Why First-Order Logic?

- Propositional logic is limited in representation power. It can only represent facts (as sentences).
- First-order logic (FOL) can represent:
  - Facts
  - Objects
  - Relations
  - Functions
- First-order logic is *universal* (or general-purpose) in the sense that it can express everything that can be programmed.

Elements in FOL

- Constants:
  - Example:
    HKUST
- Predicates:
  - Relations among objects
  - Example:
    $University(HKUST, Hongkong)$
- Functions:
  - Special type of relations
  - Example:
    $Instructor(Class(COMP221, 2))$
- Terms:
  - Objects (as constants, variables, or functions)
- Atomic sentences:
  - Pure predicate or equality form
  - Examples:
    $Teach(Pong, Class(COMP221, 1))$
    $Teach(Lin, Class(COMP221, 2))$
  - Examples:
    $Instructor(Class(COMP221, 1)) = Pong$
    $Instructor(Class(COMP221, 2)) = Lin$

Elements in FOL (cont'd)

- Complex sentences:
  - Atomic sentences plus logical connectives
  - Example:
    $\mathit{Employee}(\mathit{Pong},\ \mathit{UST}) \land \mathit{University}(\mathit{UST},\ \mathit{Hongkong})$
- Quantifiers:
  - Universal quantification ($\forall$):
    - General rules (for *all* objects in universe)
    - Example:
      $\forall x\ \mathit{Enrolled}(x,\ \mathit{COMP221}) \Rightarrow \mathit{Student}(x,\ \mathit{HKUST})$
  - Existential quantification ($\exists$):
    - For *one or more* objects in universe
    - Example:
      $\exists x\ \mathit{Student}(x,\ \mathit{HKUST}) \land \neg \mathit{Enrolled}(x,\ \mathit{COMP221})$
  - Relationships between $\forall$ and $\exists$:
    - $\forall x\ \neg P \ \equiv\ \neg \exists x\ P$
    - $\neg \forall x\ P \ \equiv\ \exists x\ \neg P$
    - $\forall x\ P \ \equiv\ \neg \exists x\ \neg P$
    - $\exists x\ P \ \equiv\ \neg \forall x\ \neg P$

Some Examples

- All purple mushrooms are poisonous:

$$\forall x\ \mathit{Mushroom}(x) \land \mathit{Purple}(x) \Rightarrow \mathit{Poisonous}(x)$$

- No purple mushroom is poisonous:

$$\forall x\ \mathit{Mushroom}(x) \land \mathit{Purple}(x) \Rightarrow \neg \mathit{Poisonous}(x)$$

- All mushrooms are either purple or poisonous:

$$\forall x\ \mathit{Mushroom}(x) \Rightarrow \mathit{Purple}(x) \lor \mathit{Poisonous}(x)$$

- All mushrooms are either purple or poisonous but not both:

$$\begin{aligned} \forall x \quad & \mathit{Mushroom}(x) \Rightarrow \\ & (\mathit{Purple}(x) \land \neg \mathit{Poisonous}(x)) \lor \\ & (\neg \mathit{Purple}(x) \land \mathit{Poisonous}(x)) \end{aligned}$$

- All purple mushrooms except one are poisonous:

$$\begin{aligned} \exists x \quad & \mathit{Purple}(x) \land \mathit{Mushroom}(x) \land \neg \mathit{Poisonous}(x) \land \\ & (\forall y\ \mathit{Purple}(y) \land \mathit{Mushroom}(y) \land \neg(x = y) \Rightarrow \\ & \ \mathit{Poisonous}(y)) \end{aligned}$$

- There are exactly two purple mushrooms:

$$\begin{aligned} \exists x, y \quad & \mathit{Mushroom}(x) \land \mathit{Purple}(x) \land \\ & \mathit{Mushroom}(y) \land \mathit{Purple}(y) \land \neg(x = y) \land \\ & (\forall z\ \mathit{Mushroom}(z) \land \mathit{Purple}(z) \Rightarrow (z = x) \lor (z = y)) \end{aligned}$$