Quicksort

Revision of August 25, 2016

THE DEPARTMENT OF COMPUTER SCIENCE
計算機科學系 HKUST

Outline

Reference: Chapter 7 of CLRS

Outline

Reference: Chapter 7 of CLRS

Outline:

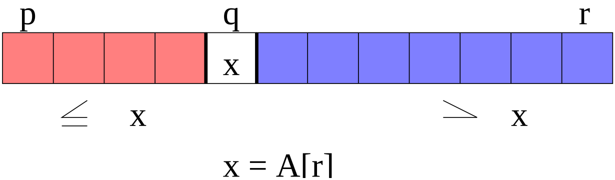
- Partitions
- Quicksort
- Analysis of Quicksort

Partition

Given: An array of numbers

Partition: Rearrange the array $A[p..r]$ in place into two (possibly empty) subarrays $A[p..q-1]$ and $A[q+1..r]$ such that

$$A[u] < A[q] < A[v], \quad \text{for any } p \leq u \leq q-1 \text{ and } q+1 \leq v \leq r$$

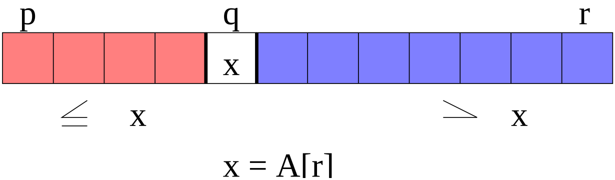


Partition

Given: An array of numbers

Partition: Rearrange the array $A[p..r]$ in place into two (possibly empty) subarrays $A[p..q-1]$ and $A[q+1..r]$ such that

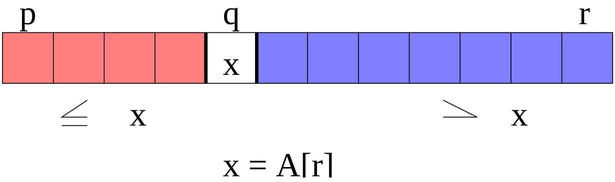
$$A[u] < A[q] < A[v], \quad \text{for any } p \le u \le q-1 \text{ and } q+1 \le v \le r$$



x is called the pivot. Assume $x = A[r]$; if not, swap first

Given: An array of numbers
Partition: Rearrange the array $A[p..r]$ in place into two (possibly empty) subarrays $A[p..q-1]$ and $A[q+1..r]$ such that

$$A[u] < A[q] < A[v], \quad \text{for any } p \le u \le q-1 \text{ and } q+1 \le v \le r$$



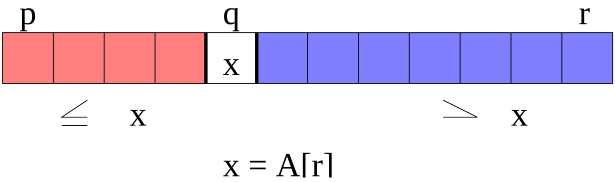
x is called the pivot. Assume $x = A[r]$; if not, swap first
Quicksort works by:

Partition

Given: An array of numbers

Partition: Rearrange the array $A[p..r]$ in place into two (possibly empty) subarrays $A[p..q-1]$ and $A[q+1..r]$ such that

$$A[u] < A[q] < A[v], \quad \text{for any } p \le u \le q-1 \text{ and } q+1 \le v \le r$$



x is called the pivot. Assume $x = A[r]$; if not, swap first

Quicksort works by:

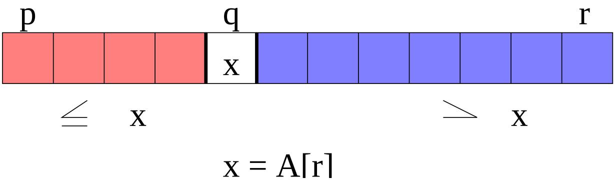
1. calling partition first

Partition

Given: An array of numbers

Partition: Rearrange the array $A[p..r]$ in place into two (possibly empty) subarrays $A[p..q-1]$ and $A[q+1..r]$ such that

$$A[u] < A[q] < A[v], \quad \text{for any } p \leq u \leq q-1 \text{ and } q+1 \leq v \leq r$$



x is called the pivot. Assume $x = A[r]$; if not, swap first

Quicksort works by:

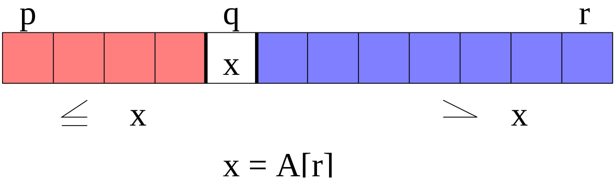
1. calling partition first
2. recursively sorting $A[\quad]$ and $A[\quad]$

Partition

Given: An array of numbers

Partition: Rearrange the array $A[p..r]$ in place into two (possibly empty) subarrays $A[p..q-1]$ and $A[q+1..r]$ such that

$$A[u] < A[q] < A[v], \quad \text{for any } p \le u \le q-1 \text{ and } q+1 \le v \le r$$



x is called the pivot. Assume $x = A[r]$; if not, swap first

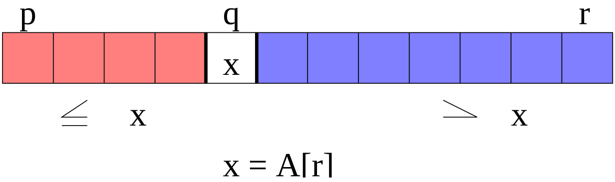
Quicksort works by:

1. calling partition first
2. recursively sorting $A[p..q-1]$ and $A[\qquad]$

Given: An array of numbers

Partition: Rearrange the array $A[p..r]$ in place into two (possibly empty) subarrays $A[p..q-1]$ and $A[q+1..r]$ such that

$$A[u] < A[q] < A[v], \quad \text{for any } p \le u \le q-1 \text{ and } q+1 \le v \le r$$



x is called the pivot. Assume $x = A[r]$; if not, swap first

Quicksort works by:

1. calling partition first
2. recursively sorting $A[p..q-1]$ and $A[q+1..r]$

Partitioning $A[p..r]$ with extra memory

- Copy $A[p..r]$ to another array $B[p..r]$

Partitioning $A[p..r]$ with extra memory

- Copy $A[p..r]$ to another array $B[p..r]$
- With $p - r$ comparisons find the *rank* R of $x = A[r]$ in $B[p..r]$

Partitioning $A[p..r]$ with extra memory

- Copy $A[p..r]$ to another array $B[p..r]$
- With $p - r$ comparisons find the *rank* R of $x = A[r]$ in $B[p..r]$
- Copy the items in $B[p..r]$ back to $A[p..r]$ placing
  - items smaller than x into first $R - 1$ locations
  - x into location $p + R - 1$
  - items larger than x into last $r - R$ locations

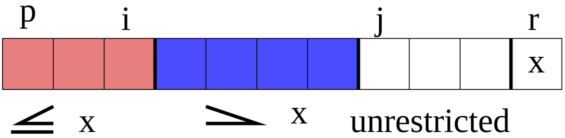
Partitioning $A[p..r]$ with extra memory

- Copy $A[p..r]$ to another array $B[p..r]$
- With $p - r$ comparisons find the *rank* R of $x = A[r]$ in $B[p..r]$
- Copy the items in $B[p..r]$ back to $A[p..r]$ placing
  - items smaller than x into first $R - 1$ locations
  - x into location $p + R - 1$
  - items larger than x into last $r - R$ locations
- $O(r - p)$ time but needs extra space.

Partition(A, p, r) without extra memory

Use $A[r]$ as the pivot, and grow partition from left to right.

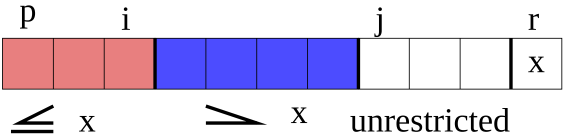
- i will be largest index of processed item $\leq x$.
- j will be smallest index of unprocessed item.



Use $A[r]$ as the pivot, and grow partition from left to right.

i will be largest index of processed item $\leq x$.

j will be smallest index of unprocessed item.



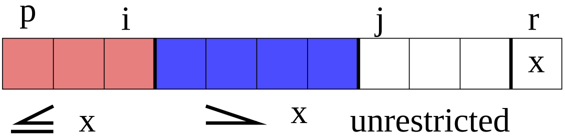
1. Initially $(i, j) = (p - 1, p)$

Partition(A, p, r) without extra memory

Use $A[r]$ as the pivot, and grow partition from left to right.

i will be largest index of processed item $\leq x$.

j will be smallest index of unprocessed item.



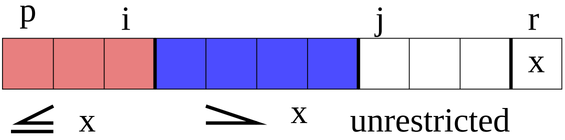
1. Initially $(i, j) = (p - 1, p)$
2. Increase j by 1 each time to find a place for $A[j]$
   At the same time increase i when necessary

Partition(A, p, r) without extra memory

Use $A[r]$ as the pivot, and grow partition from left to right.

i will be largest index of processed item $\le x$.

j will be smallest index of unprocessed item.



1. Initially $(i, j) = (p - 1, p)$
2. Increase j by 1 each time to find a place for $A[j]$
   At the same time increase i when necessary
3. Stops when $j = r$

One Iteration of the Procedure Partition

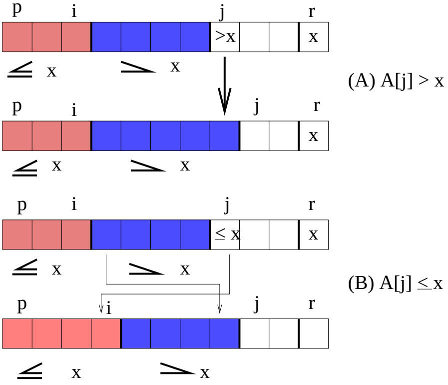
Increase j by 1 each time to find a place for $A[j]$

At the same time increase i when necessary

One Iteration of the Procedure Partition

Increase j by 1 each time to find a place for $A[j]$

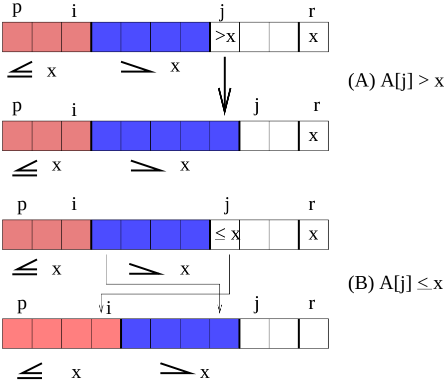
At the same time increase i when necessary



One Iteration of the Procedure Partition

Increase j by 1 each time to find a place for $A[j]$

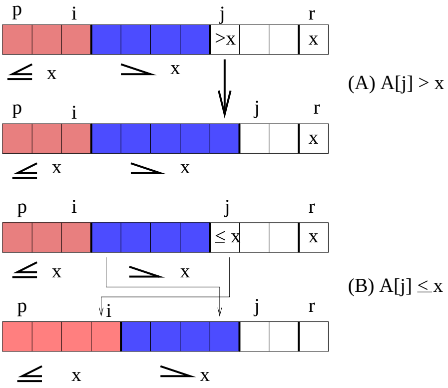
At the same time increase i when necessary



1. Only increase j by 1

Increase j by 1 each time to find a place for $A[j]$

At the same time increase i when necessary

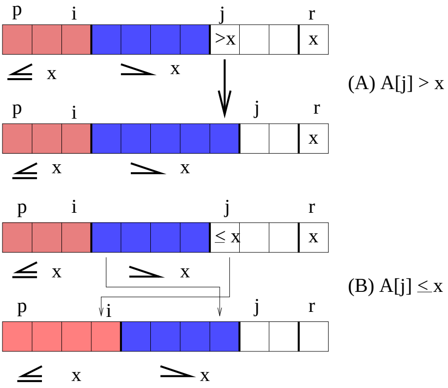


1. Only increase j by 1
2. $i = i + 1$.

One Iteration of the Procedure Partition

Increase j by 1 each time to find a place for $A[j]$

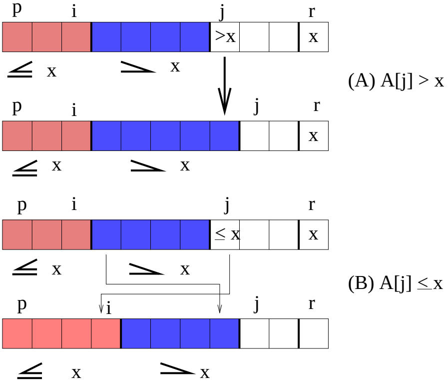
At the same time increase i when necessary



1. Only increase j by 1
2. $i = i + 1$. $A[i] \leftrightarrow A[j]$.

Increase j by 1 each time to find a place for $A[j]$

At the same time increase i when necessary



1. Only increase j by 1
2. $i = i + 1$. $A[i] \leftrightarrow A[j]$. $j = j + 1$

i	p, j							r	
	2	8	7	1	3	5	6	4	(1)
	p, i	j						r	
	2	8	7	1	3	5	6	4	(2)
	p, i		j					r	
	2	8	7	1	3	5	6	4	(3)
	p, i			j				r	
	2	8	7	1	3	5	6	4	(4)
	p	i			j			r	
	2	1	7	8	3	5	6	4	(5)
	p		i			j		r	
	2	1	3	8	7	5	6	4	(6)
	p		i				j	r	
	2	1	3	8	7	5	6	4	(7)
	p		i					j, r	
	2	1	3	8	7	5	6	4	(8)
	p		i					j, r	
	2	1	3	4	7	5	6	8	(9)

The Partition(A, p, r) Algorithm

Partition(A, p, r)

begin

$x = A[r]$; `// ` $A[r]$ `is the pivot element`

The Partition(A, p, r) Algorithm

Partition(A, p, r)

begin

$x = A[r]$; `// A[r] is the pivot element`

$i = p - 1$;

for $j = p$ *to* $r - 1$ **do**

Partition(A, p, r)

begin
  $x = A[r]$; `// A[r] is the pivot element`
  $i = p - 1$;
  for $j = p$ *to* $r - 1$ **do**
    if $A[j] \leq x$ **then**

The Partition(A, p, r) Algorithm

Partition(A, p, r)

```
begin
    x = A[r]; // A[r] is the pivot element
    i = p − 1;
    for j = p to r − 1 do
        if A[j] ≤ x then
            i = i + 1;
            exchange A[i] and A[j];
        end
    end
```

Partition(A, p, r)

```
begin
    x = A[r]; // A[r] is the pivot element
    i = p − 1;
    for j = p to r − 1 do
        if A[j] ≤ x then
            i = i + 1;
            exchange A[i] and A[j];
        end
    end
                                  ; // put pivot in position
```

The Partition(A, p, r) Algorithm

Partition(A, p, r)

```
begin
    x = A[r]; // A[r] is the pivot element
    i = p − 1;
    for j = p to r − 1 do
        if A[j] ≤ x then
            i = i + 1;
            exchange A[i] and A[j];
        end
    end
    exchange A[i + 1] and A[r]; // put pivot in position
```

The Partition(A, p, r) Algorithm

Partition(A, p, r)

```
begin
    x = A[r]; // A[r] is the pivot element
    i = p − 1;
    for j = p to r − 1 do
        if A[j] ≤ x then
            i = i + 1;
            exchange A[i] and A[j];
        end
    end
    exchange A[i + 1] and A[r]; // put pivot in position
    return
```

The Partition(A, p, r) Algorithm

Partition(A, p, r)

```
begin
    x = A[r]; // A[r] is the pivot element
    i = p - 1;
    for j = p to r - 1 do
        if A[j] ≤ x then
            i = i + 1;
            exchange A[i] and A[j];
        end
    end
    exchange A[i + 1] and A[r]; // put pivot in position
    return i + 1 // q = i + 1
end
```

Running Time of Partition(A, p, r)

Partition(A, p, r)

```
begin
    x = A[r];
    i = p − 1;
    for j = p to r − 1 do
        if A[j] ≤ x then
            i = i + 1;
            exchange A[i] and A[j]; // O(r − p)
        end
    end
    exchange A[i + 1] and A[r];
    return i + 1
end
```

Running Time of Partition(A, p, r)

Partition(A, p, r)

```
begin
    x = A[r];
    i = p − 1;
    for j = p to r − 1 do
        if A[j] ≤ x then
            i = i + 1;
            exchange A[i] and A[j]; //  O(r − p)
        end
    end
    exchange A[i + 1] and A[r];
    return i + 1
end
```

Running time is $O(\quad)$

Partition(A, p, r)

```
begin
    x = A[r];
    i = p − 1;
    for j = p to r − 1 do
        if A[j] ≤ x then
            i = i + 1;
            exchange A[i] and A[j]; //  O(r − p)
        end
    end
    exchange A[i + 1] and A[r];
    return i + 1
end
```

Running time is $O(r - p)$

Running Time of Partition(A, p, r)

Partition(A, p, r)

```
begin
    x = A[r];
    i = p − 1;
    for j = p to r − 1 do
        if A[j] ≤ x then
            i = i + 1;
            exchange A[i] and A[j]; // O(r − p)
        end
    end
    exchange A[i + 1] and A[r];
    return i + 1
end
```

Running time is $O(r - p)$

- linear in the length of the array $A[p..r]$

Partition(A, p, r)

```
begin
    x = A[r];
    i = p − 1;
    for j = p to r − 1 do
        if A[j] ≤ x then
            i = i + 1;
            exchange A[i] and A[j]; // O(r − p)
        end
    end
    exchange A[i + 1] and A[r];
    return i + 1
end
```

Running time is $O(r-p)$

- linear in the length of the array $A[p..r]$

Quicksort

Quicksort(A, p, r)

```
begin
  if p < r then
    q = Partition(A, p, r);
    Quicksort(A,          );
    Quicksort(A,          );
  end
end
```

Quicksort(A, p, r)

```
begin
    if p < r then
        q = Partition(A, p, r);
        Quicksort(A, p, q − 1);
        Quicksort(A,           );
    end
end
```

Quicksort(A, p, r)

```
begin
    if p < r then
        q = Partition(A, p, r);
        Quicksort(A, p, q − 1);
        Quicksort(A, q + 1, r);
    end
end
```

Quicksort(A, p, r)

```
begin
    if p < r then
        q = Partition(A, p, r);
        Quicksort(A, p, q − 1);
        Quicksort(A, q + 1, r);
    end
end
```

- If we could always partition the array into halves, then we have the recurrence $T(n) \leq 2T(n/2) + O(n)$, hence $T(n) = O(n \log n)$
- However, if we always get unlucky with very unbalanced partitions, then $T(n) \leq T(n-1) + O(n)$, hence $T(n) = O(n^2)$

Outline:

- Partition
- Quicksort
- Average Case Analysis of Quicksort

Average Case Analysis of Quicksort

Measuring running time:

Average Case Analysis of Quicksort

Measuring running time:

- The running time is dominated by the time spent in partition.

Average Case Analysis of Quicksort

Measuring running time:

- The running time is dominated by the time spent in partition.
- The running time of the partition procedure can be measured by the number of key comparisons.

Measuring running time:

- The running time is dominated by the time spent in partition.
- The running time of the partition procedure can be measured by the number of key comparisons.
- Need to specify m, the size of the left partition block.

$T(n)$: running time on array of size n.
Recurrence: $T(n) =$

Average Case Analysis of Quicksort

Measuring running time:

- The running time is dominated by the time spent in partition.
- The running time of the partition procedure can be measured by the number of key comparisons.
- Need to specify m, the size of the left partition block.

$T(n)$: running time on array of size n.
Recurrence: $T(n) = T(m) +$

Average Case Analysis of Quicksort

Measuring running time:

- The running time is dominated by the time spent in partition.
- The running time of the partition procedure can be measured by the number of key comparisons.
- Need to specify m, the size of the left partition block.

$T(n)$: running time on array of size n.
Recurrence: $T(n) = T(m) + T(n - m - 1) +$

Average Case Analysis of Quicksort

Measuring running time:

- The running time is dominated by the time spent in partition.
- The running time of the partition procedure can be measured by the number of key comparisons.
- Need to specify m, the size of the left partition block.

$T(n)$: running time on array of size n.

Recurrence: $T(n) = T(m) + T(n - m - 1) + O(n)$

Average Case Analysis of Quicksort

Measuring running time:

- The running time is dominated by the time spent in partition.
- The running time of the partition procedure can be measured by the number of key comparisons.
- Need to specify m, the size of the left partition block.

$T(n)$: running time on array of size n.
Recurrence: $T(n) = T(m) + T(n-m-1) + O(n)$
Worst Case:

Average Case Analysis of Quicksort

Measuring running time:

- The running time is dominated by the time spent in partition.
- The running time of the partition procedure can be measured by the number of key comparisons.
- Need to specify m, the size of the left partition block.

$T(n)$: running time on array of size n.
Recurrence: $T(n) = T(m) + T(n-m-1) + O(n)$
Worst Case:

$$T(n) = T(0) + T(n-1) + O(n)$$

Average Case Analysis of Quicksort

Measuring running time:

- The running time is dominated by the time spent in partition.
- The running time of the partition procedure can be measured by the number of key comparisons.
- Need to specify m, the size of the left partition block.

$T(n)$: running time on array of size n.
Recurrence: $T(n) = T(m) + T(n - m - 1) + O(n)$
Worst Case:

$$\begin{aligned} T(n) &= T(0) + T(n-1) + O(n) \\ T(n) &= O(\quad) \end{aligned}$$

Average Case Analysis of Quicksort

Measuring running time:

- The running time is dominated by the time spent in partition.
- The running time of the partition procedure can be measured by the number of key comparisons.
- Need to specify m, the size of the left partition block.

$T(n)$: running time on array of size n.
Recurrence: $T(n) = T(m) + T(n-m-1) + O(n)$
Worst Case:

$$\begin{aligned} T(n) &= T(0) + T(n-1) + O(n) \\ T(n) &= O(n^2) \end{aligned}$$

Measuring running time:

- The running time is dominated by the time spent in partition.
- The running time of the partition procedure can be measured by the number of key comparisons.
- Need to specify m, the size of the left partition block.

$T(n)$: running time on array of size n.
Recurrence: $T(n) = T(m) + T(n-m-1) + O(n)$
Worst Case:

$$\begin{aligned} T(n) &= T(0) + T(n-1) + O(n) \\ T(n) &= O(n^2) \end{aligned}$$

What inputs give worst case performance?

Measuring running time:

- The running time is dominated by the time spent in partition.
- The running time of the partition procedure can be measured by the number of key comparisons.
- Need to specify m, the size of the left partition block.

$T(n)$: running time on array of size n.
Recurrence: $T(n) = T(m) + T(n-m-1) + O(n)$
Worst Case:

$$\begin{aligned} T(n) &= T(0) + T(n-1) + O(n) \\ T(n) &= O(n^2) \end{aligned}$$

What inputs give worst case performance?

We will analyze average case running time.

Average Case Analysis

1. Worst-case doesn't make sense: for any given input, the worst case is very unlikely to happen

Average Case Analysis

1. Worst-case doesn't make sense: for any given input, the worst case is very unlikely to happen
2. Use Average Case Analysis

Average Case Analysis

1. Worst-case doesn't make sense: for any given input, the worst case is very unlikely to happen
2. Use Average Case Analysis
3. Assume every possible input permutation of the n items are equally likely.
4. $n!$ permutations so each one has probability $\frac{1}{n!}$ of ocurring
5. If S_n is set of all permutations, $\sigma \in S_n$ is a possible input permutation, then average running time is

$$\frac{1}{n!} \sum_{\sigma \in S_n} C(\sigma)$$

Let **A** be the set of items in $A[p..r]$ and σ a random permutation of **A**.

1. $A[r]$ is equally likely to be any item in **A**.
2. After running the partition algorithm on $A[p..r]$, the input to the new left and right subproblems are again random permutations (need to argue why).

Recall that if X is a random variable and $E_1, E_2, \ldots, E_n$ are events that partition the probability space then we can write the expectation of X in terms of the Expectation of X conditioned on E_i. That is

$$E(X) = \sum_i E(X|E_i)\Pr(E_i).$$

Average Case Analysis

Assume that the input to is a random permutation of N items.

- Let C_N be the average amount of work performed on the input
- $C_0 = C_1 = 0$.
- Partition requires $N - 1$ comparisons
- Each item has probability $1/N$ of being pivot.
- If Item k is pivot, the two remaining subproblems require $C_{k-1} + C_{N-k}$ average time

$$\begin{aligned} C_N &= N - 1 + \frac{1}{N} \sum_{1 \le k \le N} (C_{k-1} + C_{N-k}) \\ &= N - 1 + \frac{2}{N} \sum_{1 \le k \le N} C_{k-1} \end{aligned}$$

Average Case Analysis

Multiplying both sides of previous equation by N and then rewriting the equation for $N-1$ yields

$$NC_N = N(N-1) + 2\sum_{1\le k\le N} C_{k-1}, \qquad (N-1)C_{N-1} = (N-1)(N-2) + 2\sum_{1\le k\le N-1} C_{k-1}$$

Subtracting the 2nd from the 1st and simplifying yields

$$NC_N = (N+1)C_{N-1} + 2N - 2$$

Dividing both sides by $N(N+1)$ gives

$$\frac{C_N}{N+1} = \frac{C_{N-1}}{N} + \frac{2}{N+1} - \frac{2}{N(N+1)}.$$

Average Case Analysis

Telescoping the recurrence down to $N = 3$ and recalling that $C_1 = 0$ yields

$$\begin{aligned}
\frac{C_N}{N+1} &= \frac{C_{N-1}}{N} + \frac{2}{N+1} - \frac{2}{N(N+1)} \\
&= \frac{C_{N-2}}{N-1} + \left(\frac{2}{N} - \frac{2}{(N-1)N}\right) + \left(\frac{2}{N+1} - \frac{2}{N(N+1)}\right) \\
&= \ldots \\
&= \frac{C_1}{2} + 2\sum_{i=3}^{N} \frac{1}{i+1} - \sum_{i=3}^{N} \frac{2}{i(i+1)} \\
&= 2H_{N+1} - 2H_3 + O(1) = 2H_N + O(1)
\end{aligned}$$

where $H_N = \sum_{i=1}^{N} 1/i$ and we are using the fact that $\sum_{i=1}^{\infty} 1/i(i=1)$ is bounded.

We just saw that

$$\frac{C_N}{N+1} = 2H_N + O(1).$$

H_N is called the Nth *Harmonic number* and it is well known that

$$H_n = \ln n + O(1).$$

So, we have just proven that the average number of operations performed running Quicksort on a random permutation of N items is

$$C_N = 2(N+1)H_N + O(N) = 2N \ln N + O(N).$$

Odds and Ends

- Quicksort is a divide and conquer algorithm.
- The Quicksort code can be *tuned*
  - When N is small, call Insertion Sort rather than Quicksort (on very small N, Insertion sort is faster.
  - Instead of using last item $A[r]$ as pivot, set pivot to be median of first, last and middle item. (Why should this help?)
- *qsort* under UNIX was an extremely popular sorting routine for decades. It was a finely tuned version of Quicksort
- Quicksort was published by Tony Hoare in the Communications of the ACM **4**(7), 1961.