Logical Inference

- Inference is the process of deriving new sentences from old ones.
- Logical inference is governed by the *proof theory*, which is a set of inference rules for deducing the entailments of a set of sentences.
- An inference process is *sound* if it derives only true conclusions given true premises.
- An inference process is *complete* if it can derive all true conclusions from a set of premises.
- Two common kinds of logic:
  - Propositional logic (also called Boolean logic)
  - First-order predicate logic (or simply first-order logic)

Propositional Logic

- Sentence:
  - Atomic sentence
  - Complex sentence
- Atomic sentence:
  - A single symbol
  - `True`
  - `False`
- Complex sentence:
  - With parentheses: (Sentence)
  - With logical connective: Sentence Connective Sentence
  - With negation: $\neg$Sentence
- Logical connective:
  - And: $\wedge$
  - Or: $\vee$
  - Equivalent: $\Leftrightarrow$
  - Implies: $\Rightarrow$
- Order of precedence (from highest to lowest):

$$\neg \;\; \wedge \;\; \vee \;\; \Rightarrow \; \Leftrightarrow$$

Truth tables for logical connectives

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
False	*False*	*True*	*False*	*False*	*True*	*True*
False	*True*	*True*	*False*	*True*	*True*	*False*
True	*False*	*False*	*False*	*True*	*False*	*False*
True	*True*	*False*	*True*	*True*	*True*	*True*

Example: Truth tables

- If a man marries, then his life is miserable.

  If a man does not marry, then his life is miserable.

- Let p be Man Marries
  q be Life is miserable

- Premises
  $p \Rightarrow q$
  $\neg p \Rightarrow q$

p	q	p⇒q	¬p⇒q	(p⇒q)∧(¬p⇒q)	[(p⇒q)∧(¬p⇒q)] ⇒ q
T	T	T	T	T	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

- Conclusion
  q

  i.e. A man's life is miserable!

Taulologies

- p : A is a bird

  q : A has wings

  r : A can fly

p	q	r	$[p \Rightarrow (q \wedge r)] \Leftrightarrow [(p \Rightarrow q)] \wedge (p \Rightarrow r)]$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

- "A is a bird implies A has wings and A can fly." is the same as saying that "A is a bird implies A has wings and A is a bird implies A can fly."

- **Taulologies** - A sentence is true under all possible interpretations.

Contradiction

- p : $a > 1$

  q : $a < 0$

p	q	$[\neg (p \vee q)] \Leftrightarrow [\neg (\neg p \wedge \neg q]$
T	T	
T	F	
F	T	
F	F	

- $[\neg (p \vee q)]$ **contradicts** $[\neg (\neg p \wedge \neg q]$

Truth Tables

- An example sentence to be tested:

$$((P \vee H) \wedge \neg H) \Rightarrow P$$

- Truth table:

P	H	$P \vee H$	$(P \vee H) \wedge \neg H$	$((P \vee H) \wedge \neg H) \Rightarrow P$
False	*False*	*False*	*False*	*True*
False	*True*	*True*	*False*	*True*
True	*False*	*True*	*True*	*True*
True	*True*	*True*	*False*	*True*

- Implication:
  - If a machine has some premises and a possible conclusion, it can determine if the conclusion is true (by building a truth table for the sentence *Premises* $\Rightarrow$ *Conclusion* and checking if all rows have a *True* value).
- This method of making inferences is computationally impractical, as it requires enumerating 2^n rows of a truth table if the proof involves n symbols.

Inference Rules

- Inference rules are more convenient than truth tables for making logical inferences.
- A logical proof is a sequence of applications of inference rules.
- Examples:
  - Modus ponens (or implication-elimination):
    From an implication and the premise of the implication, we can infer the conclusion.
  - Resolution:
    Implication is transistive.
- The worst-case complexity of the proof procedure for propositional logic using inference rules is still intractable, although not all instances of such proof (or inference) take time proportional to 2^n.

Inference Rules

- Inference rules for propositional logic:
  - Modus ponens:
$$\frac{\alpha \Rightarrow \beta,\ \alpha}{\beta}$$
  - And-elimination:
$$\frac{\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n}{\alpha_i}$$
  - And-introduction:
$$\frac{\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n}{\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n}$$
  - Or-introduction:
$$\frac{\alpha_i}{\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n}$$
  - Double-negation elimination:
$$\frac{\neg\neg\alpha}{\alpha}$$
  - Unit resolution:
$$\frac{\alpha \vee \beta,\ \neg\beta}{\alpha}$$
  - Resolution:
$$\frac{\alpha \vee \beta,\ \neg\beta \vee \gamma}{\alpha \vee \gamma}$$

    Or equivalently
$$\frac{\neg\alpha \Rightarrow \beta,\quad \beta \Rightarrow \gamma}{\neg\alpha \Rightarrow \gamma}$$

Modus Ponens

- Modus Ponens:
  If $\mathbf{x}$ and $\mathbf{x} \Rightarrow \mathbf{y}$ are true, then we can infer that the sentence $\mathbf{y}$ is true.

- Example:

  John is an uncle.

  If John is an uncle, then John is male.

  $\Rightarrow$ John is male.

x	y	x ⇒ y	x ∧ (x ⇒ y)	[x ∧ (x ⇒ y)] ⇒ y
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

An Agent for the Wumpus World

- The agent's knowledge after the third move. The current percept is [stench,None,None,None,None].

1,4	2,4	3,4	4,4
1,3 **W!**	2,3	3,3	4,3
1,2 $\boxed{\text{A}}$ **S** **OK**	2,2 **OK**	3,2	4,2
1,1 **V** **OK**	2,1 **B** **V** **OK**	3,1 **P!**	4,1

$\boxed{\text{A}}$ = *Agent*
B = *Breeze*
G = *Glitter, Gold*
OK = *Safe square*
P = *Pit*
S = *Stench*
V = *Visited*
W = *Wumpus*

- Let $S_{i,j}$ means "There is a stench in $[i, j]$"
  Let $B_{i,j}$ means "There is a stench in $[i, j]$"
  Let $W_{i,j}$ means "There is a Wumpus in $[i, j]$"
- The percept has led to the following facts:
  $\neg S_{1,1}$, $\neg S_{2,1}$, $S_{1,2}$
  $\neg B_{1,1}$, $B_{2,1}$, $\neg B_{1,2}$
- The agent knows that if a square has no smell, then neither the square nor any of its adjacent squares can house a wumpus:
  $R_1 : \neg S_{1,1} \Rightarrow \neg W_{1,1} \wedge \neg W_{1,2} \wedge \neg W_{2,1}$
  $R_2 : \neg S_{2,1} \Rightarrow \neg W_{1,1} \wedge \neg W_{2,1} \wedge \neg W_{2,2} \wedge \neg W_{3,1}$
  $R_3 : \neg S_{1,2} \Rightarrow \neg W_{1,1} \wedge \neg W_{1,2} \wedge \neg W_{2,2} \wedge \neg W_{1,3}$
- If there is a stench in $[1, 2]$, then there must be a wumpus in $[1, 2]$ or one of the adjacent squares:
  $R_4 : S_{1,2} \Rightarrow W_{1,3} \vee W_{1,2} \vee W_{2,2} \vee W_{1,1}$

Finding the Wumpus

First, we will show that the wumpus is not in one of the other squares, and then conclude by elimination that it must be in $[1, 3]$:

- Applying Modus Ponens with $\neg S_{1,1}$ and R_1:

$$\neg W_{1,1} \wedge \neg W_{1,2} \wedge \neg W_{2,1}$$

- Applying And-Elimination:

$$\neg W_{1,1}, \neg W_{1,2}, \neg W_{2,1}$$

- Applying Modus Ponens to $\neg S_{2,1}$ and R_2, and then And-Elimination to the result:

$$\neg W_{1,1}, \neg W_{2,1}, \neg W_{2,2}, \neg W_{3,1}$$

- Applying Modus Ponens with $\neg S_{1,2}$ and R_4:

$$W_{1,3} \vee W_{1,2} \vee W_{2,2} \vee W_{1,1}$$

- Applying the unit resolution rule, where α is $W_{1,3} \vee W_{1,2} \vee W_{2,2}$ and β is $W_{1,1}$:

$$W_{1,3} \vee W_{1,2} \vee W_{2,2}$$

- Applying unit resolution again with $W_{1,3} \vee W_{1,2}$ as α and $W_{2,2}$ as β:

$$W_{1,3} \vee W_{1,2}$$

- One more resolution with $W_{1,3}$ as α and $W_{1,2}$ as β:

$$W_{1,3}$$