The Probabilistic Complexity of the Voronoi Diagram of Points on a Polyhedron*

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ABSTRACT

It is well known that the complexity, i.e., the number of vertices, edges and faces, of the 3-dimensional Voronoi diagram of n points can be as bad as $\Theta(n^2)$. Interest has recently arisen as to what happens, both in deterministic and probabilistic situations, when the 3-dimensional points are restricted to lie on the surface of a 2-dimensional object. In this paper we consider the situation when the points are drawn from a 2-dimensional Poisson distribution with rate n over a fixed union of triangles in $\mathbb{R}^3$. We show that with high probability the complexity of their Voronoi diagram is $\tilde{O}(n)$.

This implies, for example, that the complexity of the Voronoi diagram of points chosen from the surface of a general fixed polyhedron in $\mathbb{R}^3$ will also be $\tilde{O}(n)$ with high probability.

Categories and Subject Descriptors

G.3 [**Mathematics of Computing**]: Probability And Statistics; I.3.5 [**Computing Methodologies**]: Computer Graphics—*Computational Geometry and Object Modeling*

General Terms

Algorithms

1. INTRODUCTION

Let P be a set of 3-dimensional points. The *Voronoi Diagram* of the points and its dual, the Delaunay triangulation, are extremely well studied structures. The *complexity*, $|VD(P)|$ of the Voronoi diagram is the number of lower dimensional pieces of which it is composed, i.e., the total number of vertices, edges and faces and regions that it contains. It is well known that, in the worst case, the complexity can be as high as $\Theta(n^2)$ [6]. It has also been observed that, if the points are sampled from some types of restricted point sets, the complexity tends, in practice, to be much lower.

The problem of understanding the structure of the 3-dimensional Voronoi diagram of point sets from 2-dimensional surfaces has begun to be of interest in recent years. This is because, as described in [1] and [3], Voronoi diagrams and Delaunay triangulations are of use in several geometric problems, e.g., surface reconstruction, mesh generation and surface modeling. In these problems a 2-dimensional surface is often sampled and then modeled, at least initially, by the Delaunay triangulation of the sample. Many parameters of such algorithms such as their running times and the complexity of their representations, then depend upon the complexity of the Delaunay triangulation (which is the same as that of the Voronoi diagram).

The two results [1] and [3] mentioned above seem to be the first to try and formally analyze the complexity of such Voronoi diagrams. In [1] Attali and Boissonnat prove that if n "well-sampled" points are chosen from a polyhedral surface then the complexity of their Voronoi diagram is $O(n^{7/4})$ where "well-sampled" is defined using the concept of local feature size; if the points are drawn in the same way from the surface of a *convex* polytope this reduces down to $O(n^{3/2})$.[1] In [3] Erikson proves that there is a set of n "well-sampled" points from the cylinder with Voronoi diagram complexity $\Omega(n^{3/2})$.

Working from a probabilistic perspective the authors of this work showed [4] that if points are drawn from a 2-dimensional Poisson distribution with rate n from the surface of a fixed *convex* polytope then the expected complexity of the Voronoi diagram of the points would be $O(n)$ (with the same result also holding if n points were chosen IID uniformly from the surface of the polytope).

The major result of this paper is to prove a high probability theorem when points are drawn from the surface of a collection of triangles. More specifically

THEOREM 1. *Let $\mathcal{F} = \{F_1, \ldots, F_k\}$ be a collection of k triangles in $\mathbb{R}^3$. Let P_n be a set of points drawn from a 2-dimensional Poisson distribution on $\mathcal{F}$ with rate n. Then* $\Pr\left(|VD(P_n)| = \tilde{O}(n)\right) = 1 - n^{-\Omega(\log n)}$.

[1]Just prior to submission we learnt of new work [2] by Attali and Boissonnat that proves a linear bound on the complexity of the Delaunay Triangulation of n points well-sampled from a polyhedral surface (using a slightly revised definition of well-sampled). This seems to be the deterministic analogue of the result in this current paper.

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Note that since a polyhedron can be decomposed into a finite set of triangles this immediately implies that if P_n is a set of points drawn from the standard 2-dimensional Poisson distribution with rate n on the surface of a fixed polyhedron then $\Pr\left(|\mathrm{VD}(P_n)| = \tilde{O}(n)\right) = 1 - n^{-\Omega(\log n)}$.

We start by defining some of the terms used in the theorem. A *triangle* will denote a *closed* triangle that contains its interior and edges.

A *Poisson distribution on $\mathcal{F}$ with rate n* is the distribution in which, for every measurable $M \subset \cup_i F_i$, we have that $N(M)$, the number of points in M, satisfies $\Pr(N(M) = k) = \frac{(n\mathrm{Area}(M))^k e^{-(n\mathrm{Area}(M))}}{k!}$ and for $M, M' \subset F$ with $M \cap M' = \emptyset$, $N(M)$ and $N(M')$ are independent. The expected number of points in P_n will be $\mathrm{Area}(\mathcal{F}) \cdot n$.

$g(n) = \tilde{O}(f(n))$ denotes that $g(n) = O(f(n)\log^i n)$ for some fixed i., i.e., it "hides" $\log n$ factors. In Theorem 1 as proved in this paper the $\tilde{O}(n)$ is actually a $O(n \log^6 n)$ term. The reason for using the tilde notation is that it vastly simplifies the proofs by allowing us to lump many cases together and makes them more readible.

Also, with slight modifications to the proofs, all of the probabilistic results given in this paper will hold if we change the distribution so that P_n, instead of being drawn from a Poisson distribution, is a set of n points independently identically distributed uniformly from $\mathcal{F}$.

In the next section we discuss how to reduce the proof of Theorem 1 to a case-by-case analysis of a simpler problem. In Section 3 we introduce some tools that we will need in our proof and in Section 4 we perform the case-by-case analysis.

Before ending this section we review some notation that we will be using: For a point $p \in \mathbb{R}^3$ and any set $X \subseteq \mathbb{R}^3$ extend the Euclidean distance function so that $d(p, X) = \inf_{q \in X} d(p, q)$. Also set $NN(p, X)$ to be a nearest neighbor to p in X, i.e., a $q \in X$ such that $d(p, q) = d(p, X)$. In this paper X will always be a closed polygonal piece so $NN(p, X)$ will be unique.

For $r > 0$ define

$$S(p, r) = \{q \in \mathbb{R}^3 \,:\, d(q, p) \leq r\}$$

to be the closed ball of radius r around p.

Let Π be a plane and $p \in \Pi$. Then define $C_\Pi(p, r) = \{q \in \Pi \,:\, d(q, p) \leq r\}$ to be the closed disc of radius r on Π centered at p.

Finally, for F_1, F_2 triangles and Π_1, Π_2 their supporting planes, we say that $F_1 \parallel F_2$ if Π_1 is parallel to Π_2 and $F_1 \not\parallel F_2$ otherwise.

Important Note: In this extended abstract we only provide the upper level description of some of the proofs and omit some details.

2. REDUCTIONS

Instead of calculating the complexity $|VD(P_n)|$ directly we will instead bound *the number of Voronoi spheres corresponding to Voronoi faces*:

A Voronoi sphere is a sphere that has at least one point of P_n on its boundary and no other points of P_n in its interior. We count the number of *combinatorially different* spheres where spheres are considered to be different only if *they have different set of points of P_n on the boundary.* Also, since the event of points in a P_n chosen from the Poisson distribution being in general position has probability 1, we can assume that each Voronoi sphere has at most 4 points on its surface. Thus every vertex/edge/face/region of $\mathrm{VD}(P_n)$ corresponds to Voronoi sphere $S = S(p, r)$ with 4/3/2/1 points of P_n on its boundary, i.e., the complexity of $\mathrm{VD}(P_n)$ is bounded by the number of possible Voronoi spheres having different 4/3/2/1 tuple of points of P_n on its boundary.

Noting that every edge corresponds to a triple of points chosen from the 4 points corresponding to some Voronoi vertex and every face corresponds to a pair of points chosen from the 4 points corresponding to some Voronoi vertex, we see the complexity of $\mathrm{VD}(P_n)$ is proportional to the number of Voronoi vertices. It therefore suffices to count the number of Voronoi vertices.

Furthermore, as recently pointed out by Attali and Boissonnat [2], Euler's relations imply that the number of tetrahedra in the 3-D Delaunay triangulation of n sites is linear in the number of edges in this triangulation; by taking the dual we have that the number of Voronoi vertices in the 3-D Voronoi diagram is actually linear in the number of Voronoi faces. So, the size of $\mathrm{VD}(P_n)$ is bounded by the number of Voronoi spheres defining Voronoi faces, i.e., the Voronoi spheres defined by exactly two points. To simplify matters, in the rest of this paper we will therefore assume that $VD(P_n)$ is not the full set of Voronoi spheres but only those corresponding to Voronoi faces, i.e., those defined by two points in P_n.

To further simplify matters we will also assume that the pair $p, p' \in P_n$ defining the Voronoi spheres are not both on the same triangle. The justification for this assumption is that if p, p' were on the same triangle F_i then the intersection of their empty Voronoi sphere with F_i would be an empty circle in F_i. This implies that p, p' are Voronoi neighbors, i.e., define an edge, in the *two-dimensional* Voronoi diagram of $P_n \cap F_i$ on Π_i, the supporting plane of F_i. The two dimensional Voronoi diagram is linear in the number of sites so the number of such faces is $O(|P_n \cap F_i|)$. Summing over all F_i we have that the total number of Voronoi spheres defined by p, p' with both points on the same triangle is $\sum_i O(|P_n \cap F_i|) = O(|P_n|)$. Let $A = \mathrm{Area}(\mathcal{F}) = \sum_i \mathrm{Area}(F_i)$. From the Poisson distribution it is easy to work out that $\Pr(|P_n| \geq 2An) = n^{-\Omega(\log n)}$ so, with probability $1 - n^{-\Omega(\log n)}$, the number of such faces is $O(n)$. Thus, in the sequel, we will assume that $\mathrm{VD}(P_n)$ is the set of Voronoi spheres defined by two points in P_n such that the two points do not lie on the same triangle F_i.

Bounding the probabilistic complexity of $\mathrm{VD}(P_n)$ *directly* can be quite difficult since the conditionality of assuming that certain spheres are Voronoi skews the rest of the distribution. In this section we show how to reduce the problem to a more manageable one. This will require introducing new definitions and utility lemmas. In everything that follows it is implicitly assumed that n is fixed.

Definition 1. *Let* $F \subset \mathbb{R}^3$ *be a triangle,* $p \in \mathbb{R}^3$ *and* $r \geq 0$. *Then* $S(p,r)$ *is an* F-good sphere *if* $Area(S(p,r) \cap F) \leq \frac{\log^2 n}{n}$.

If $\mathcal{F} = \{F_1, \ldots, F_k\}$ *is a collection of triangles then* $S(p,r)$ *is an* $\mathcal{F}$-good sphere *if* $S(p,r)$ *is an* F_i*-good sphere for every triangle* F_i.

The important observation is

Lemma 1. *Let* $\mathcal{F} = \{F_1, \ldots, F_k\}$ *be a collection of* k *triangles in* $\mathbb{R}^3$. *Let* P_n *be a set of points drawn from a 2-dimensional Poisson distribution on* $\mathcal{F}$ *with rate* n. *Then*

$$\begin{aligned}&\Pr(\textit{all Voronoi spheres in } VD(P_n) \textit{ are } \mathcal{F}\textit{-good}) \\ &\quad = 1 - n^{-\Omega(\log n)}.\end{aligned}$$

The inituition behind this proof is that if a given sphere is not $\mathcal{F}$-good then the probability that it contains no points of the Poisson distributed P_n is no more than $e^{-\frac{\log^2 n}{n} n} = n^{-\Omega(\log n)}$. This intuition can be formalized into a rigorous proof (a similar lemma was proved in [4] for $\mathcal{F}$ the boundary of a convex polygon.

We now define something easier to bound than Voronoi spheres:

Definition 2. *Let* $\mathcal{F} = \{F_1, \ldots, F_k\}$ *be a collection of triangles,* $X \subseteq \mathbb{R}^3$ *and* $P \subseteq \cup_i F_i$. *Now define*

$$D(P) = \left\{ \{p_1,\, p_2\} \;:\; \begin{array}{r} p_1, p_2 \textit{ are not on the same face } F_i \\ \textit{and } \{p_1,\, p_2\} \subseteq P \cap S(p,r) \\ \textit{for some } \mathcal{F}\textit{-good sphere } S(p,r) \end{array} \right\}$$

$$D_X(P) = \left\{ \{p_1,\, p_2\} \;:\; \begin{array}{r} p_1, p_2 \textit{ are not on the same face } F_i \\ \textit{and } \{p_1,\, p_2\} \subseteq P \cap S(p,r) \textit{for some} \\ \mathcal{F}\textit{-good sphere } S(p,r) \textit{ with } p \in X \end{array} \right\}$$

Note that if some Voronoi Sphere $S = S(p,r)$ in $\mathrm{VD}(P_n)$ containing 2 points $\{p_1,\, p_2\}$ on its boundary is a $\mathcal{F}$-good sphere then $\{p_1,\, p_2\} \in D(P_n)$ (recall that we defined $\mathrm{VD}(P_n)$ so that the two defining points can not lie on the same triangle). So, if *all* Voronoi spheres in $VD(P_n)$ are $\mathcal{F}$-good spheres then $|VD(P_n)| \leq |D(P_n)|$. Combining this with Lemma 1 gives that

$$\begin{aligned}&\Pr\left(|D(P_n)| = \tilde{O}(n)\right) = 1 - n^{-\Omega(\log n)} \\ &\text{implies } \Pr\left(|\mathrm{VD}(P_n)| = \tilde{O}(n)\right) = 1 - n^{-\Omega(\log n)}.\end{aligned}$$

We will now devote ourselves to proving

Theorem 2. *Let* $\mathcal{F} = \{F_1, \ldots, F_k\}$ *be a collection of* k *triangles in* $\mathbb{R}^3$. *Let* P_n *be a set of points drawn from a 2-dimensional Poisson distribution on* $\mathcal{F}$ *with rate* n. *Then*

$$\Pr\left(|D(P_n)| = \tilde{O}(n)\right) = 1 - n^{-\Omega(\log n)}.$$

From the discussion above, proving Theorem 2 implies Theorem 1.

We can actually go one step further and notice that, since (i) if $S(p,r)$ is a $\mathcal{F}$-good sphere with respect to P_n then $S(p,r)$ is a $\{F_{i_1}, F_{i_2}\}$-good sphere for any $1 \leq i_1, i_2 \leq k$ and (ii) $D(P_n)$ is defined by a pair of points, we have

$$D(P_n) \subseteq \bigcup_{1 \leq i_1, i_2 \leq k} D(P_n \cap (F_{i_1} \cup F_{i_2})).$$

so

$$|D(P_n)| \leq \sum_{1 \leq i_1, i_2 \leq k} |D(P_n \cap (F_{i_1} \cup F_{i_2}))| \qquad (1)$$

Suppose we had

Theorem 3. *Let* $\mathcal{F} = \{F_1, F_2\}$ *be a pair of triangles in* $\mathbb{R}^3$. *Let* P_n *be a set of points drawn from a 2-dimensional Poisson distribution on* $\mathcal{F}$ *with rate* n. *Then*

$$\Pr\left(|D(P_n)| = \tilde{O}(n)\right) = 1 - n^{-\Omega(\log n)}.$$

One of the properties of the Poisson distribution is that if P_n is a set of points drawn from a 2-dimensional Poisson distribution on $\mathcal{F}$ with rate n and $X \subset \mathcal{F}$ is measurable, then $P_n \cap X$ has the same distribution as a set of points drawn from a 2-dimensional Poisson distribution on X with rate n. Therefore, combining Theorem 3 and (1) would imply Theorem 2. Thus, to prove Theorem 2 it suffices to prove Theorem 3.

From now on we will therefore assume that $\mathcal{F} = \{F_1, F_2\}$ is composed of two triangles. Our approach to proving Theorem 3 will be to split $D(P_n)$ up into managable pieces. We do this as follows:

Definition 3. *Let* $F \subset \mathbb{R}^3$ *be a triangle. Set* $V(F)$, $E(F)$ *and* $I(F)$, *to be, respectively, the vertices, edges and interior of* F. *That is,* $V(F)$ *are the three vertices of* F, $E(F)$ *the three edges minus the vertices and* $I(F)$, *the triangle without its edges.*

For any $p \in \mathbb{R}^3$ set

$$r_p = \max\{r \,:\, S(p,r) \text{ is a } \mathcal{F}\text{-good sphere}\}.$$

For $i = 1, 2$, define $L_i(p) \in \{\emptyset, V, E, I\}$ such that

$$L_i(p) = \begin{cases} \emptyset & \text{if } S(p,r_p) \cap F_i = \emptyset \\ V & \text{if } S(p,r_p) \cap F_i \neq \emptyset \text{ and } NN(p,F_i) \in V(F_i) \\ E & \text{if } S(p,r_p) \cap F_i \neq \emptyset \text{ and } NN(p,F_i) \in E(F_i) \\ I & \text{if } S(p,r_p) \cap F_i \neq \emptyset \text{ and } NN(p,F_i) \in I(F_i) \end{cases}$$

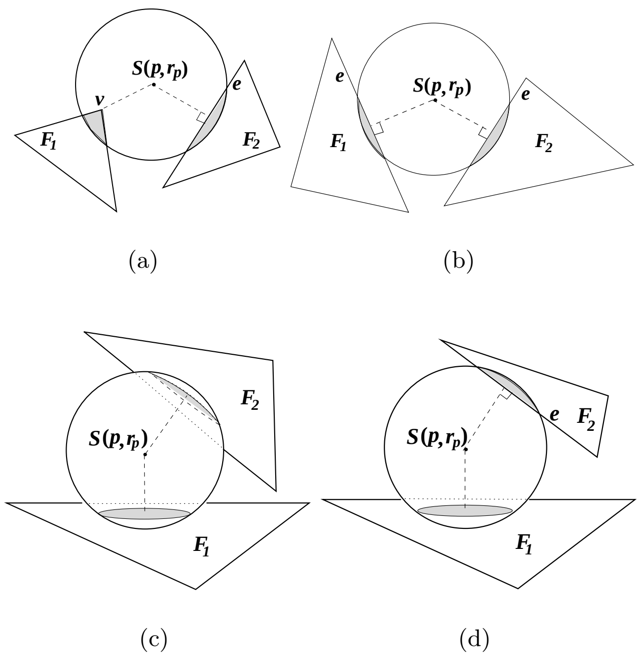

Figure 1: Different ways for spheres to intersect triangles: (a) $L = (V, E)$; **(b)** $L = (E, E)$; **(c)** $L = (I, I)$; **(d)** $L = (I, E)$. **The cases** $L = (V, I)$ **and** $L = (V, V)$ **are not illustrated.**

$L = (L_1, L_2)$ will be the *label* of p. Figure 1 illustrates ways in which a sphere can intersect two triangles and their labels. For any given label $L = (L_1, L_2) \in \{\emptyset, V, E, I\}^2$ we can define

$$R_L = \left\{p \in \mathbb{R}^3 \,:\, L(p) = L\right\}.$$

Since every $p \in \mathbb{R}^3$ has a *unique* label $L(p)$ the R_L form a partition of $\mathbb{R}^3$ into 15 regions (label $(\emptyset, \emptyset)$ can trivially not occur). Note that this partition depends upon $\mathcal{F}$ and n but not on the Poisson distribution.

Now, let P_n be any finite set of points in $\mathbb{R}^3$ (not necessarily a random one). Since every pair $\{p_1, p_2\} \in D(P_n)$ is contained in at least one $\mathcal{F}$-good sphere $S(p, r)$ that intersects both F_1 and F_2 and every p is in exactly one R_L, we have

$$D(P_n) = \bigcup_{L \in \{V,E,I\}^2} D_{R_L}(P_n)$$

so $|D(P_n)| \leq \sum_{L \in \{V,E,I\}^2} |D_{R_L}(P_n)|$. Our major theorem is:

Theorem 4. *Let $\mathcal{F} = \{F_1, F_2\}$ be a pair of triangles in $\mathbb{R}^3$ and P_n a set of points drawn from a 2-dimensional Poisson distribution on $\mathcal{F}$ with rate n. Let $L \in \{V, E, I\}^2$ be an arbitrary fixed label. Then*

$$\Pr\left(|D_{R_L}(P_n)| = \tilde{O}(n)\right) = 1 - n^{-\Omega(\log n)}. \tag{2}$$

Since this is true for all labels L and there are only 9 different labels, summing them gives

$$\Pr\left(|D(P_n)| = \tilde{O}(n)\right) = 1 - n^{-\Omega(\log n)}$$

proving Theorem 3 which, as discussed before, implies Theorem 1.

We have therefore just demonstrated that to prove Theorem 1 it suffices to prove Theorem 4. The sequel of this paper will be devoted to proving this.

3. TOOLS

In this section we introduce the major tools that we will use in our proof. In what follows we always assume that $\mathcal{F} = \{F_1, F_2\}$ is a pair of triangles.

Definition 4. *Let $F \subset \mathbb{R}^3$ be a triangle, $p' \in F$ and $X \subseteq F$. Define*

$$\mathcal{G}_F(p') = \left\{ S(p,r) \,:\, \begin{array}{l} NN(p,F) = p' \text{ and} \\ S(p,r) \text{ is an } F\text{-good sphere} \end{array} \right\}$$

$$M_F(p') = \bigcup_{S(p,r) \in \mathcal{G}_F(p')} (S(p,r) \cap F)$$

$$M_F(X) = \bigcup_{p' \in X} M_F(p').$$

Intuitively $M_F(p')$ is the set of all points in F that can belong to some F-good sphere such that if p is the center of the sphere then $NN(p, F) = p'$.

Definition 5. *Let $p \in \mathbb{R}^3$ be fixed. For $i = 1, 2$, set*

$$G_i(p) = \begin{cases} M_{F_i}(NN(p, F_i)) & L_i(p) \neq \emptyset \\ \emptyset & L_i(p) = \emptyset \end{cases}$$

and define $G(p) = \cup_{i=1}^2 G_i(p)$.

For $X \subseteq \mathbb{R}^3$ set $G_i(X) = \cup_{p \in X} G_i(p)$ and $G(X) = \cup_{i=1}^2 G_i(X) = \cup_{p \in X} G(p)$.

The important thing to notice is that $G(p)$ contains *all* points that can be contained by some F_i-good sphere centered at p. More formally, if $S(p, r)$ is an F_i-good sphere then, by definition, $S(p, r) \cap F_i \subseteq G_i(p)$; thus, if $\{p_1, p_2\} \subset S(p, r)$ for some $\mathcal{F}$-good sphere then $\{p_1, p_2\} \subset G(p)$. This in turn implies

Lemma 2. *Let $P_n \subset \mathbb{R}^3$ be finite and $X \subseteq \mathbb{R}^3$. Then*

$$|D_X(P_n)| \leq |P_n \cap G(X)|^2.$$

Noting the fact that if $X \subset \mathbb{R}^3$ is measurable in $\mathbb{R}^3$ then $G(X)$ is measurable in $\mathcal{F}$ and using the definition of the Poisson distribution we can then prove:

Corollary 3. *Let $X \subset \mathbb{R}^3$ be measurable and $f > 0$ such that Area$(G(X)) \leq f/n$. Let P_n be a set of points drawn from a 2-dimensional Poisson distribution on $\mathcal{F}$ with rate n. Then*

$$\Pr\left(|D_X(P_n)| \geq (f \log n)^2\right) = n^{-\Omega(\log n)}. \tag{3}$$

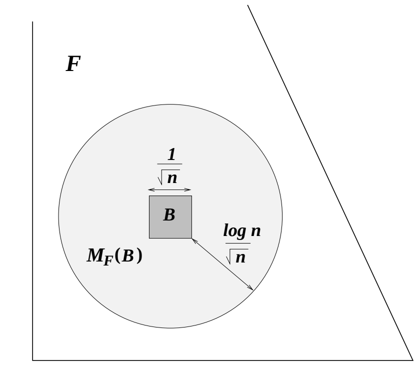

Figure 2: Illustration of Lemma 4 (1) and (2) for B far from the boundary of F: $\forall$ good sphere $S(p,r)$ such that $NN(p,F)=p'\in B$, $S(p,r)\cap F$ must be contained in the weakly shaded region.

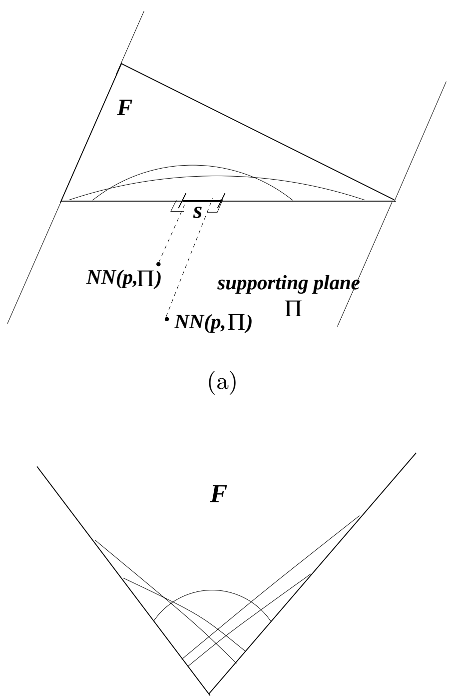

Figure 3: Illustration of Lemma 4 (3) and (4): Each solid circular arc is the boundary of $S(p,r)\cap F$ for some good sphere $S(p,r)$ such that in (a) $NN(p,F)=p'\in s$ and in (b) $NN(p,F)=v$, respectively. Note that the centers of the arcs are $NN(p,\Pi)$ where Π is the supporting plane of F. The union of all possible regions $S(p,r)\cap F$, i.e., $M_F(s)$ in (a) and $M_F(v)$ in (b), respectively, has area $\le c_F\frac{\log^3 n}{n}$.

This Corollary will be our major tool in proving Theorem 4. To employ it properly we will need to be able to efficiently bound $G(X)$. We will do this using the following:

LEMMA 4. *Let $F\subset\mathbb{R}^3$ be a triangle. Then There exists constant $c_F>0$ such that*

(1) *Let $p'\in I(F)$. Then*

$$Area(M_F(p'))=\frac{\log^2 n}{n}.$$

(2) *Let $B\subset I(F)$ be an $\frac{1}{\sqrt{n}}\times\frac{1}{\sqrt{n}}$ square. Then*

$$Area(M_F(B))\le c_F\frac{\log^2 n}{n}.$$

(3) *Let $s\subset E(F)$ be an edge segment [2] with $length(s)\le\frac{\log n}{\sqrt{n}}$. Then*

$$Area(M_F(s))\le c_F\frac{\log^3 n}{n}.$$

(4) *Let $v\in V(F)$ be a vertex of F. Then*

$$Area(M_F(v))\le c_F\frac{\log^3 n}{n}.$$

In this extended abstract we do not give the full proof of this lemma. To provide the intuition as to why it is correct suppose that $p'\in I(F)$. Let Π be the supporting plane of F. Then $M_F(p')$ is *exactly* $C_\Pi(p',r)\cap F$ where r is the unique value such that $Area(C_\Pi(p',r)\cap F)=\frac{\log^2 n}{n}$. This proves (1). To prove (2) we examine the union of all such discs whose center can be in the small square B. See Figure 2. The proofs of (3) and (4) are much more tedious and require a detailed case-by-case analysis. This analysis can be found in the proof of Lemma 3 in [5]. See Figure 3.

As a first consequence of Lemma 4 let $X=S(p',r)$ be any ball; we will find a general bound on $|D_X(P_n)|$. For all i let Π_i be the supporting plane of F_i and $p_i''=NN(p',\Pi_i)$. Now let $p\in X$. If $NN(p,F_i)\in I(F_i)$ then $NN(p,F_i)\in C_{\Pi_i}(p_i'',r)$, i.e., the projection of X on Π_i. See Figure 4. Such a disk can be covered by $4nr^2$ squares of size $\frac{1}{\sqrt{n}}\times\frac{1}{\sqrt{n}}$. So, from part (2) of the Lemma,

$$\begin{aligned}&\text{Area}\left(\bigcup\{G_i(p)\,:\,p\in X\text{ and }NN(p,F_i)\in I(F_i)\}\right)\\&\quad\le 4nr^2c_{F_i}\tfrac{\log^2 n}{n}=4c_{F_i}r^2\log^2 n.\end{aligned}$$

If $NN(p,F_i)\in E(F_i)$ it is not hard to see that all such nearest neighbors must be contained in the projection of X on the three edges of F_i which can be composed of at most three segments, each of length at most $2r$. These can be partitioned into $\frac{6r\sqrt{n}}{\log n}$ segments of length $\le\frac{\log n}{\sqrt{n}}$. From part (3) of the Lemma,

$$\begin{aligned}&\text{Area}\left(\bigcup\{G_i(p)\,:\,p\in X\text{ and }NN(p,F_i)\in E(F_i)\}\right)\\&\quad\le\tfrac{6r\sqrt{n}}{\log n}c_{F_i}\tfrac{\log^3 n}{n}=\tfrac{6rc_{F_i}\log^2 n}{\sqrt{n}}.\end{aligned}$$

[2] an edge segment can be either closed, open, or half open, half closed.

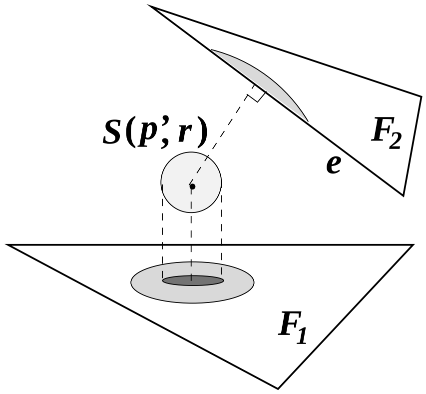

Figure 4: Illustrating $D_X(P_n)$ for $X = S(p', r)$ with $r = \tilde{O}\left(\frac{1}{\sqrt{n}}\right)$. Recall that $D_X(P_n)$ is the set of pairs $(p_1, p_2) \in P_n$ s.t. $p_1 \in F_1$, $p_2 \in F_2$ and (p_1, p_2) are on some $\mathcal{F}$ good sphere $S(p'', r'')$ with $p'' \in X$. The claim is that p_1, p_2 must be in the shaded regions on, respectively, F_1, F_2. The smaller disk in F_1 is the projection of $S(p', r)$ onto F, more precisely $X_1 = \cup_{p \in S(p',r)} NN(p, F_1)$; the bigger disk in F_1 is $M_{F_1}(X_1)$. The shaded region in F_2 is $M_{F_2}(X_2)$ where $X_2 = \cup_{p \in S(p',r)} NN(p, F)$.

Finally, from part (4) of the Lemma and the fact that $V(F_i)$ contains only 3 vertices,

$$\begin{aligned} &\text{Area}\left(\bigcup \{G_i(p) : p \in X \text{ and } NN(p, F_i) \in V(F_i)\}\right) \\ &\quad \le 3c_{F_i} \frac{\log^3 n}{n}. \end{aligned}$$

Combining the above we have

$$\text{Area}(G_i(X)) = O\left(r^2 \log^2 n + \frac{r \log^2 n}{\sqrt{n}} + \frac{\log^3 n}{n}\right).$$

In particular, if $r = \tilde{O}\left(\frac{1}{\sqrt{n}}\right)$ then $\text{Area}(G_i(X)) = \tilde{O}\left(\frac{1}{n}\right)$ and $\text{Area}(G(X)) \le \sum_{i=1}^{2} \text{Area}(G_i(X)) = \tilde{O}\left(\frac{1}{n}\right)$ as well. Applying Corollary 3, we have just proven:

Lemma 5. *If $X = S(p', r)$ with $r = \tilde{O}\left(\frac{1}{\sqrt{n}}\right)$ then*

$$\Pr\left(|D_X(P_n)| = \tilde{O}(1)\right) = 1 - n^{-\Omega(\log n)}.$$

4. PROOF OF THEOREM 4

We will now prove Theorem 4 by doing a case-by-case analysis for the 9 different possible labels $L = (L_1, L_2) \in \{V, E, I\}^2$ and proving (2) for each one. As we will see, after symmetry and other reductions, there will only be three distinct cases.

In what follows let Π_1 and Π_2, respectively, be the supporting planes of F_1 and F_2.

Case 1: $L_1 = V$. (The case $L_2 = V$ is symmetric.)

This case is very simple and can be proven directly without using the tools developed in the previous section. Suppose $(p_1, p_2) \subseteq P_n \cap S(p, r)$ for some $\mathcal{F}$-good sphere $S(p, r)$ with $L_1(p) = V$ and $L_2(p) \in \{V, E, I\}$.

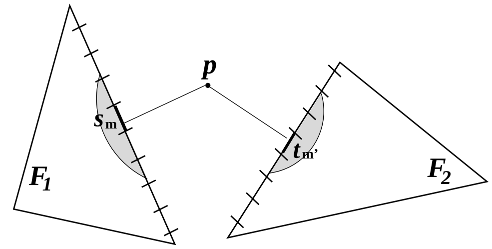

Figure 5: The case $L = (E, E)$. Knowing $NN(p, F_1)$ is on segment s_m and $NN(p, F_2)$ is on segment $t_{m'}$ restricts the locations of p_1 and p_2 to, respectively, the shaded regions $M_{F_1}(s_m)$ and $M_{F_2}(t_{m'})$.

By definition we have that $p_1 \in M_{F_1}(V(F_1))$. From Lemma 4 (4), we know that $Area(\cup_{v \in V(F_1)} M_{F_1}(v)) \le 3c_{F_1} \frac{\log^3 n}{n}$. Plugging into the formula for the Poisson distribution, this gives

$$\Pr(|P_n \cap M_{F_1}(V(F_1))| \ge 6c_{F_1} \log^3 n) = n^{-\Omega(\log n)}.$$

Again directly from the formula for the Possion distribution and simple calculations we get that $\Pr(|P_n \cap F_2| \ge 2nArea(F_2)) = n^{-\Omega(\log n)}$.

Therefore, with probability $1 - n^{-\Omega(\log n)}$, the total number of such (p_1, p_2) is $\le \left(6c_{F_1} \log^3 n\right)(2nArea(F_2)) = \tilde{O}(n)$ and we are done.

Case 2: $L_1 = L_2 = E$.

See Figure 5. Partition the edges of F_1, F_2 into, respectively, $l_1 = O\left(\frac{\sqrt{n}}{\log n}\right)$ smaller segments $s_1, s_2, \ldots, s_{l_1}$ and $l_2 = O\left(\frac{\sqrt{n}}{\log n}\right)$ segments $t_1, t_2, \ldots, t_{l_2}$, each of length $\le \frac{\log n}{\sqrt{n}}$. $\forall m_1 \le l_1, \forall m_2 \le l_2$, set

$$X_{m_1,m_2} = \{p \in R_L : NN(p, F_1) \in s_{m_1}, NN(p, F_2) \in t_{m_2}\}.$$

Since $R_L = \cup_{1 \le m_1 \le 1_1, 1 \le m_2 \le 1_2} X_{m_1,m_2}$ we have

$$|D_{R_L}(P_n)| \le \sum_{1 \le m_1 \le 1_1, 1 \le m_2 \le 1_2} |D_{X_{m_1,m_2}}(P_n)|.$$

By definition $G_1(X_{m_1,m_2}) \subseteq M_{F_1}(s_{m_1})$ and $G_2(X_{m_1,m_2}) \subseteq M_{F_2}(t_{m_2})$. By Lemma 4,

$$\text{Area}(G(X_{m_1,m_2})) \le \sum_{i=1}^{2} \text{Area}(G_i(X_{m_1,m_2})) = O\left(\frac{\log^3 n}{n}\right),$$

so applying Corollary 3 gives

$$\Pr\left(|D_{X_{m_1,m_2}}(P_n)| = \tilde{O}(1)\right) = 1 - n^{-\Omega(\log n)}.$$

Summing over all m_1, m_2 yields

$$\Pr\left(|D_{R_L}(P_n)| = \tilde{O}(n)\right) = 1 - n^{-\Omega(\log n)}.$$

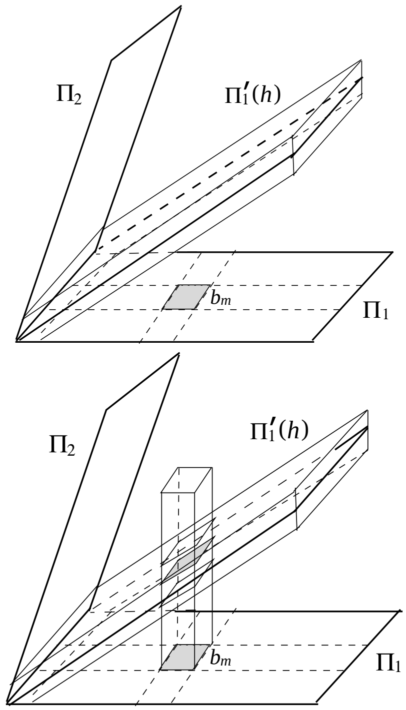

Figure 6: In the top figure we see half of the plane Π_1' which is one of the bisecting planes of Π_1 and Π_2. $\Pi_1'(h)$ is the *sandwich* of width $2h$ around Π_1', In the bottom figure we take a square b_m on Π_1 and look at X_m^1, the intersection of the infinite square prism through b_m perpendicular to Π_1 and the sandwich. If $NN(p, F_1) \in b_m$ and there is some r such that $S(p,r)$ is good sphere that intersects both F_1 and F_2, then $p \in X_m^1$.

Case 3(a): $L_1 = L_2 = I$ and $F_1 \not\parallel F_2$.

See Figure 6. Recall that Π_1, Π_2 are the supporting planes of F_1 and F_2. Define Π_1' and Π_2' to be the *bisecting planes* of Π_1, Π_2, i.e., the two planes that contain the line $\Pi_1 \cap \Pi_2$ and satisfy

$$\Pi_1' \cup \Pi_2' = \{p \,:\, d(p, \Pi_1) = d(p, \Pi_2)\}.$$

(Π_1', Π_2' exist because Π_1 and Π_2 are not parallel). For arbitrary plane Π define the *h-sandwich* around Π as

$$\Pi(h) = \{p \,:\, d(p, \Pi) \le h\}.$$

Straightforward geometric arguments (omitted in this extended abstract) show that

$$\text{if } p \notin \Pi_1'\left(\tfrac{\log^2 n}{\sqrt{n}}\right) \cup \Pi_2'\left(\tfrac{\log^2 n}{\sqrt{n}}\right), \quad \text{then } |d(p,\Pi_1) - d(p,\Pi_2)| = \Omega\left(\tfrac{\log^2 n}{\sqrt{n}}\right).$$

Now suppose that p is such that $L_1(p) = L_2(p) = I$ and $p \notin \Pi_1'\left(\frac{\log^2 n}{\sqrt{n}}\right) \cup \Pi_2'\left(\frac{\log^2 n}{\sqrt{n}}\right)$. If $S(p,r)$ is *any* sphere that intersects *both* F_1 and F_2, then

$$r \ge \max(d(p,\Pi_1),\, d(p,\Pi_2)).$$

This would imply that $S(p,r)$ would have to contain at least one of the two closed disks $C_{\Pi_1}(NN(p,F_1), r')$ or $C_{\Pi_2}(NN(p,F_2), r')$ for some $r' = \Omega\left(\frac{\log^2 n}{\sqrt{n}}\right)$. By definition, $S(p,r)$ can therefore not be *both* F_1-good *and* F_2-good and can therefore not be $\mathcal{F}$-good.

We have therefore just seen that if $S(p,r)$ is a $\mathcal{F}$-good sphere then $p \in \Pi_1'\left(\frac{\log^2 n}{\sqrt{n}}\right) \cup \Pi_2'\left(\frac{\log^2 n}{\sqrt{n}}\right)$. In particular, this implies

$$R_L \in \Pi_1'\left(\frac{\log^2 n}{\sqrt{n}}\right) \cup \Pi_2'\left(\frac{\log^2 n}{\sqrt{n}}\right).$$

We now cover the interior of F_1 by $l = O(n)$ squares $b_1, b_2, \ldots, b_l$, each of size $\frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{n}}$. $\forall m \le l$, set

$$X_m = \{p \in R_L \,:\, NN(p, F_1) \in b_m\}.$$

By definition, $R_L = \cup_{m=1}^{l} X_m$, so

$$|D_{R_L}(P_n)| \le \sum_{m=1}^{l} |D_{X_m}(P_n)|.$$

From the argument above we know that $X_m \subset \Pi_1'\left(\frac{\log^2 n}{\sqrt{n}}\right) \cup \Pi_2'\left(\frac{\log^2 n}{\sqrt{n}}\right)$. For $j = 1, 2$, set $X_m^j = X_m \cap \Pi_j'\left(\frac{\log^2 n}{\sqrt{n}}\right)$. Each X_m^j is contained in the intersection of an infinite square prism with $\frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{n}}$ crossection and a sandwich of width $\frac{\log^2 n}{\sqrt{n}}$ so there exists p_m^j, r_m^j with $r_m^j = O\left(\frac{\log^2 n}{\sqrt{n}}\right)$ such that $X_m^j \subset S(p_m^j, r_m^j)$. Applying Lemma 5 shows that

$$\Pr\left(|D_{X_m^j}(P_n)| = \tilde{O}(1)\right) = 1 - n^{-\Omega(\log n)}.$$

Since $|D_{R_L}(P_n)| \le \sum_{m,j} |D_{X_m^j}(P_n)|$, summing over all m, j yields

$$\Pr\left(|D_{R_L}(P_n)| = \tilde{O}(n)\right) = 1 - n^{-\Omega(\log n)}.$$

Case 3(b): $L_i = L_j = I$ and $F_i \| F_j$.

As before, let Π_1, Π_2 be the supporting planes of F_1, F_2. Now, let Π_3 be the plane parallel to Π_1, Π_2 that is equidistant from both. If p has $L_1(p) = L_2(p) = I$ then a straightforward geometric argument similar to the one in Case 3 (a) shows that if, for some $r \ge 0$, $S(p,r)$ is $\mathcal{F}$-good, then $p \in \Pi_3\left(\frac{\log^2 n}{\sqrt{n}}\right)$.

We now repeat the same procedure. Ccover the interior of F_1 by $l = O(n)$ squares $b_1, b_2, \ldots, b_l$ each of size $\frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{n}}$. $\forall m \le l$, set $X_m = \{p \in R_L \,:\, NN(p, F_1) \in b_m\}$. Since $R_L = \cup_{m=1}^{l} X_m$, we have

$$|D_{R_L}(P_n)| \le \sum_{m=1}^{l} |D_{X_m}(P_n)|.$$

Note that, from our discussion above, $X_m \in \Pi_3\left(\frac{\log^2 n}{\sqrt{n}}\right)$ so X_m is contained in the intersection of an infinite rectangular prism with $\frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{n}}$ crossection and a sandwich

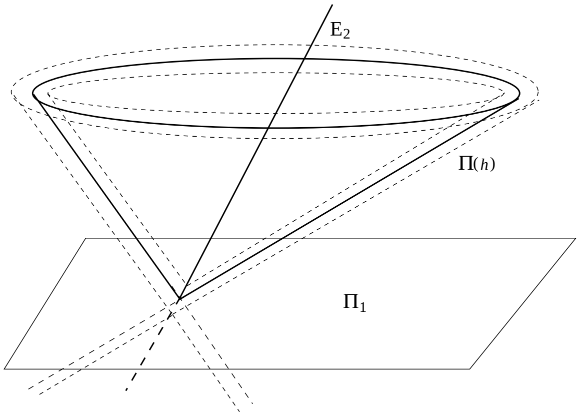

Figure 7: The cone that is the bisecting surface of Π_1 and E_2 and its h-sandwich.

of width $\frac{\log^2 n}{\sqrt{n}}$. It therefore must be contained in some ball of radius $O\left(\frac{\log^2 n}{\sqrt{n}}\right)$. Applying Lemma 5 shows that $\Pr\left(|D_{X_m}(P_n)| = \tilde{O}\left(1\right)\right) = 1 - n^{-\Omega(\log n)}$. Summing over all m yields

$$\Pr\left(|D_{R_L}(P_n)| = \tilde{O}\left(n\right)\right) = 1 - n^{-\Omega(\log n)}.$$

<u>**Case 4:**</u> $L_1 = I$ and $L_2 = E$. (The case $L_1 = E$ and $L_2 = I$ is symmetric)

See Figure 7. Fix e to be one of the three edges of F_2 and assume that $NN(p, F_2) \in e$.

We will prove the theorem for each fixed e separately; summing over the three edges will prove it for the full case

Let E_2 be the full line of which e is a segment. Define the *bisecting surface* (Figure 7) of Π_1 and E_2 to be

$$\Pi := \{p : d(p, \Pi_1) = d(p, E_2)\}.$$

If $\Pi_1 \parallel E_2$, then Π is a paraboloid; otherwise ($\Pi_1 \not\parallel E_2$), Π is an infinite double cone passing through the intersection point $\Pi_1 \cap E_2$. Define the *h-sandwich* around Π to be

$$\Pi(h) = \{p : d(p, \Pi) \le h\}.$$

Using straightforward geometric arguments (omitted in this extended abstract) we can show that $\forall p$ such that $L_1(p) = I$, $L_2(p) = E$, and $NN(p, F_2) \in e$, if $p \notin \Pi\left(\frac{\log n}{\sqrt{n}}\right)$ then $|d(p, \Pi_1) - d(p, E_2)| = \Omega\left(\frac{\log n}{\sqrt{n}}\right)$. So $\forall p$ such that $L_1(p) = I$, $L_2(p) = E$ and $NN(p, F_2) \in e$, if $p \notin \Pi\left(\frac{\log n}{\sqrt{n}}\right)$ and $S(p, r)$ is *any* sphere that intersects *both* F_1 and e, then $r \ge \max(d(p, \Pi_1), d(p, E_2))$. This would imply that $S(p, r)$ would have to contain at least one of $C_{\Pi_1}(NN(p, F_1), r')$ or $C_{\Pi_2}(NN(p, E_2), r')$ for some $r' = \Omega\left(\frac{\log n}{\sqrt{n}}\right)$. $S(p, r)$ can therefore not be *both* F_1-good *and* F_2-good and can therefore not be $\mathcal{F}$-good.

We have just seen that if $S(p, r)$ is a $\mathcal{F}$-good sphere, then $p \in \Pi\left(\frac{\log n}{\sqrt{n}}\right)$. In particular, this implies $R_L \in \Pi\left(\frac{\log n}{\sqrt{n}}\right)$.

As in Case 2, we now cover the interior of F_1 by $l = O(n)$ squares $b_1, b_2, \ldots, b_l$, each of size $\frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{n}}$. $\forall m \le l$, set

$$X_m = \{p \in R_L \,:\, NN(p, F_1) \in b_m\}.$$

By definition, $R_L = \cup_{m=1}^{l} X_m$ so $|D_{R_L}(P_n)| \le \sum_{m=1}^{l} |D_{X_m}(P_n)|$.

From the argument above we know that $X_m \subset \Pi\left(\frac{\log n}{\sqrt{n}}\right)$. More precisely, each X_m is contained in the intersection of an infinite square prism with $\frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{n}}$ cross-section b_m and a sandwich of width $\frac{\log n}{\sqrt{n}}$. So there exists p_m, r_m with $r_m = O\left(\frac{\log n}{\sqrt{n}}\right)$ such that $X_m \subset S(p_m, r_m)$. Applying Lemma 5 shows that $\Pr\left(|D_{X_m}(P_n)| = \tilde{O}\left(1\right)\right) = 1 - n^{-\Omega(\log n)}$. Since $|D_{R_L}(P_n)| \le \sum_m |D_{X_m}(P_n)|$, summing over all m yields

$$\Pr\left(|D_{R_L}(P_n)| = \tilde{O}\left(n\right)\right) = 1 - n^{-\Omega(\log n)}.$$

5. ACKNOWLEDGEMENTS

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