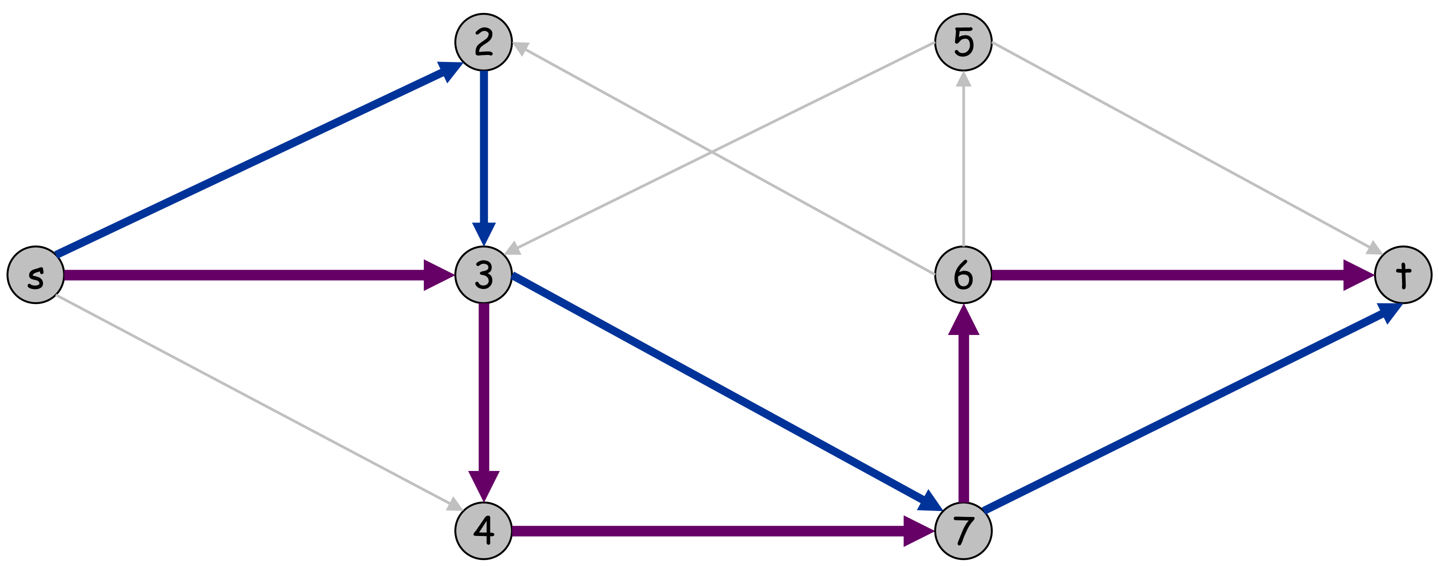
More Applications of Max Flow

Edge Disjoint Paths

Disjoint path problem. Given a directed graph $G = (V, E)$ and two nodes s and t, find the max number of edge-disjoint s-t paths.

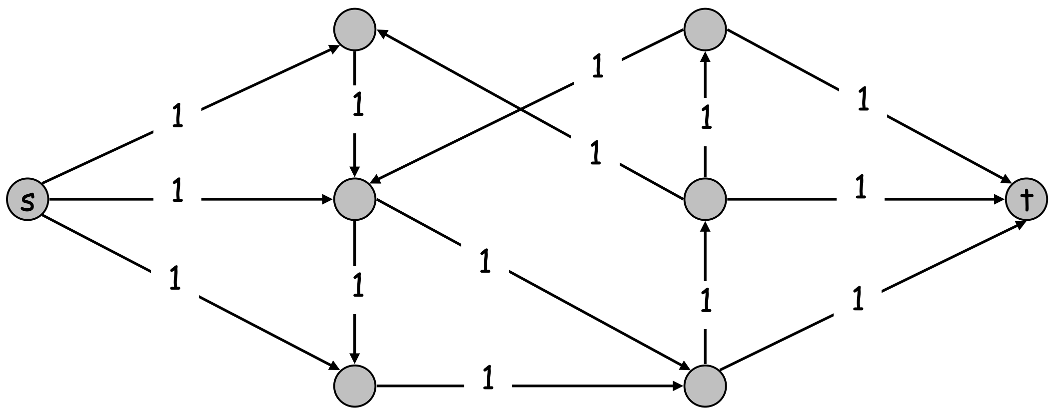
Def. Two paths are edge-disjoint if they have no edge in common.

Application: Communication networks.



Edge Disjoint Paths

Max flow formulation: assign unit capacity to every edge.



Theorem. Max number edge-disjoint s-t paths equals max flow value.

Proof. $\leq$

- Suppose there are k edge-disjoint paths $P_1, \dots, P_k$.
- Set $f(e) = 1$ if e participates in some path P_i; else set $f(e) = 0$.
- Since paths are edge-disjoint, f is a flow of value k.

Edge Disjoint Paths

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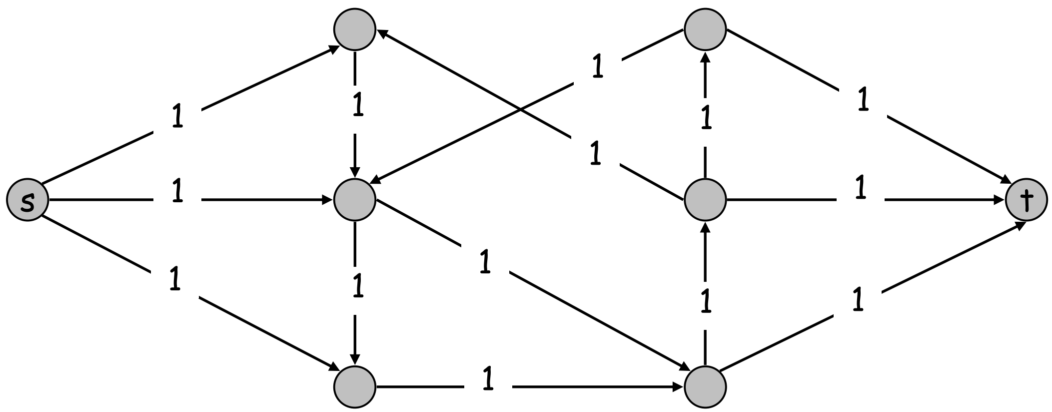


Proof. $\geq$

- Let f be a max flow in G' of value k computed by Ford-Fulkerson
- $f(e) = 1$ or 0 for every edge e (integrality property).
- Consider any edge (s, u) with $f(s, u) = 1$.
  - By conservation, there exists an edge (u, v) with $f(u, v) = 1$
  - Continue to find the next unused edge out of v until reaching t.
- After finding one path, flow value decreases by 1.
- Repeat the process k times to find k edge-disjoint paths.
- The proof above also provides an algorithm.

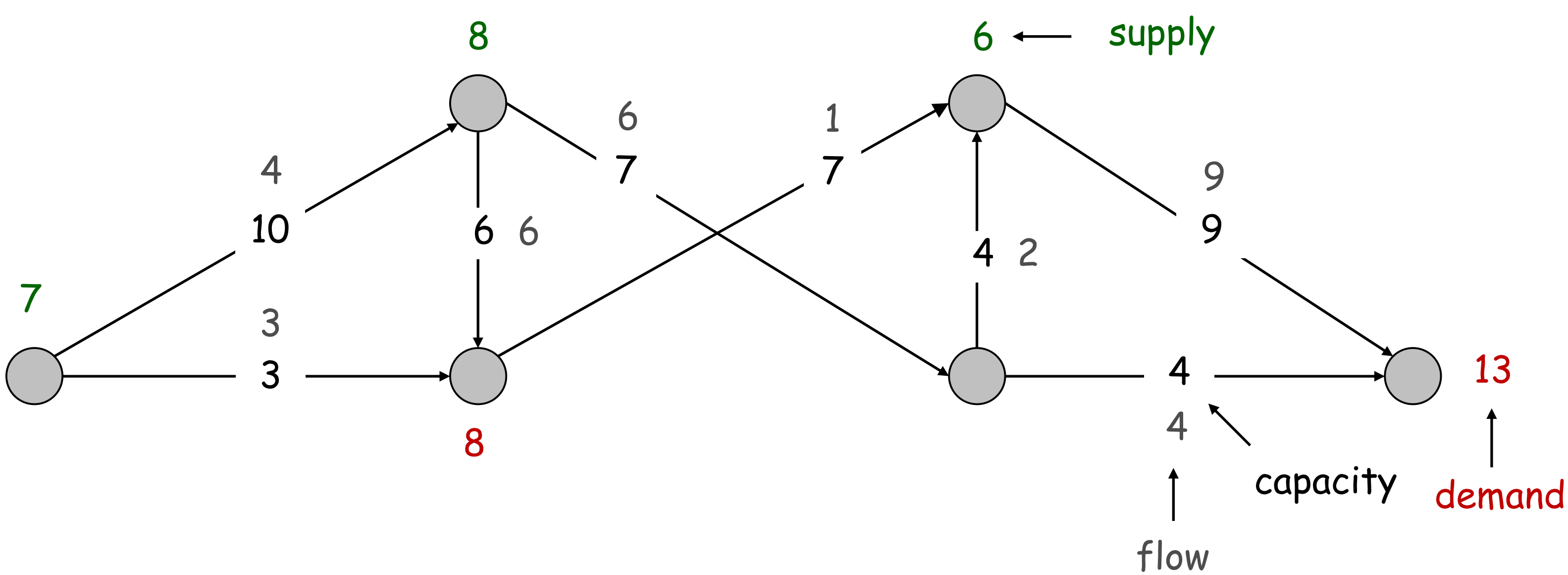
Circulation with Demands

Input: A directed connected graph $G = (V, E)$, where

- every edge $e \in E$ has a capacity $c(e)$;
- a number of source vertices $s_1, s_2, \ldots$, each with a supply of $sup(s_i)$ and a number of target vertices $t_1, t_2, \ldots$, each with a demand of $dem(t_i)$;
- $\sum_i sup(s_i) \geq \sum_i dem(t_i)$

Output: A flow f that meets capacity and conservation conditions, and

- At each source vertex s_i, $\sum_{e \text{ out of } s_i} f(e) - \sum_{e \text{ into } s_i} f(e) \leq sup(s_i)$;
- At each target vertex t_i, $\sum_{e \text{ into } t_i} f(e) - \sum_{e \text{ out of } t_i} f(e) = dem(s_i)$.

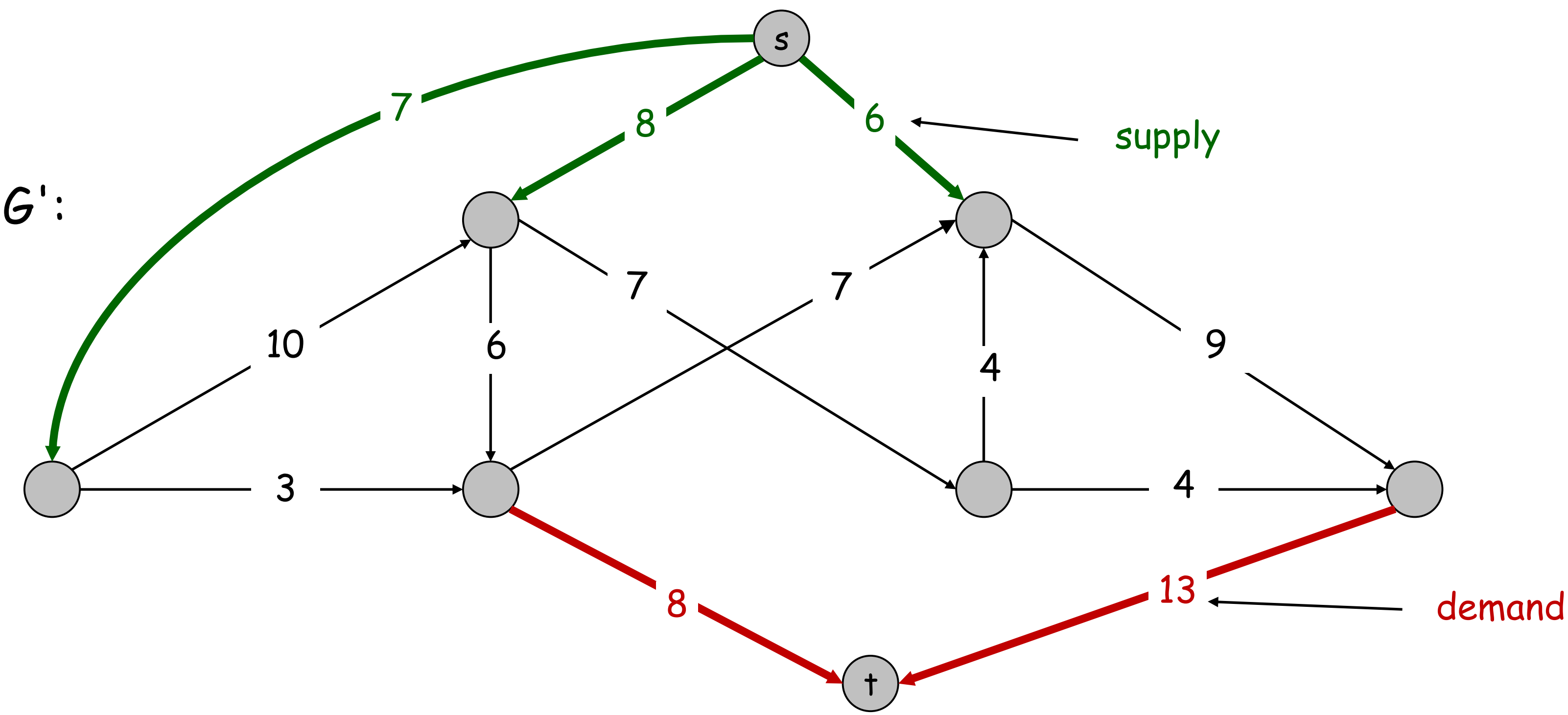


Solving Circulation with Demands using Max Flow

Algorithm:

- Add a "super source" s and a "super target" t.
- Add an edge from s to each s_i with capacity $sup(s_i)$.
- Add an edge from each t_i to t with capacity $dem(t_i)$.
- Compute the max flow f.
- If $|f| = \sum_i dem(t_i)$, then return f; else return "no solution".

G':



Baseball Elimination

Team i	Wins w_i	To play r_i	Remaining Against = r_{ij}			
			1	2	3	4
1	3	2	-	1	1	0
2	2	3	1	-	1	1
3	2	3	1	1	-	1
4	0	2	0	1	1	-

Rule: Order teams by the number of wins.

Q: Does Team 4 still have a chance to finish in the first place (tie is OK)?

A: No, obviously.

Baseball Elimination

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4	1	2	0	1	1	-

Q: Does Team 4 still have a chance to finish in the first place (tie is OK)?

A: No, because

- Team 4 has to win both remaining games against team 2 and 3.
- Team 1 has to lose both remaining games against team 2 and 3.
- Then 2 and 3 will both have 3 wins.
- The game between team 2 and 3 will give one of them one more win.

Suppose you need to do this for MLB / Premier League…

Baseball Elimination: Formal Definition

Input:

- n teams: $1, 2, \ldots, n$
- One particular team, say n (without loss of generality)
- Team i has won w_i games already
- Team i and j still need to play r_{ij} games, $r_{ij} = 0$ or 1.
- Team i has a total of $r_i = \sum_j r_{ij}$ games to play

Output:

- "Yes", if there is an outcome for each remaining game such that team n finishes with the most wins (tie is OK).
- "No", if no such possibilities.

Brute-force algorithm:

- For each remaining game, consider two possible outcomes.
- Try all 2^r possible combinations, where $r = \sum_{i,j} r_{ij}$

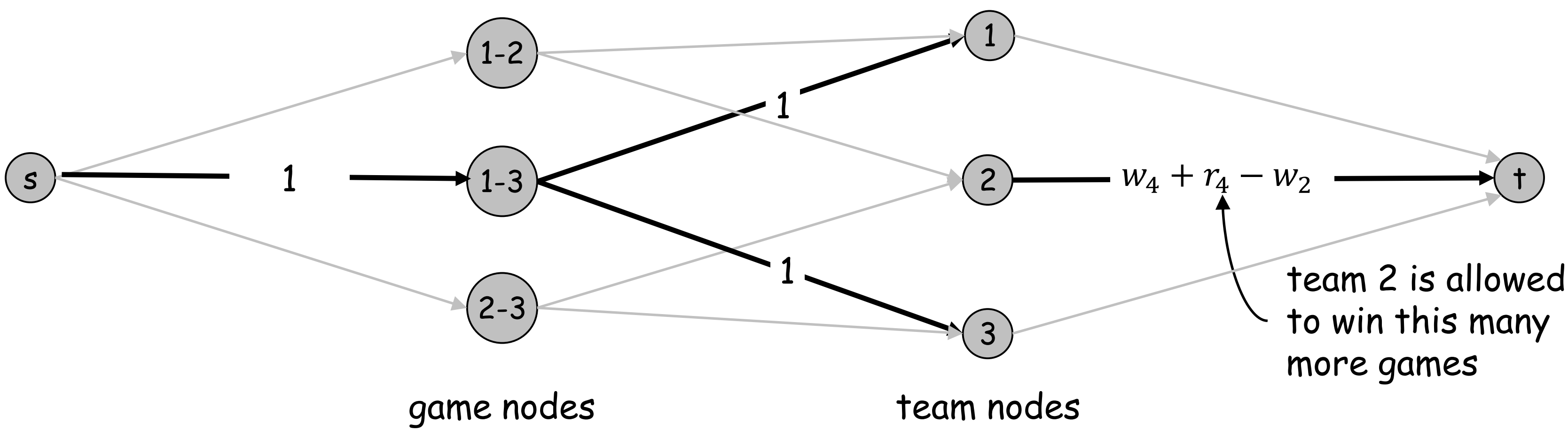
Baseball Elimination: Max Flow Formulation

Can team n finish with most wins?

- Assume team n wins all remaining games $\Rightarrow w_n + r_n$ wins.
- All other teams must have $\leq w_n + r_n$ wins.

Flow network construction:

- A source s and a target t
- A node for each remaining game (i, j); and an edge from s to it with capacity 1
- A node for each team $i = 1, 2, \ldots, n-1$; and an edge from it to t with capacity $w_n + r_n - w_i$
- Game node (i, j) has edges to team node i and j, with capacity 1



Baseball Elimination: Max Flow Formulation

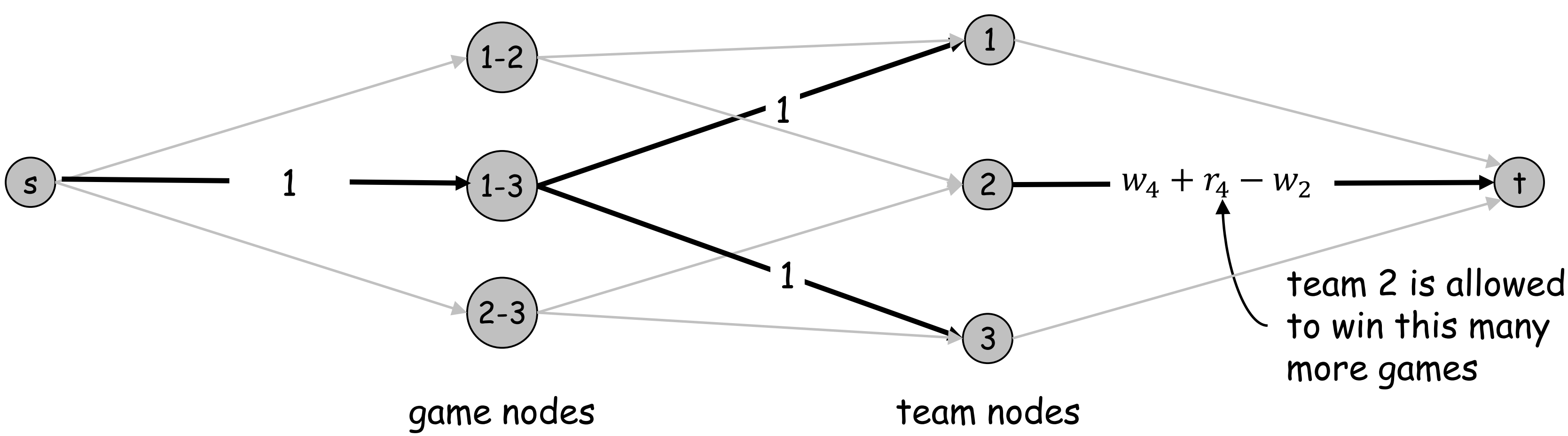
Claim: There is a way for team n to finish in the first place iff the max flow has value $r = \sum_{i,j} r_{ij}$.

Proof: "⇒": Suppose there is an outcome for each remaining game such that team n finishes the first. First set $f\big(s,(i,j)\big) = 1$ for all (i,j).

For each remaining game (i,j):

- if i wins, set $f\big((i,j),i\big) = 1$ and $f\big((i,j),j\big) = 0$;
- if j wins, set $f\big((i,j),j\big) = 1$ and $f\big((i,j),i\big) = 0$.

Team i wins $\leq w_n + r_n - w_i$ games, so it can send all incoming flow to t.



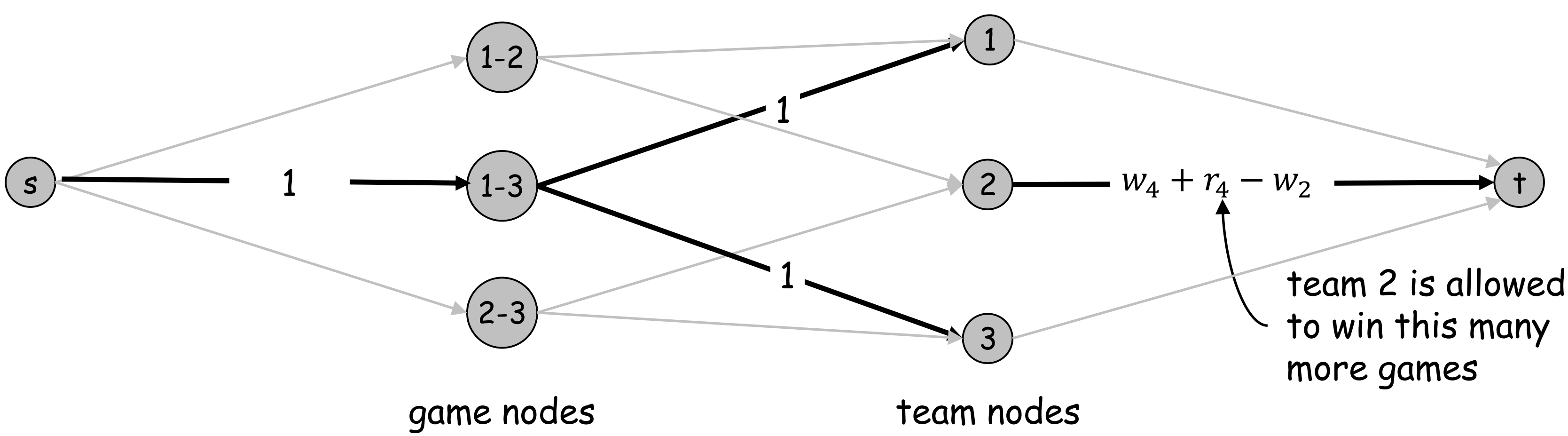
Baseball Elimination: Max Flow Formulation

Proof: "$\Leftarrow$": Suppose the max flow f has $|f| = r$. It must saturate all edges out of s.

Look at each game node (i, j). Exactly one of its outgoing edges must have 1 unit of flow (integrality property):

- If $f\big((i,j), i\big) = 1$, let i win the game;
- If $f\big((i,j), j\big) = 1$, let j win the game.

Team node i receives $\leq w_n + r_n - w_i$ units of flow, each corresponding to one win, so it cannot beat team n.



Baseball Elimination: Extensions

Q: What if r_{ij} can be more than 1?

Q: Can this be used for football (soccer) leagues?

- Using the old rule: Winner takes 2 points, loser 0 point; each team gets 1 point in case of a tie.