Sorting and Searching

Lecture 2: Priority Queues, Heaps, and Heapsort

Priority Queue: Motivating Example

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Consider the printing pool at this moment.

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Average finish time for shortest-job-first service:

$$(1+11+111) \; / \; 3 = 41 \text{ time units}$$

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- Insert new printing jobs as they arrive

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A queue capable of supporting two operations: Insert and Extract-Min?

Priority Queue

Priority queue is an abstract data structure that supports two operations

- Insert: inserts the new element into the queue
- Extract-Min: removes and returns the smallest element from the queue

Possible Implementations

- unsorted list + a pointer to the smallest element
  - Insert in $O(1)$ time
  - Extract-Min in $O(n)$ time, since it requires a linear scan to find the new minimum
- sorted array
  - Insert in $O(n)$ time
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- sorted doubly linked list
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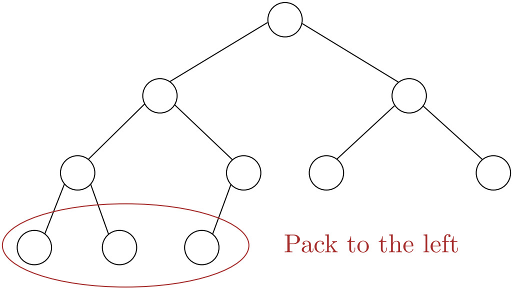
Question

Is there any data structure that supports both these priority queue operations in $O(\log n)$ time?

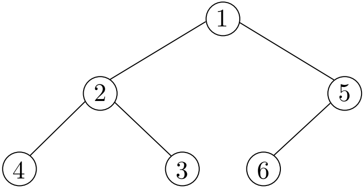
(Binary) Heap



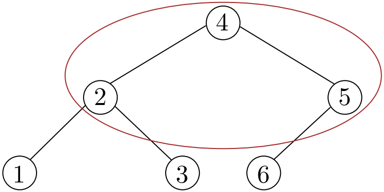
Heaps are "almost complete binary trees"

- All levels are full except possibly the lowest level
- If the lowest level is not full, then nodes must be packed to the left

Heap-order Property



A min-heap



Not a heap

Heap-order property:

The value of a node is at least the value of its parent — Min-heap

Heap Properties

- If the heap-order property is maintained, heaps support the following operations efficiently (assume there are n elements in the heap)
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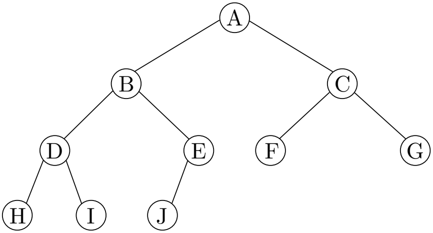
  - The structure is so regular, it can be represented in an array and no links are necessary !!!

Array Implementation of Heap

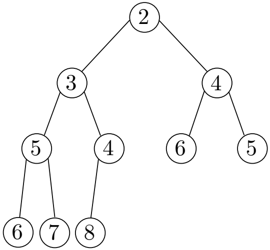


1	2	3	4	5	6	7	8	9	10
A	B	C	D	E	F	G	H	I	J

- The root is in array position 1
- For any element in array position i
  - The left child is in position $2i$
  - The right child is in position $2i + 1$
  - The parent is in position $\lfloor i/2 \rfloor$
- We will draw the heaps as trees, with the understanding that an actual implementation will use simple arrays

Insertion

- Add the new element to the next available position at the lowest level
- Restore the min-heap property if violated
  - General strategy is percolate up (or bubble up): if the parent of the element is larger than the element, then interchange the parent with child.



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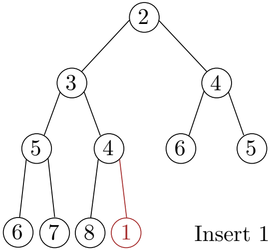

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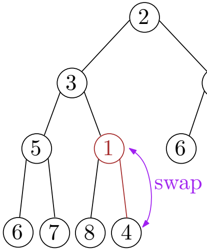

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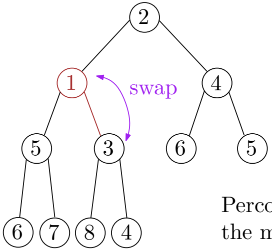

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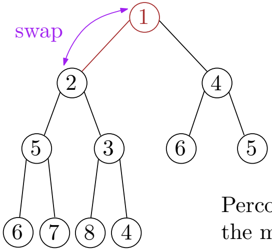

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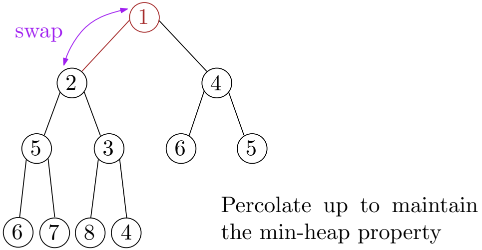

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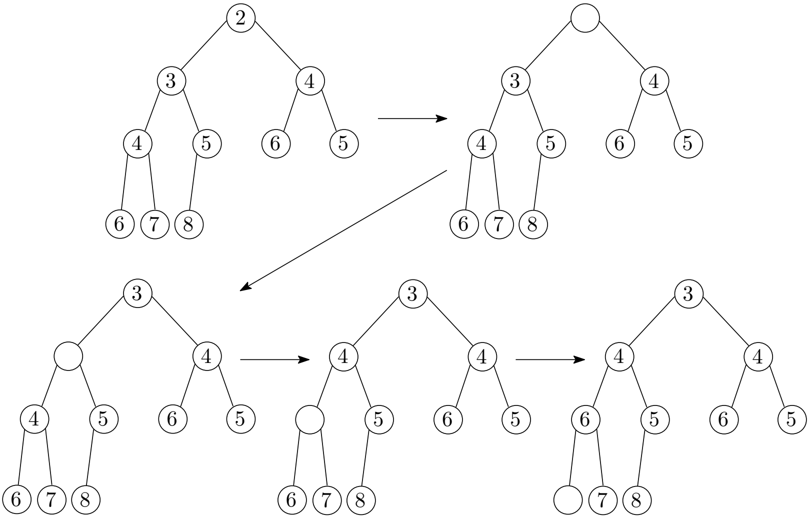
Extract-Min: First Attempt



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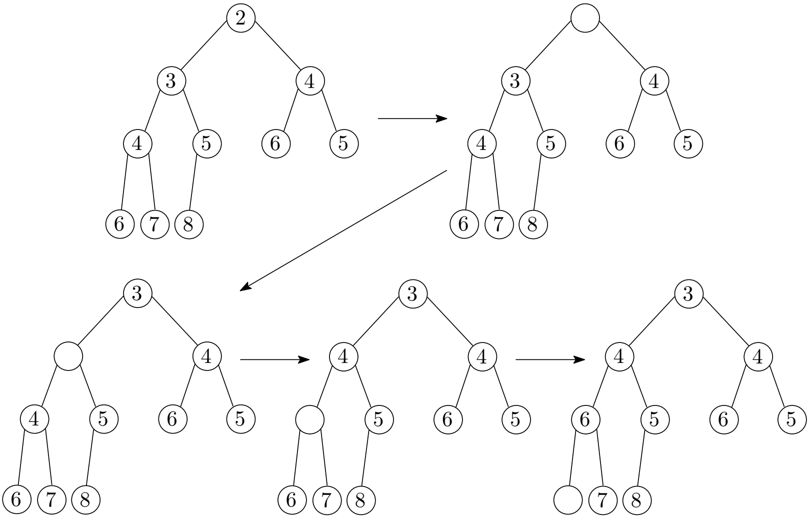

Min-heap property preserved, but completeness not preserved!

Extract-Min

- Copy the last element to the root (i.e., overwrite the minimum element stored there)
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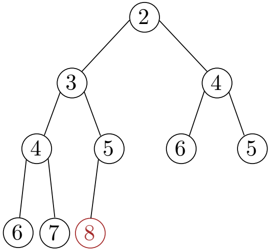

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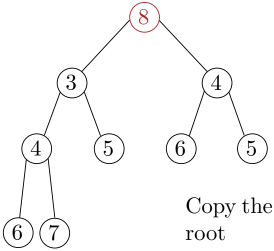

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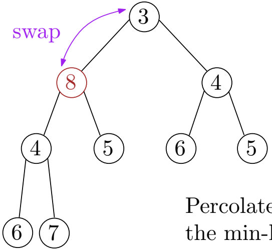

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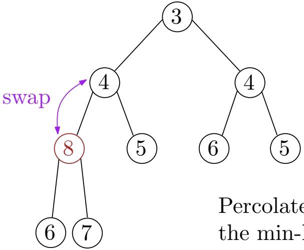

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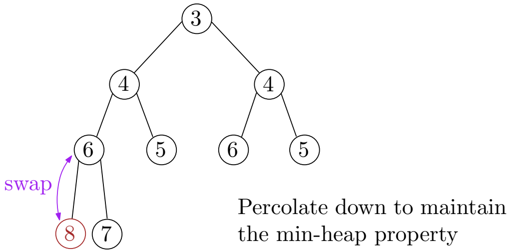

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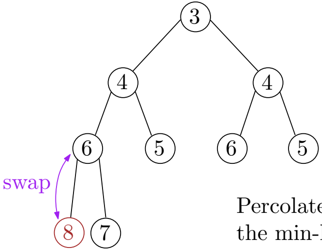

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- Total time complexity: $O(n \log n)$

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- If priority queues are implemented using heaps, then these two operations are supported in $O(\log n)$ time.
- Heapsort takes $O(n \log n)$ time, which is as efficient as merge sort and quicksort.

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Question

How can heaps be modified to support Decrease-Key in $O(\log n)$ time?

Going Further

- For some algorithms, there are other desirable Priority Queue operations, e.g., *Delete* an arbitrary item and *Meld*ing or taking the union of two priority queues
- There is a tradeoff between the costs of the various operations. Depending upon where the data structure is used, different priority queues might be better.
- Most famous variants are *Binomial Heaps* and *Fibonacci Heaps*