COMP 3711H
Lecture 1: Introduction

Computational Problems and Algorithms

- A computational problem is a specification of the desired input-output relationship.
- An instance of a problem is all the inputs needed to compute a solution to the problem.
- An algorithm is a well defined computational procedure that transforms inputs into outputs, achieving the desired input-output relationship.
- A correct algorithm halts with the correct output for every input instance. We then say that the algorithm solves the problem.

Example of a Problem and an Instance

Computational Problem: Sorting

- **Input:** Sequence of n numbers $\langle a_1, \cdots, a_n \rangle$.
- **Output:** Permutation (reordering)

$$\langle a'_1, a'_2, \cdots, a'_n \rangle$$

such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$.

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Instance of Problem Sorting

- **Input:** Permutation

$$\langle 8, 3, 6, 7, 1, 2, 9 \rangle$$

- **Output:** Permutation (reordering)

$$\langle 1, 2, 3, 6, 7, 8, 9 \rangle$$

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A is an array of numbers

for $j = 2$ to length(A)
$\{$ key $= A[j]$;
$i = j - 1$;
while ($i \geq 1$ and $A[i] >$ key)
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Pause:

How does it work?

Insertion Sort: an Incremental Approach

To sort an array of length n: n steps

ith step sorts the array of the first i items by inserting ith item properly into sorted array of the first $i - 1$ items (created in previous step)

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Step 3: $\langle 2, 3, 6, 4 \rangle$

Step 4: $\langle 2, 3, 4, 6 \rangle$

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Predict resource utilization

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Remark: Depends on model of computation, e.g.,

sequential vs. parallel or
internal memory vs. external memory.

In this class we usually assume
sequential and *internal memory*.

Analyzing Algorithms (Continued)

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Input size: rigorous definition given later.

1. **Sorting:** number of items to be sorted
2. **Multiplication:** number of bits, number of digits.
3. **Graphs:** number of vertices and edges.

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Remark: All cases are relative to the algorithm under consideration.

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Average Case: $\Theta(n^2)$ assuming that each of the $n!$ instances are equally likely.

Further thoughts on algorithm design

- *Algorithm Design*, as taught in this class, is mainly about designing algorithms that have small big $O()$ running times.

- **"All other things being equal"**,
  $O(n \log n)$ algorithms will run more quickly than $O(n^2)$ ones and
  $O(n)$ algorithms will beat $O(n \log n)$ ones.

- Being able to do good algorithm design lets you identify the *hard parts* of your problem and deal with them effectively.

- Too often, programmers try to solve problems using brute force techniques and end up with slow complicated code! A few hours of abstract thought devoted to algorithm design could have speeded up the solution substantially *and* simplified it.

Final Note

Note: After algorithm design one can continue on to *Algorithm tuning* which would further concentrate on improving algorithms by cutting cut down on the *constants* in the big $O()$ bounds. This needs a good understanding of both algorithm design principles and efficient use of data structures.

In this course we will not go further into algorithm tuning. For a good introduction, see Chapter 9 in *Programming Pearls, 2nd ed* by Jon Bentley.