Sorting: Lower Bounds and Linear Time

Last Revision: August 25, 2016

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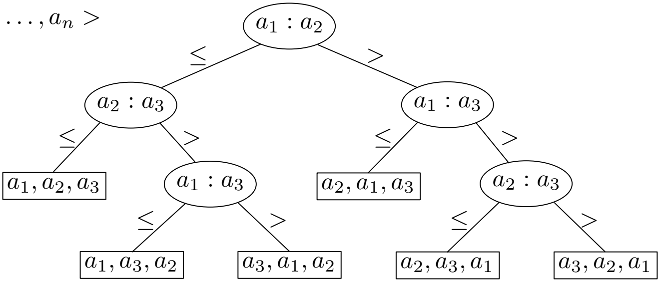
Goal

We will prove that any comparison-based sorting algorithm has a worst-case running time $\Omega(n \log n)$.

Decision-tree Example

Sort $< a_1, a_2, \ldots, a_n >$



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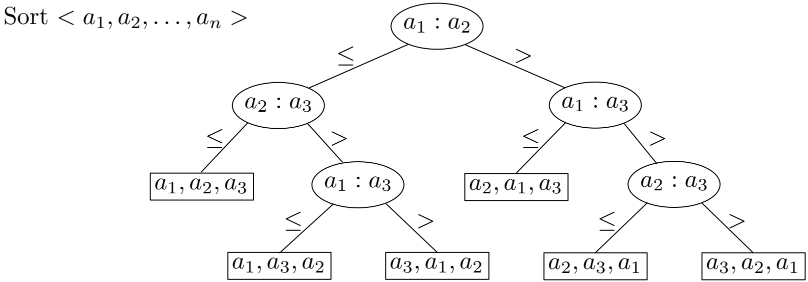

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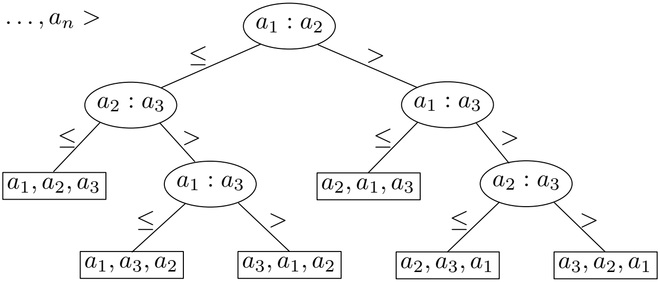

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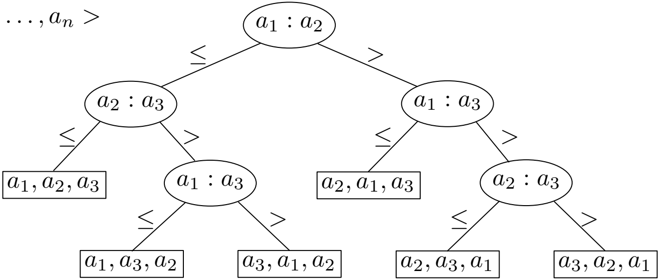

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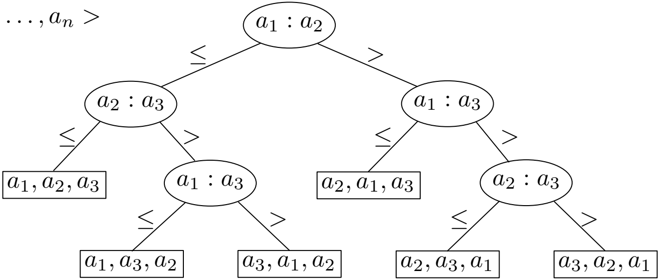

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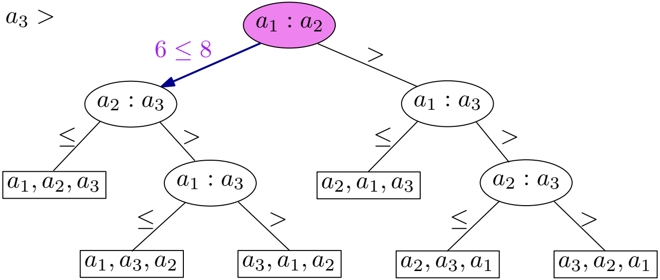

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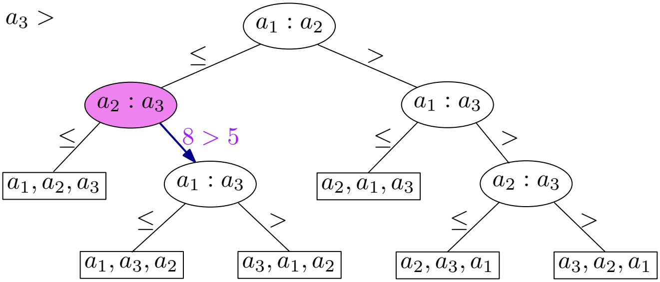

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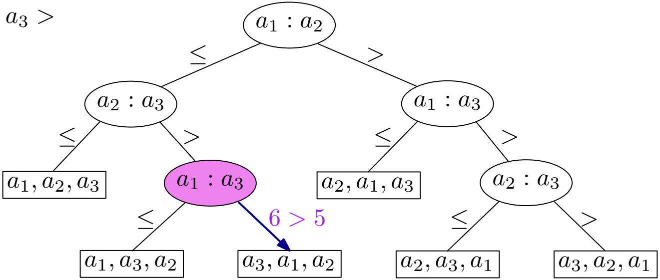

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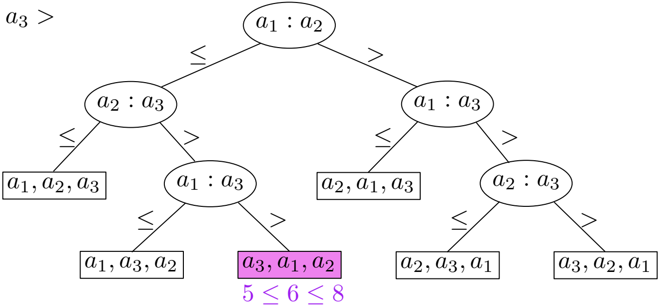

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- Worst-case running time = height of tree

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Any comparison-based sorting algorithm requires $\Omega(n \log n)$ *comparisons in the worst case.*

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Corollary

Heapsort and merge sort are asymptotically optimal comparison-based sorting algorithms.

Lower Bound for Average Running Time of Sorting

- We just proved that worst case number of comparisons used is $\Omega(n \log n)$
- Suppose that each of the $n!$ input permutations is equally likely. What can be said about the average case running time?

Note: Average is taken by adding up individual running time of algorithm on each possible input and dividing total by $n!$.

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- The EPL of a binary tree with m leaves is at least $m \log_2 m + O(m)$.
- The comparison tree has $m = n!$ leaves
  $\Rightarrow$ its external path length is $n! \log_2 n! + O(n!)$
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  $\Rightarrow$ average number of comparisons used is $\log_2 n! + O(1)$.
- We already saw $\log_2 n! = \Omega(n \log n)$.

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- Counting sort
  - Assumes items are in set $\{1, 2, \ldots, k\}$.
  - Is a *stable* sort (defined soon).
- Radix sort
  - Assumes items are stored in fixed size words using finite alphabet

Counting-sort(A, B, k)

```
Input: A[1...n], where A[j] ∈ {1, 2, ..., k}
Output: B[1...n], sorted
let C[1...k] be a new array;
for i ← 1 to k do
  | C[i] ← 0;
end
for j ← 1 to n do
  | C[A[j]] ← C[A[j]] + 1; // C[i] = |{key = i}|
end
for i ← 2 to k do
  | C[i] ← C[i] + C[i − 1]; // C[i] = |{key ≤ i}|
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for j ← n to 1 do
  | B[C[A[j]]] ← A[j];
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end
```

Example: Counting Sort

A

1	2	3	4	5
4	2	1	4	2

C

1	2	3	4

B

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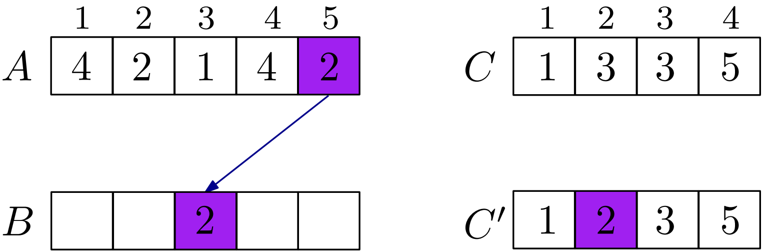

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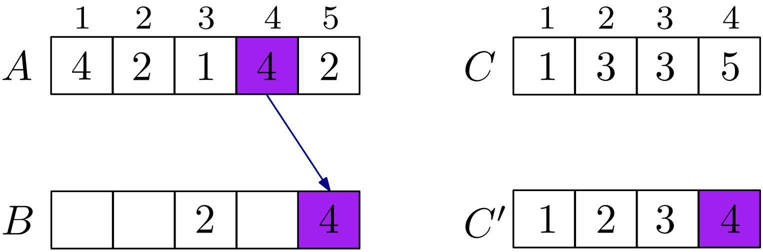

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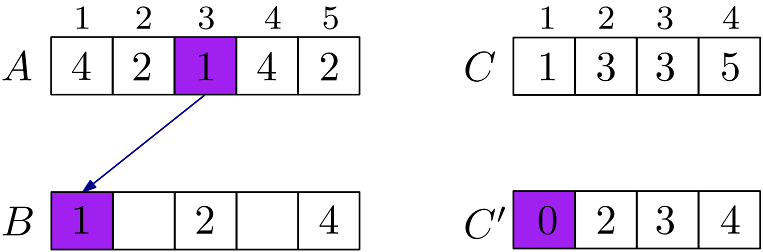

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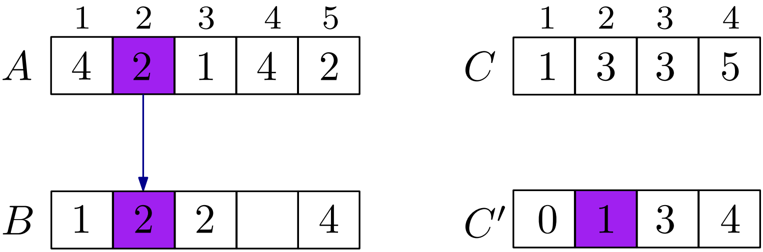

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Input: $A[1 \dots n]$, where $A[j] \in \{1, 2, \dots, k\}$
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let $C[1 \dots k]$ be a new array;
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Total: $O(n + k)$

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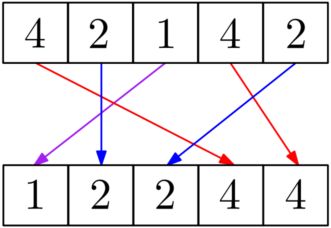
- No, actually we proved that any comparison-based sorting algorithm takes $\Omega(n \log n)$ time.
- Note that counting sort is *not* a comparison-based sorting algorithm.
- In fact, it makes no comparisons at all!

Stable Sorting

Counting sort is a stable sort

- it preserves the input order among equal elements.



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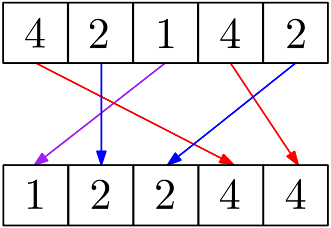

Exercise

What other sorts have this property?

Radix Sort

- Sort on *least significant* digit first using stable sort

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2 3 2 9		2 7 2 **0**		2 7 **2** 0		2 **3** 2 9		**2** 3 2 9
5 4 5 7		5 3 5 **5**		2 3 **2** 9		5 **3** 5 5		**2** 7 2 0
3 6 5 7		3 4 3 **6**		3 4 **3** 6		3 **4** 3 6		**3** 4 3 6
5 8 3 9	→	5 4 5 **7**	→	5 8 **3** 9	→	5 **4** 5 7	→	**3** 6 5 7
3 4 3 6		3 6 5 **7**		5 3 **5** 5		3 **6** 5 7		**5** 3 5 5
2 7 2 0		2 3 2 **9**		5 4 **5** 7		2 **7** 2 0		**5** 4 5 7
5 3 5 5		5 8 3 **9**		3 6 **5** 7		5 **8** 3 9		**5** 8 3 9

Radix Sort: Correctness

Induction on digit position

- Assume that the numbers are sorted by their low-order $i-1$ digits
- Sort on digit i

2720	→	2329
2329		5355
3436		3436
5839		5457
5355		3657
5457		2720
3657		5839

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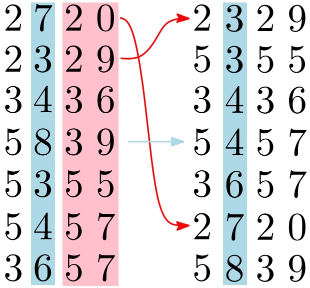

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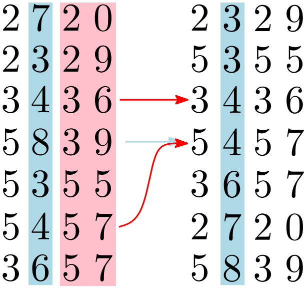
- Sort on digit i
  - Two numbers that differ on digit i are correctly sorted by their low-order i digits
  - Two numbers equal on digit i are put in the same order as the input $\Rightarrow$ correctly sorted by their low-order i digits



Radix Sort: Running Time & Application

Lemma

Given n d-digit numbers in which each digit can take on up to k possible values, radix sort correctly sorts these numbers in $O(d(n+k))$ time if the stable sort it uses takes $O(n+k)$ time.

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- running time is $O(d(n+k)) = O(b(n + 2^{\log n})) = O(bn)$
- since b is a constant, the running time is $O(n)$.