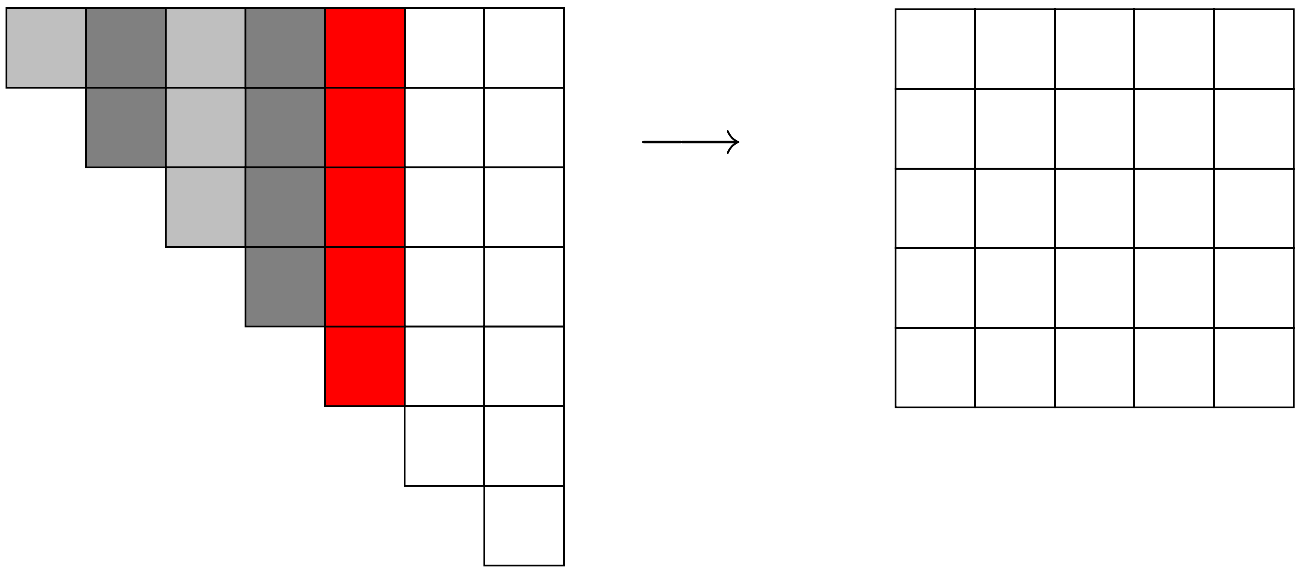

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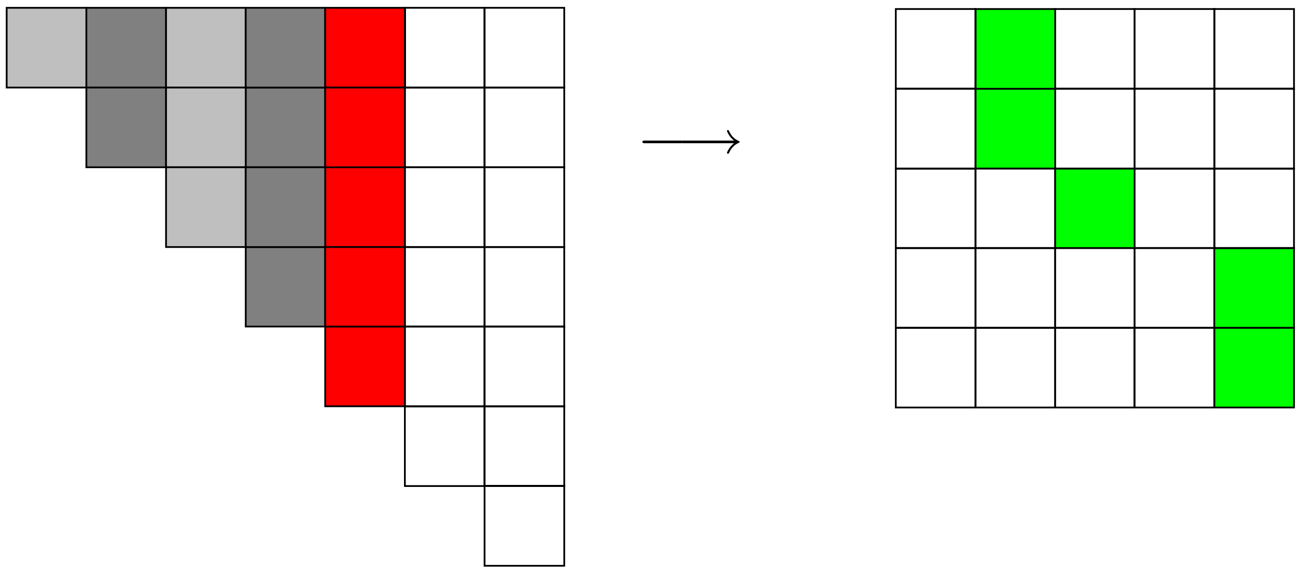

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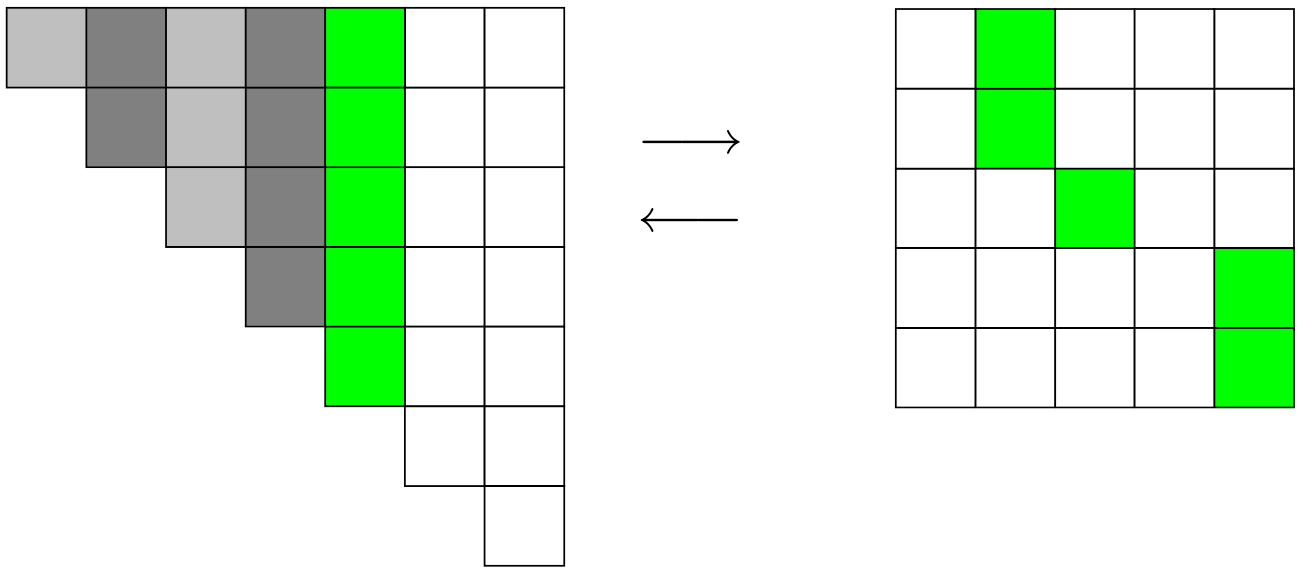

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  - Define $(n-d+1) \times (n+1)$ matrix D^d

    $$D^d_{i,j} = \begin{cases} w(i,i+d) + \{B_{i,j-1} + B_{j,i+d}\} & \text{if } 0 \leq i < j \leq i+d \leq n \\ \infty & \text{otherwise} \end{cases}$$

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  D^d is Monge, for each $1 \le d < n$.
- For fixed d, SMAWK can be used to find all the $B_{i,i+d}$ in $O(n)$ time.
  - $\Rightarrow O(n^2)$ time for all D^d.

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  - $O(n)$ time for each column $\Rightarrow O(n^2)$ in total.

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Finding row minima in totally monotone matrices with limited dependency.

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Entries of column j can depend on the row minima of rows i where $M_{i,j} = \infty$.

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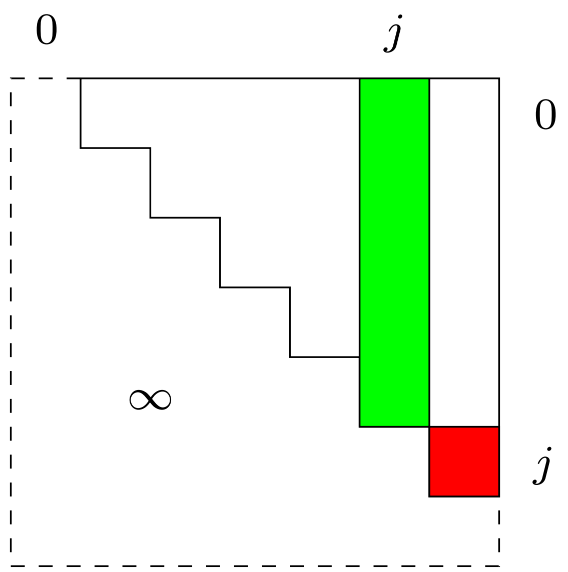

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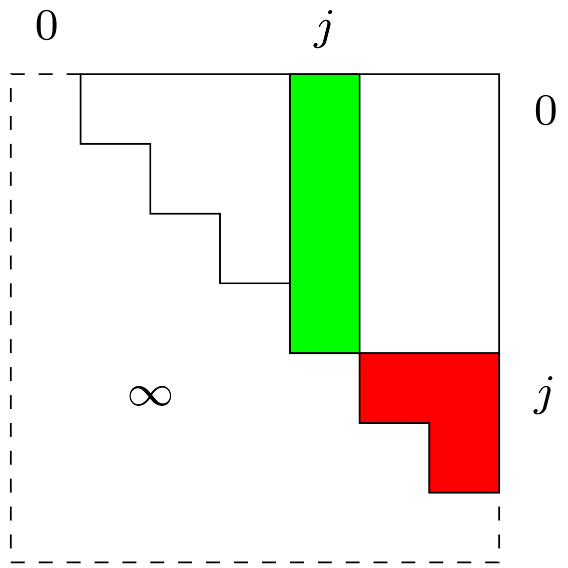

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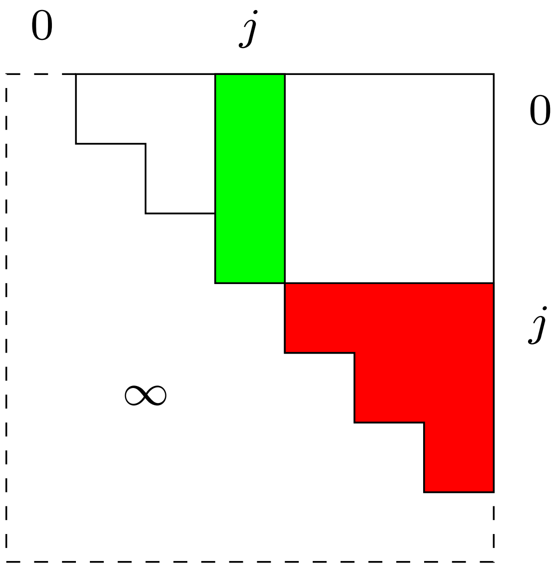

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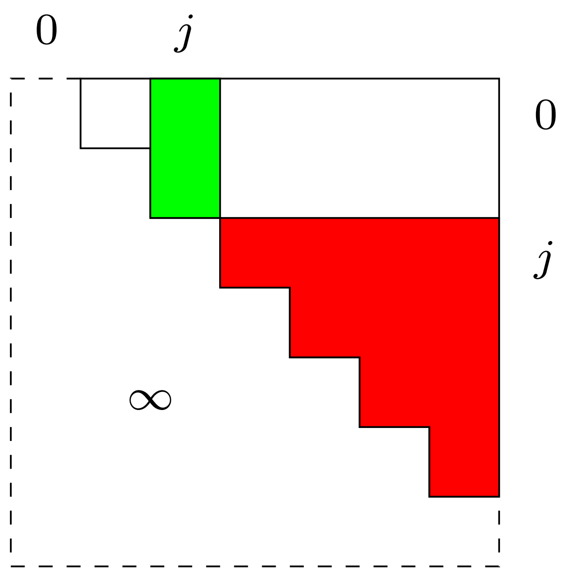

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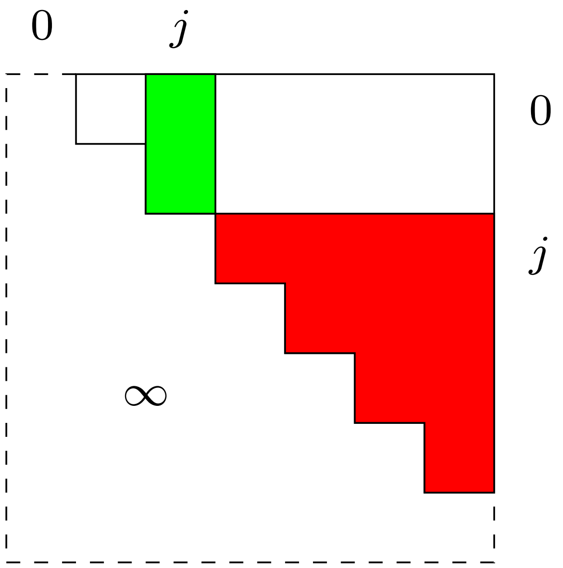

R^m satisfies the condition of LARSCH.

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  Input $=$ (Key$_l, \ldots,$ Key$_r$)

	1	2	3	4	5	6
1						
2		0	75	141	250	
3			0	43	119	
4				0	44	
5					0	
6						

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	1	2	3	4	5	6
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2		0	75	141	250	357
3			0	43	119	204
4				0	44	121
5					0	52
6						0

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- Using L^m and R^m decomposition
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