A Generic Top-Down Dynamic-Programming Approach to Prefix-Free Coding

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This talk: a simple technique for speeding up the DP for many prefix-free coding variants.

- Introduction
  - A Quick Review of Prefix-Free Coding
  - New Results

- The Basic Top-Down Dynamic Programming Technique

- The Speedup

- Conclusion & Comments

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- W is prefix-free if $\forall w, w' \in W$, w is not a prefix of w'.

  E.g., $\begin{matrix} 10 \\ 11 \\ 01 \end{matrix}$ is prefix free; $\begin{matrix} 1 \\ 11 \\ 01 \end{matrix}$ is not prefix-free;

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  - Same problem as finding a tree with n leaves weighted by P that minimizes weighted external path length.

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Correspondence between codes on $\Sigma = \{0, 1\}$ and binary trees.

(or codes on general Σ and $|\Sigma|$-ary trees)

Let 0 denote a left edge and 1 a right edge.

Codewords are leaves; Create paths to all codewords.

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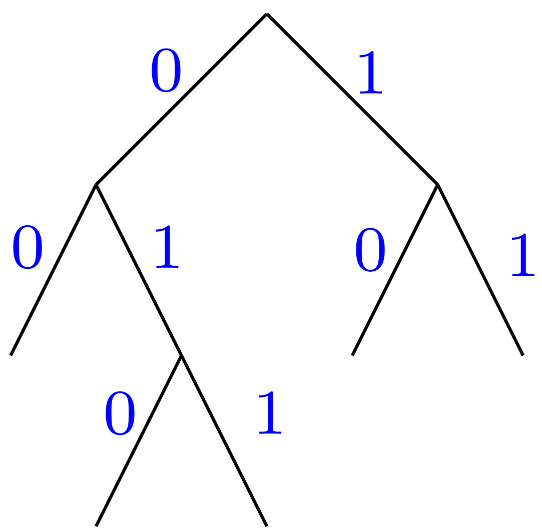

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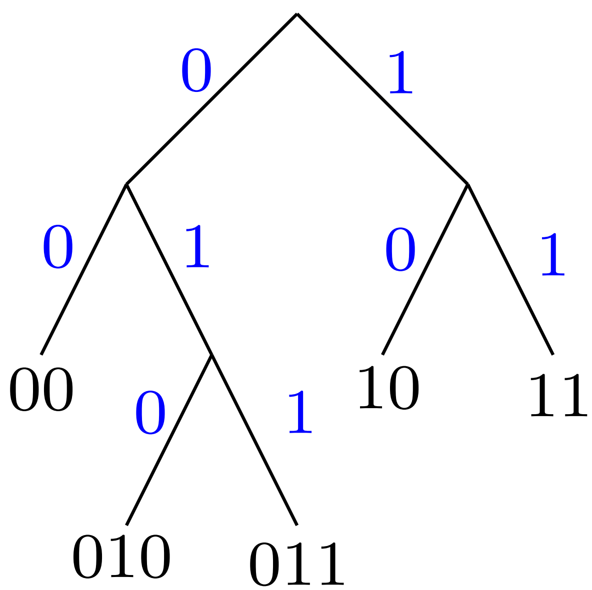

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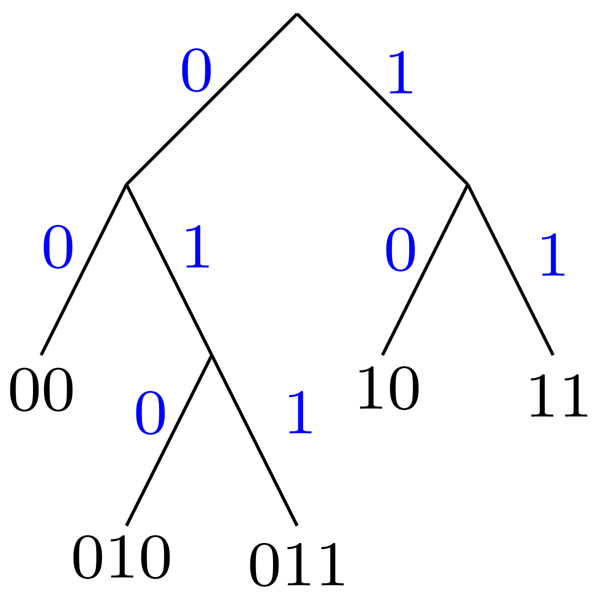

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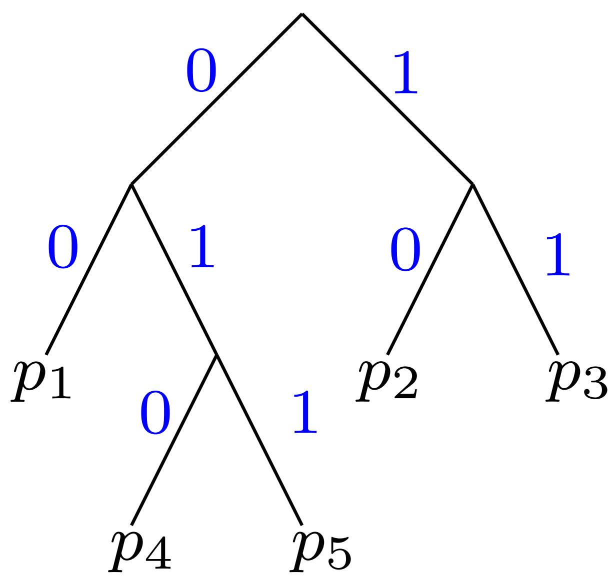

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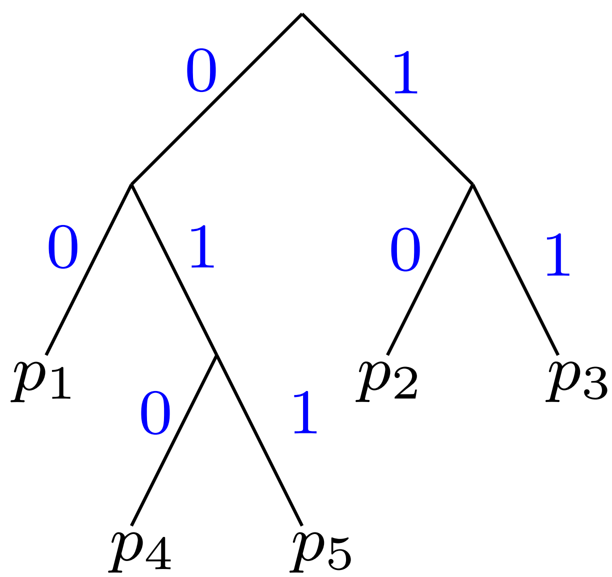

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Weighted external path length is

$$\sum_{i=1}^{n} |w_i| p_i$$

which is $Cost(W, P)$.

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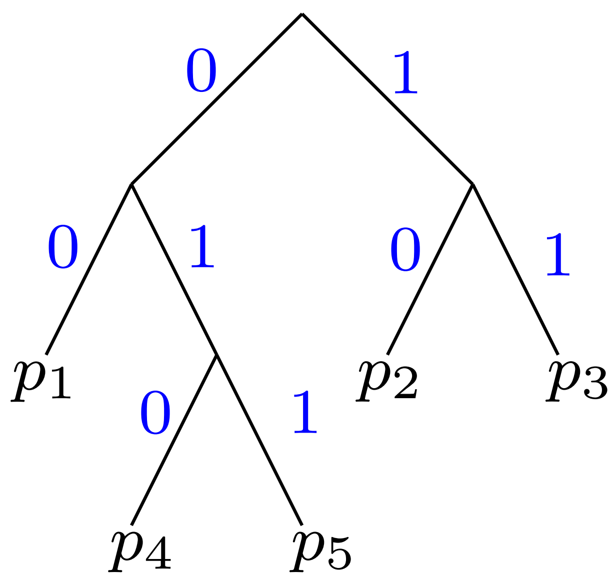

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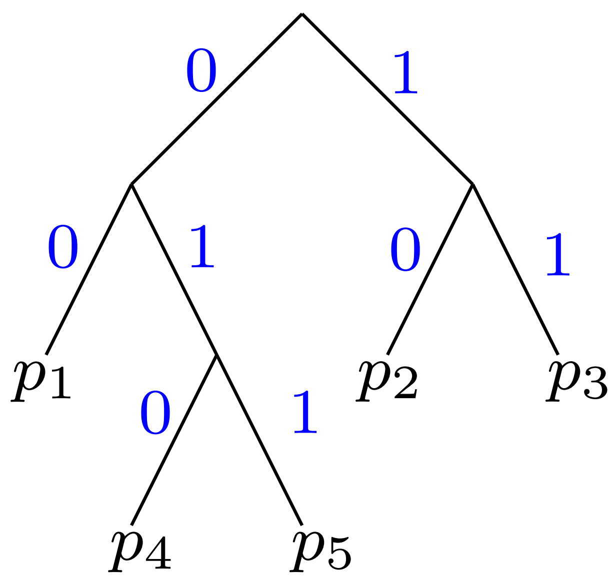

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Change problem to

Given P,
Find Min-Cost Tree

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- Mixed-Radix Coding:
  Size of alphabet depends upon position of character within codeword,
  e.g., arity of node depends upon level in the tree.

New Results

With exception of Length-Limited Coding (which takes advantage of Schieber's (1998) min-cost length-limited paths in Monge-graphs result) we improve the DP-based algorithms for all problems on previous page.

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[1]: Chu and Gill (1992)
[2]: Baer (2008)
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DP creates a search space and calculates *optimal* cost
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This leads to lower amortized time per optimal-cost calculation.

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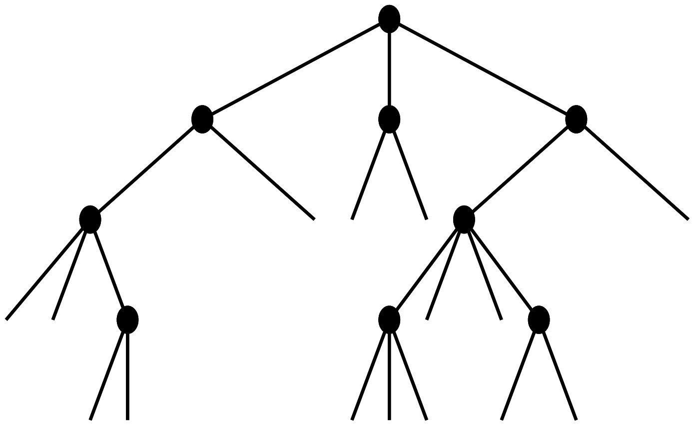

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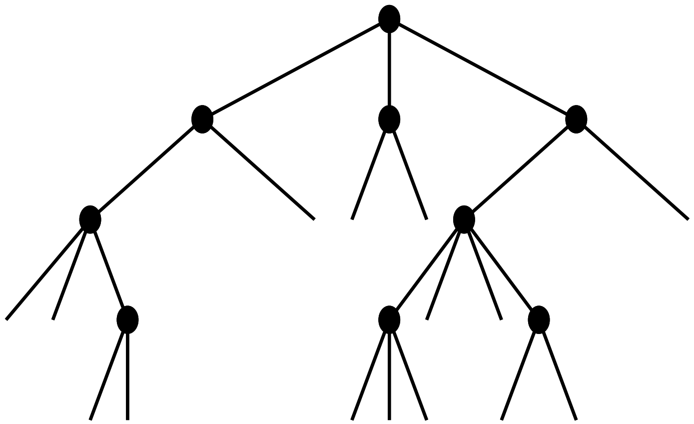

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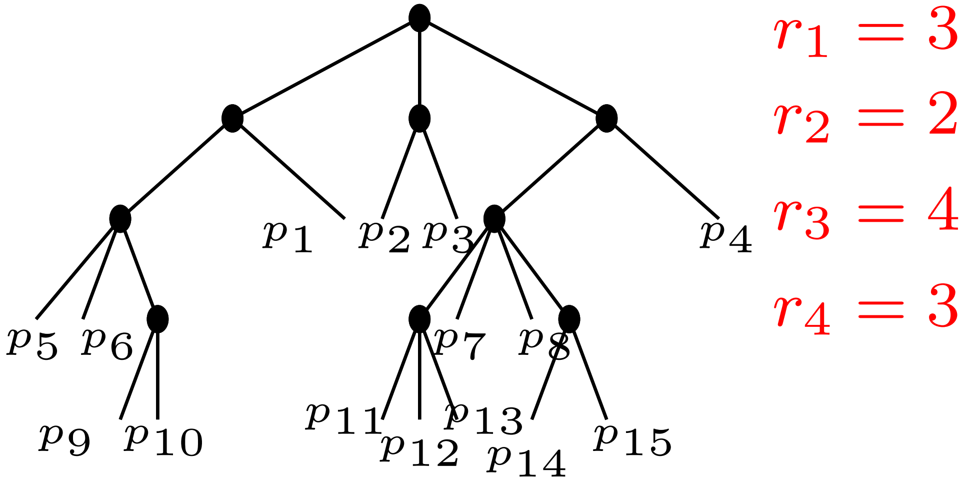

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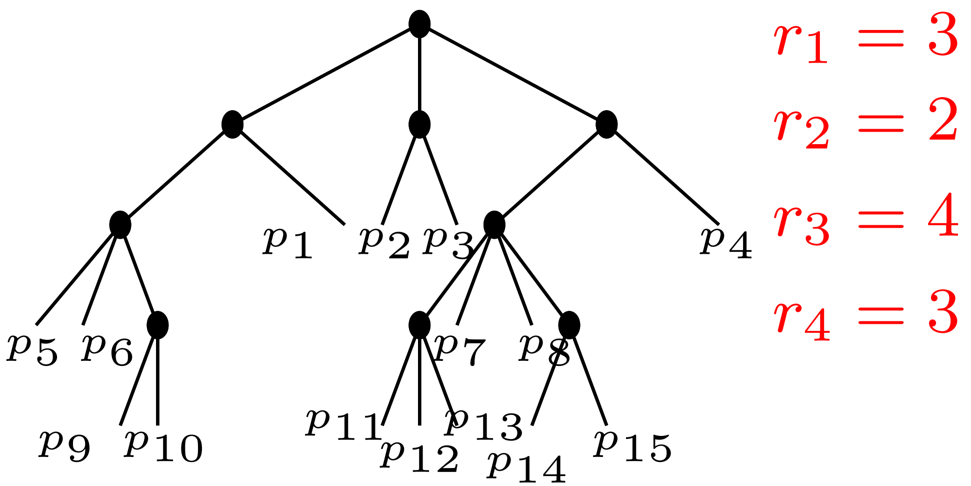

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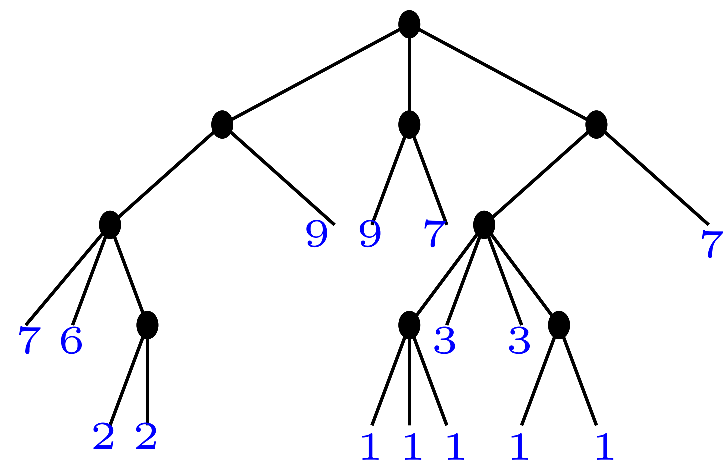

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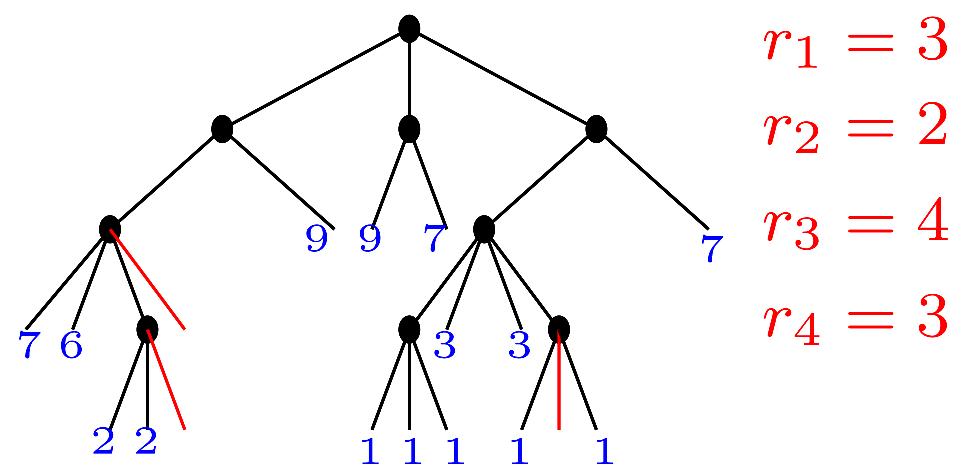

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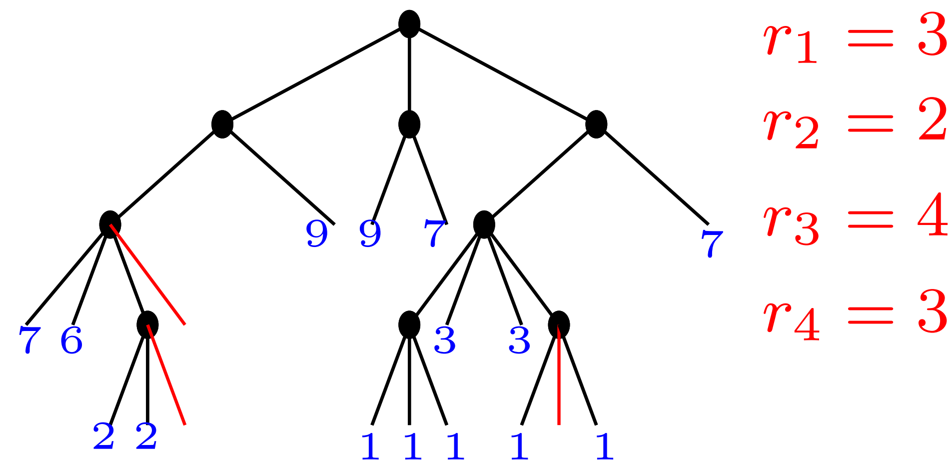

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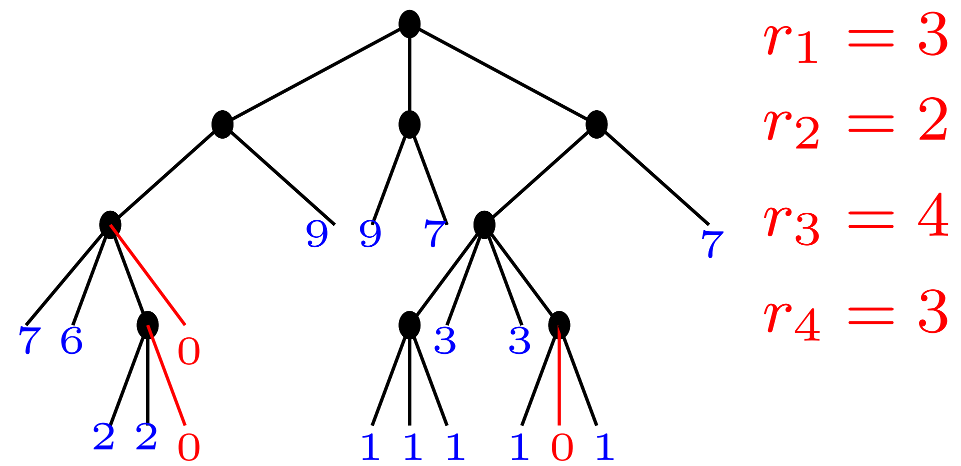

W.L.O.G. assume that the p_i are sorted in non-increasing order

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Can ensure this by padding P with arbitrarily many 0s.

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We illustrate the technique by showing how to speed up *mixed-radix* coding from $O(n^4)$ down to $O(n^3)$. The same technique, with various bells and whistles added, speeds up all of the other problems.

In mixed -radix coding, input is weight set $P = \{p_1, \ldots, p_n\}$ and *arity* list $R = \{r_1, r_2, r_3 \ldots\}$.

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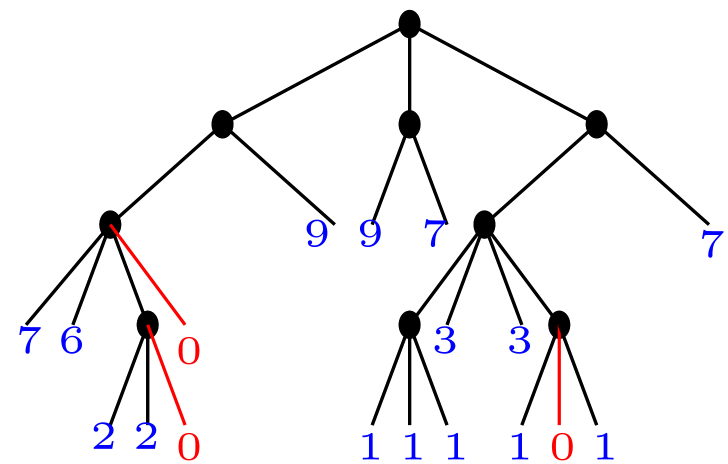

$r_1 = 3$

$r_2 = 2$

$r_3 = 4$

$r_4 = 3$

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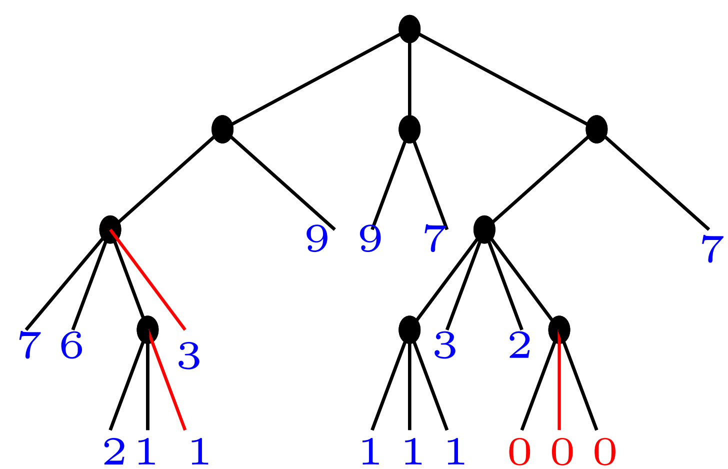

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Standard technique: e.g., Golin & Rote '98, Dolev, Korach & Yukelson '99, Chan & Golin '00, Baer '08

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	d	m	b
•	0	0	1

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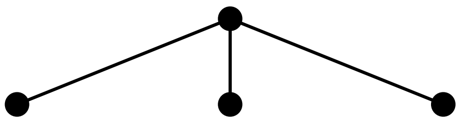

d	m	b
0	0	1
1	0	3

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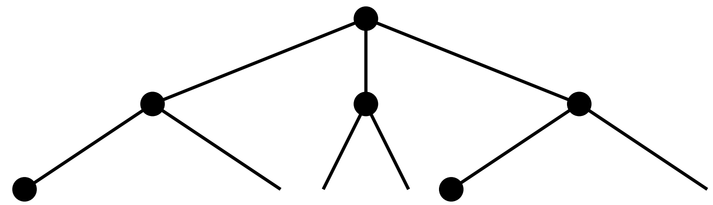

d	m	b
0	0	1
1	0	3
2	4	2

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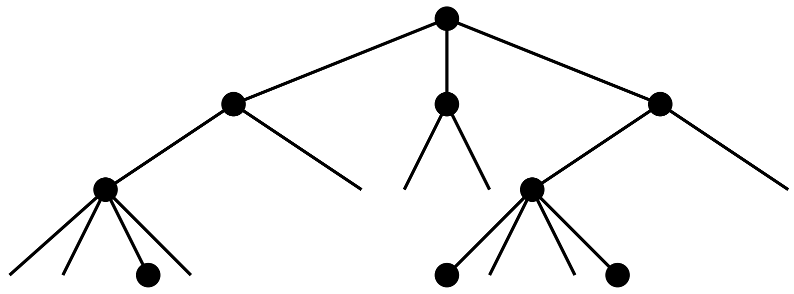

d	m	b
0	0	1
1	0	3
2	4	2
3	9	3

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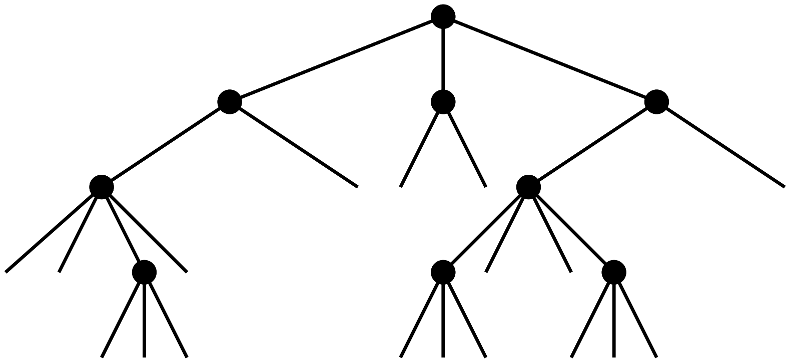

d	m	b
0	0	1
1	0	3
2	4	2
3	9	3
4	18	0

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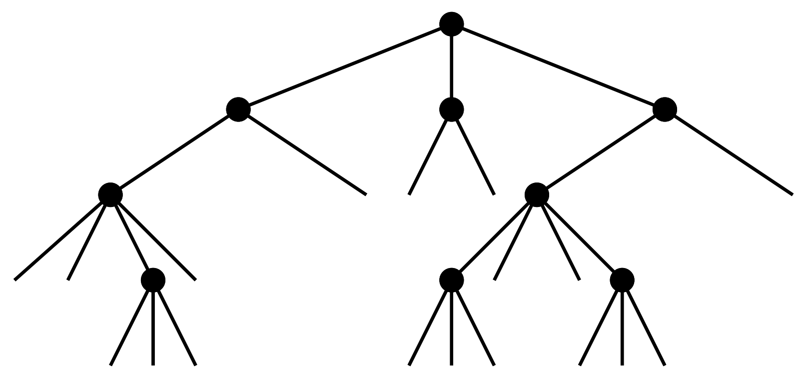

d	m	b
0	0	1
1	0	3
2	4	2
3	9	3
4	18	0

Will also keep track of cost "so far".

$\sum_{t=1}^{m} p_i d_i + i \sum_{t>m} p_t$

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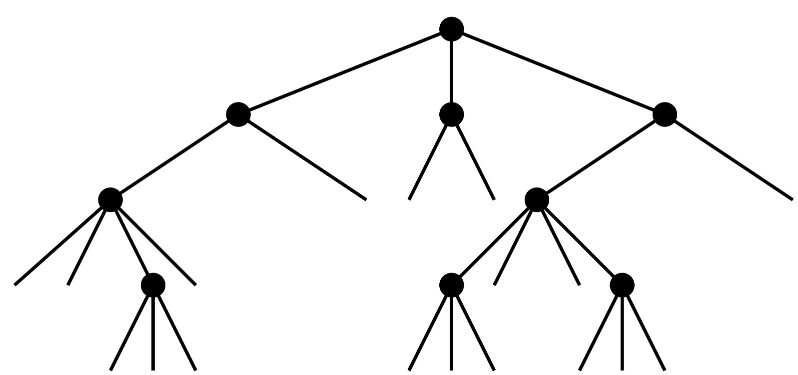

d	m	b
0	0	1
1	0	3
2	4	2
3	9	3
4	18	0

Will also keep track of cost "so far".

$$\sum_{t=1}^{m} p_i d_i + i \sum_{t>m} p_t$$

Ex: $P = \{3,3,3,3,3,\ 2,2,2,2,2,\ 1,1,1,1,1,\ 0,0,\ldots\}$

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1	0	3	
2	4	2	
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4	18	0	

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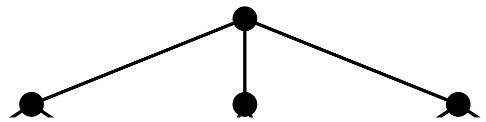

d	m	b	c
0	0	1	0
1	0	3	30
2	4	2	
3	9	3	
4	18	0	

Will also keep track of cost "so far".

$$\sum_{t=1}^{m} p_i d_i + i \sum_{t>m} p_t$$

Ex: $P = \{\underbrace{3,3,3,3,3,\ 2,2,2,2,2,\ 1,1,1,1,1,\ 0,0,\ldots}_{X1}\}$

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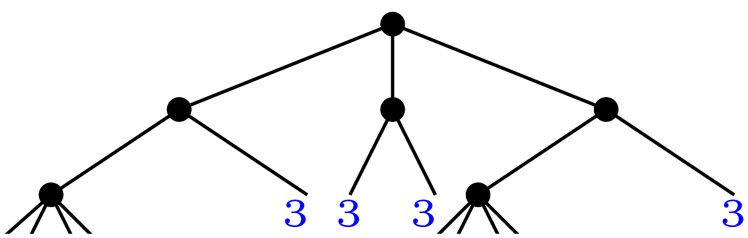

d	m	b	c
0	0	1	0
1	0	3	30
2	4	2	60
3	9	3	
4	18	0	

Will also keep track of cost "so far".

$$\sum_{t=1}^{m} p_i d_i + i \sum_{t>m} p_t$$

Ex: $P = \{3, 3, 3, 3, \underbrace{3,\ 2, 2, 2, 2, 2,\ 1, 1, 1, 1, 1,\ 0, 0, \ldots}_{X2}\}$

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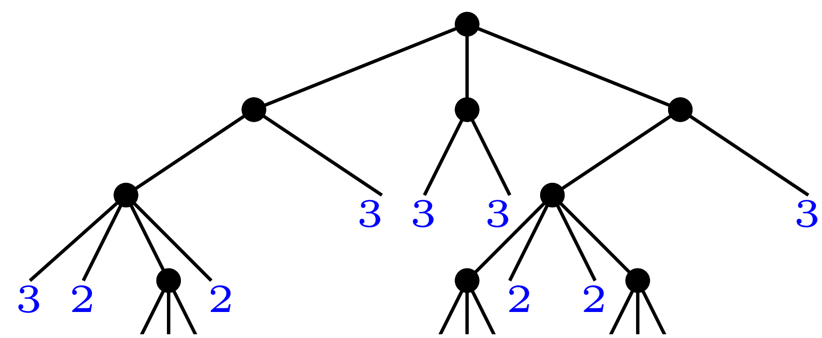

d	m	b	c
0	0	1	0
1	0	3	30
2	4	2	60
3	9	3	78
4	18	0	

Will also keep track of cost "so far".

$$\sum_{t=1}^{m} p_i d_i + i \sum_{t>m} p_t$$

Ex: $P = \{3,3,3,3,3,\ 2,2,2,2,\underbrace{2,\ 1,1,1,1,1,\ 0,0,\ldots}_{X3}.\}$

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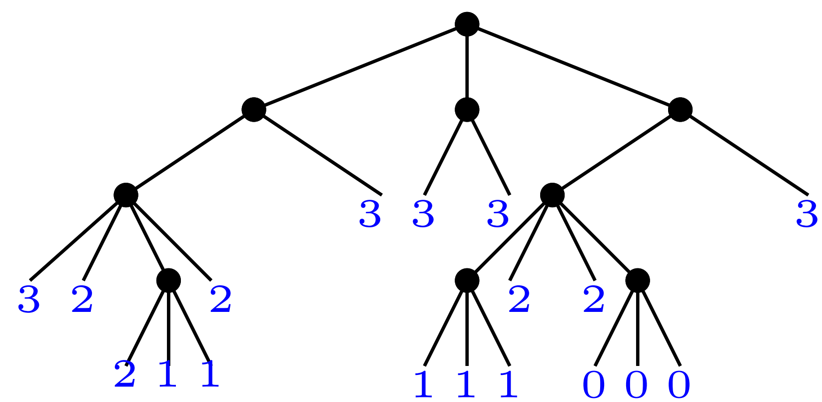

d	m	b	c
0	0	1	0
1	0	3	30
2	4	2	60
3	9	3	78
4	18	0	84

Will also keep track of cost "so far".

$\sum_{t=1}^{m} p_i d_i + i \sum_{t>m} p_t$

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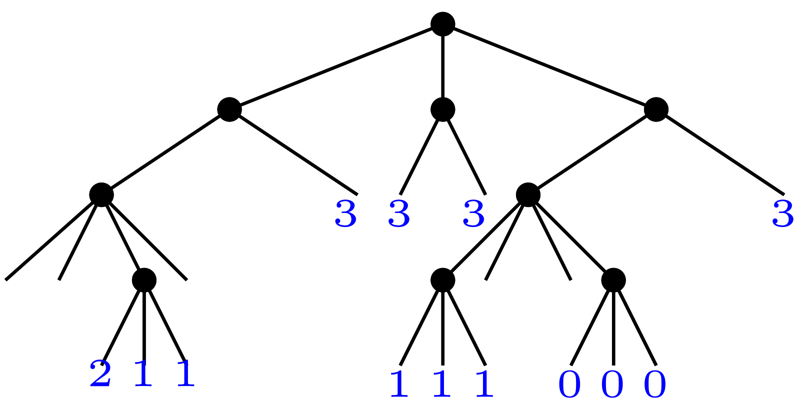


d	m	b	c
0	0	1	0
1	0	3	30
2	4	2	60
3	9	3	
4	18	0	84

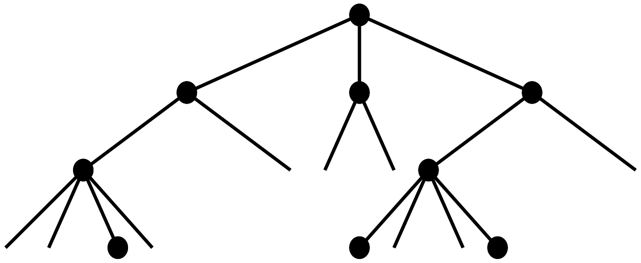
Will also keep track of cost "so far".

$$\sum_{t=1}^{m} p_i d_i + i \sum_{t>m} p_t$$

If $m \geq n$, then "cost so far" is real cost of tree

The Technique

T is an i-level tree if $d(T) \le i$.



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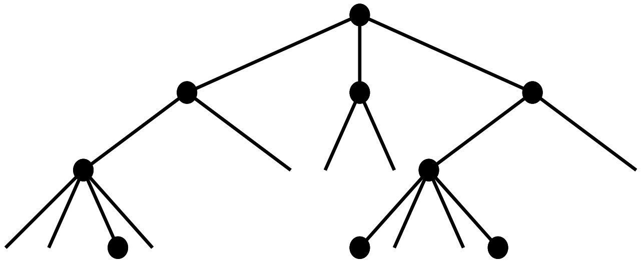
T is an i-level tree if $d(T) \leq i$.



$$sig_i(T) = (m, b)$$

$m = \#$ leaves at depth $\leq i$.

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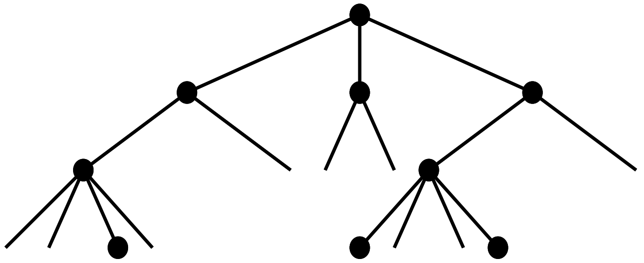

$sig_2(T) = (9, 3)$

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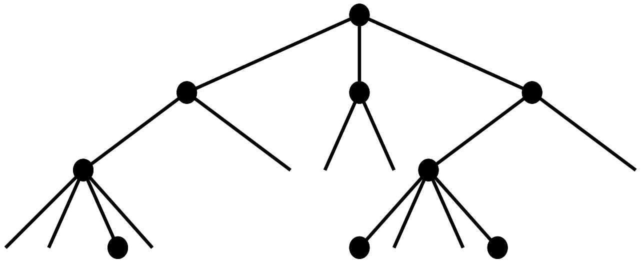

$sig_2(T) = (9, 3)$

$$OPT^i[m, b] = \min \left[cost_i(T) \mid sig_i(T) = (m, b) \right].$$

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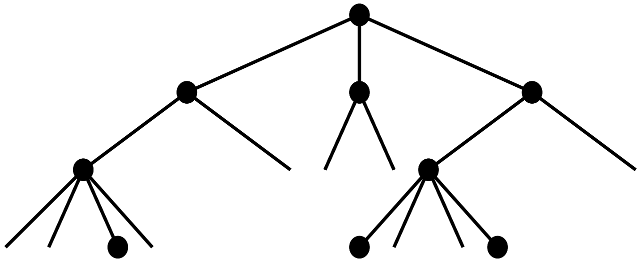

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$$OPT^i[m, b] = \min\left[cost_i(T) \mid sig_i(T) = (m, b)\right].$$

$$\min_{m \geq n}\left(OPT^i(m, 0)\right)$$

is cost of min-cost tree with at least n leaves and depth $\leq i$.

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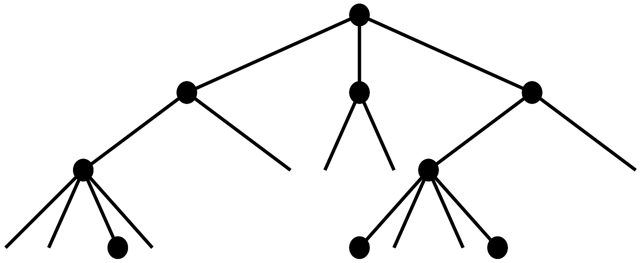

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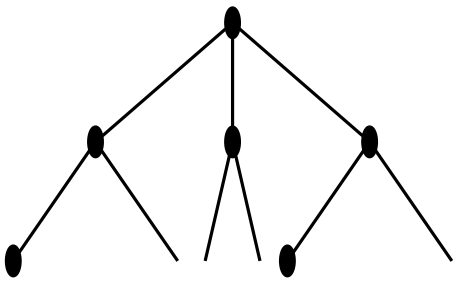
$$\min_{m \geq n}\left(OPT^i(m, 0)\right)$$

is cost of min-cost tree with at least n leaves and depth $\leq i$.

Goal: Find $\min_{m \geq n}\left(OPT^n(m, 0)\right)$ and tree that achieves it

Let T' be an $(i-1)$-level tree with $sig_{i-1}(T) = (m', b')$.



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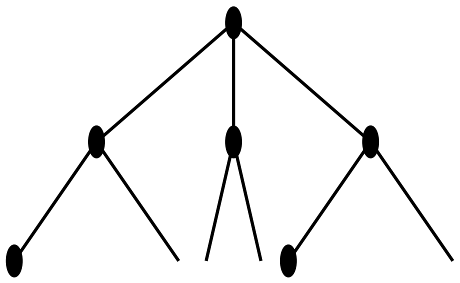

T' is *expanded* to an i level tree T by adding the $r_i b'$ children on level i and choosing b of them to be internal.

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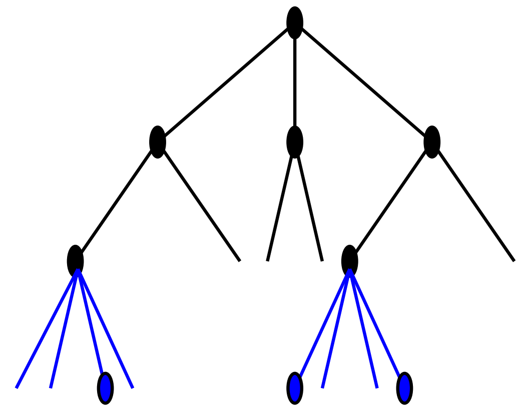

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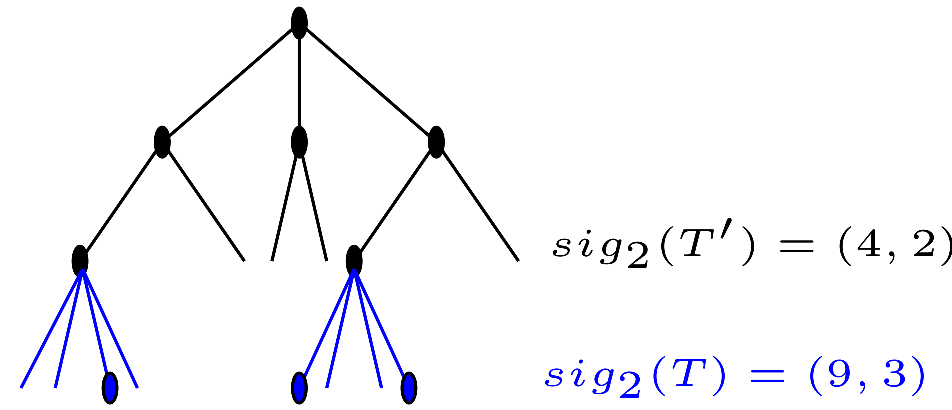

T' is *expanded* to an i level tree T by adding the $r_i b'$ children on level i and choosing b of them to be internal.

Lemma: $m = m' + b'r_i - b$ and $cost_i(T) = cost_{i-1}(T') + \sum_{t>m'} p_t$.

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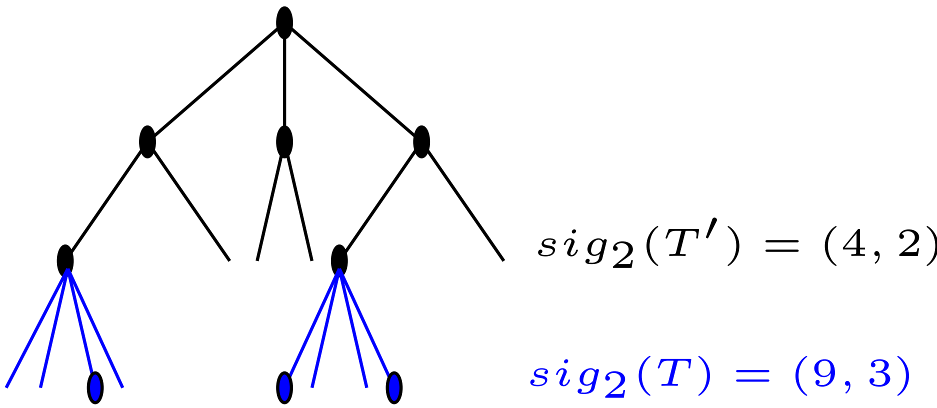

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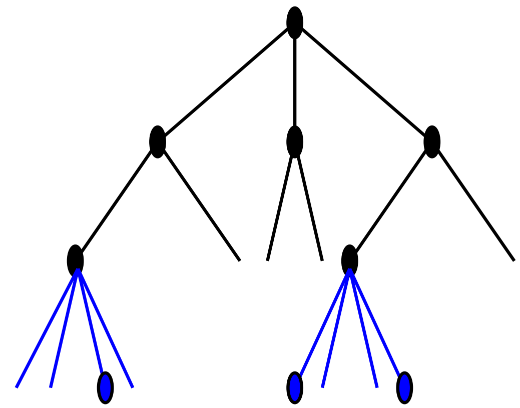
Lemma: $m = m' + b' r_i - b$ and $cost_i(T) = cost_{i-1}(T') + \sum_{t > m'} p_t$.

We say that $(m', b') \xrightarrow{i} (m, b)$ if $\exists T', T$ as above.

Let T' be an $(i-1)$-level tree with $sig_{i-1}(T) = (m', b')$.



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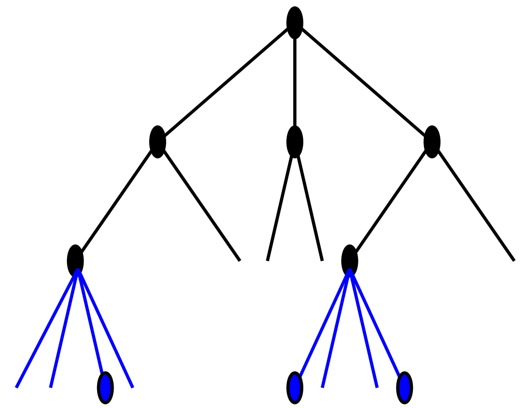

$sig_2(T') = (4, 2)$

$sig_2(T) = (9, 3)$

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$W_{m'}$

We say that $(m', b') \xrightarrow{i} (m, b)$ if $\exists T', T$ as above.

The DP recurrence is thus

$$OPT^i[m, b] = \min_{\{(m',b') \mid (m',b') \xrightarrow{i} (m,b)\}} \left\{ OPT^{i-1}[m', b'] + W_{m'} \right\}.$$

with initial condition $OPT^0[0, 1] = 0.$

- Introduction
  - A Quick Review of Prefix-Free Coding
  - New Results

- The Basic Top-Down Dynamic Programming Technique

- The Speedup

- Conclusion & Comments

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where $m = m' + b'r_i - b$ and $b \leq b'r_i$.

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where $m = m' + b'r_i - b$ and $b \le b'r_i$.

So, only need to check $O(m)$ entries to calculate given $OPT^i[m,b]$.

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where $m = m' + b'r_i - b$ and $b \le b'r_i$.

So, only need to check $O(m)$ entries to calculate given $OPT^i[m,b]$.

Not hard to prove that

if $b > 0$ then $m + b \le n$ and

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We now (finally) see how to reduce this down to $O(n^2)$.
Filling in *all* of the tables and solving the entire problem in $O(n^3)$ time.

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To fully solve the problem, we must fill in,

$OPT^1[m,b], OPT^2[m,b], \ldots, OPT^n[m,b]$.

From above this takes only $O(n^3)$ time,

improving upon the old bound of $O(n^4 \log n)$.

- Introduction
  - A Quick Review of Prefix-Free Coding
  - New Results
- The Basic Top-Down Dynamic Programming Technique
- The Speedup
- Conclusion & Comments

Problem	Previous Best Result	This paper
Mixed Radix Coding	$O(n^4 \log n)$ [1]	$O(n^3)$
Reserved Length Coding (i)	$O(gn^3)$ [2]	$O(gn^2)$
Reserved Length Coding (ii)	$O(g^3 n^3 \log^g n)$ [2]	$O(gn^2 \log n)$
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In reality, all of the above problems can be solved using a top-down DP very similar to the one for mixed-radix coding. The major problem-specific change is in the definition of signature.

Furthermore, almost the same type of batching technique, e.g., defining similar $\mathcal{I}(d)$ and $\mathcal{I}'(d)$ and showing that $OPT[m, b]$ for $(m, b) \in \mathcal{I}(d)$ only depend upon values in $\mathcal{I}'(d)$, holds for all of these problems.

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The batching technique was later shown to be applicable to other coding problems, such as 1-ended coding, that do not (at least obviously) possess the Monge property.

What other problems can this type of batching speed up?