Maximum Flow

Revision of November 7, 2016

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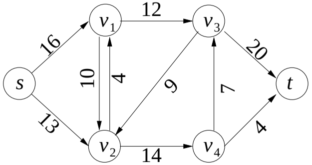
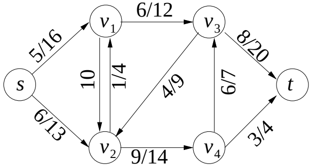
Maximum Flow

Main Reference: Sections 26.1-26.3 in CLRS.

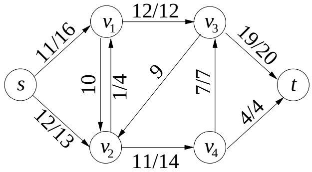
- Input: a directed graph $G = (V, E)$:
  (flow network)
- Source (producer) s and destination t.
- Internal Nodes are *warehouses*
- Edge costs are *capacities*
  Maximum amount that can be shipped over edge
- No storage at internal nodes
  All goods shipped into warehouse must leave warehouse
- **Objective:**
  Ship Maximum amount (flow) from s to t.



A Flow Network and its capacities



A flow: value 11



A max-flow: value = 23

Flow Definition: I

A flow network is a graph $G = (V, E)$.
Source $s \in V,$, sink $t \in V$.

Every edge $(u, v) \in E$ has capacity , $c(u, v) \geq 0$.
Assume that for every $v \in V$,
there is a path from s to v and from v to t.

Flow Definition: II

A **FLOW** is a function $f : V \times V \to R$ satisfying:

- Capacity Constraint:
  $\forall u, v, \in V, \quad f(u,v) \leq c(u,v).$
- Skew Symmetry:
  $\forall u, v, \in V, \quad f(u,v) = -f(v,u).$
- Flow Conservation:
  $\forall u \in V - \{s,t\}, \quad \sum_{v \in V} f(u,v) = 0.$

The **VALUE** of flow f is $\boxed{|f| = \sum_{v \in V} f(s,v)}$.

MAXIMUM-FLOW PROBLEM:

Given G, c, s, t, find f that maximizes $|f|$.

Multi-Source Multi-Sink Problem

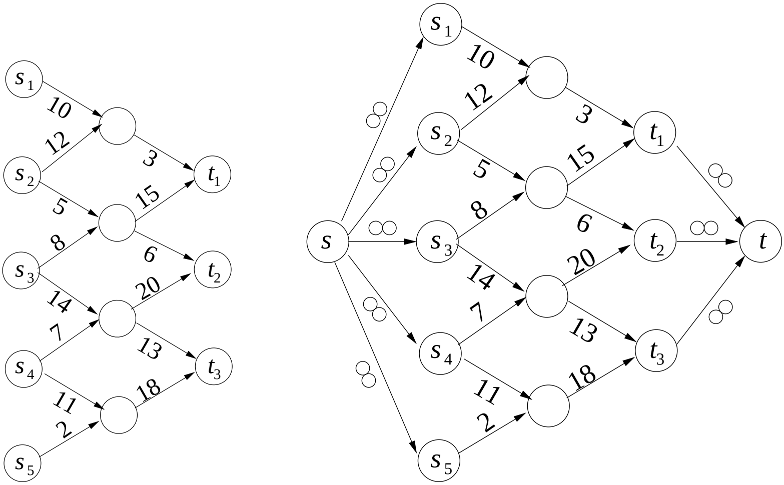
Max-Flow problem has only one source s, and one sink t. Suppose there are
multiple sources $s_1, s_2, \ldots, s_k$ and multiple sinks $t_1, t_2, \ldots, t_\ell$.

Definition of a flow remains the same except that
Flow Conservation property now becomes
$\forall u \in V - \{s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_\ell\}, \quad \sum_{v \in V} f(u, v) = 0$
and our goal is to maximize

$$|f| = \sum_{i=1}^{k} \sum_{v \in V} f(s_i, v).$$

This problem can be reduced to the original one by introducing a *supersource* s_0, a *supersink* t_0 and edges $\cup_i (s_0, s_i)$ and $\cup_j (t_j, t_0)$, all of which have capacity ∞.

A multi-source multi-sink problem and its equivalent single-source single-sink version.



Manipulating Flows

Let $X, Y \subseteq V$. We define

$$f(X,Y) = \sum_{x \in X} \sum_{y \in Y} f(x,y).$$

The *flow-conservation* constraint then just says

$$\forall u \in V - \{s,t\}, \quad f(u,V) = 0.$$

Lemma: (Proof in Homework)

$\forall X \subseteq V, \quad f(X,X) = 0.$

$\forall X, Y \subseteq V, \quad f(X,Y) = -f(Y,X).$

$\forall X, Y, Z \subseteq V$ with $X \cap Y = \emptyset$

$$f(X \cup Y, Z) = f(X,Z) + f(Y,Z) \quad \text{and}$$
$$f(Z, X \cup Y) = f(Z,X) + f(Z,Y)$$

Flow f was defined as

amount that leaves source s.

We now see that this is the same as

amount that enters sink t.

$$
\begin{aligned}
|f| &= f(s, V) && \textbf{definition} \\
&= f(V, V) - f(V - s, V) && \textbf{previous page} \\
&= -f(V - s, V) && \textbf{previous page} \\
&= f(V, V - s) && \textbf{previous page} \\
&= f(V, t) + f(V, V - s - t) && \textbf{previous page} \\
&= f(V, t) && \textbf{flow conservation}
\end{aligned}
$$

All optimization problems must deal with the question:
How to prove that solution *is* optimal (maximal/minimal)?

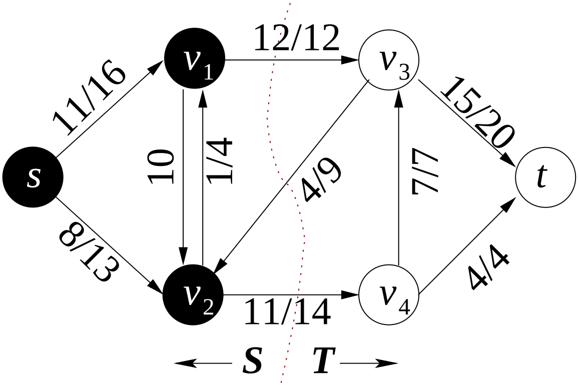
A common technique (for max problems) is to find a good upper-bound on the cost of an optimal solution and then show that our solution satisfies that bound.

A **CUT** S, T of G is a partition of the vertices
$V = S \cup T, \quad S \cap T = \emptyset, \quad s \in S,$ **and** $t \in T.$

The flow **across** the cut is $f(S, T)$.

The capacity of a cut is $C(S, T) = \sum_{x \in S, y \in T} c(x, y)$.

Note that for *any* cut, $f(S, T) \leq C(S, T)$.



Cut (S, T): $S = \{s, v_1, v_2\}$, $T = \{v_3, v_4, t\}$.
The flow value is $|f| = 19$ and $C(S, T) = 26$.
Note that, in this example, $|f| < C(S, T)$.

Lemma:

If S, T is any cut, f any flow, then
$$|f| \leq C(S,T).$$

Proof:

$$\begin{aligned}
|f| &= f(s,V) \\
&= f(s,V) + f(S-s,V) \\
&= f(S,V) \\
&= f(S,V) - f(S,S) \\
&= f(S,V-S) \\
&= f(S,T) \\
&\leq C(S,T)
\end{aligned}$$

We now develop the Ford-Fulkerson method for finding max-flows. When FF terminates it provides a flow f and a cut S, T such that $|f| = C(S,T)$, so f is maximal.

The Ford-Fulkerson Method

- Is iterative.
- Starts with flow $f = 0$, $(\forall u, v, f(u, v) = 0)$
- At each step
  - Constructs a residual network G_f of f indicating how much capacity "remains" to be used .
  - Finds an augmenting path s-t path p in G_f along which flow can be pushed.
  - pushes f' units of flow along p.
    Creates new flow $f = f + f'$.
- Stops when there is no s-t path in current G_f.
- $S =$ set of nodes reachable from s in G_f & $T = V - S$.
- At end of algorithm: $|f| = C(S, T) \Rightarrow f$ is optimal

Residual networks

Given flow f, the residual network G_f consists of the edges along which we can (still) push more flow. The amount that can (still) be pushed across (u, v) is called the ***residual capacity*** $c_f(u, v)$.

$$c_f(u, v) = c(u, v) - f(u, v).$$

If there is flow from u to v then $f(u, v) > 0$ and $c_f(u, v)$ is the remaining capacity on (u, v).

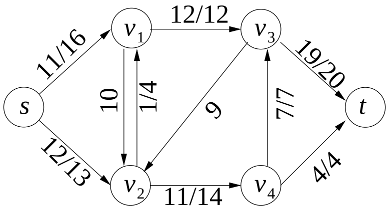
Residual Capacity: $c_f(u,v) = c(u,v) - f(u,v)$.

If there is flow from u to v then $f(u,v) > 0$
and $c_f(u,v)$ is the remaining capacity on (u,v).

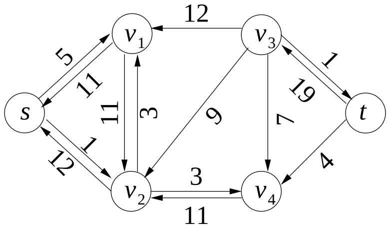
If there is flow from v to u then $f(u,v) < 0$,
and $c_f(u,v) = c(u,v) + f(v,u)$ is the capacity of (u,v)
plus amount of existing flow that can be pushed
backwards from u to v.

The *Residual Network* G_f is $G_f = (V, E_f)$ where

$$E_f = \{(u,v) \in V \times V \,:\, c_f(u,v) > 0\}$$



A Flow



Its residual network

Lemma:

Let f be a flow in $G = (V, E)$ and G_f its residual network.Let f' be a flow in G_f.

Define $f + f'$ as $(f + f')(u, v) = f(u, v) + f'(u, v)$.

Then $f + f'$ is a flow in G with value
$|f + f'| = |f| + |f'|$.

Augmenting path p is a simple s-t path in G_f.
The residual capacity of a.p. p is

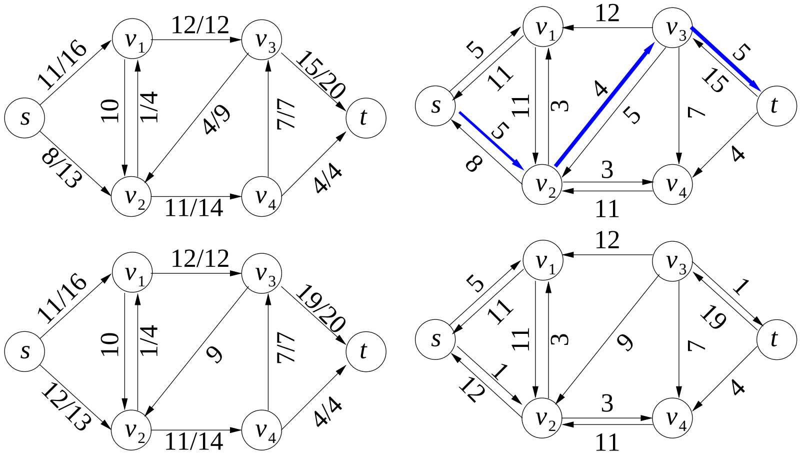
$$c_f(p) = \min\{c_f(u, v) : (u, v) \text{ on } p\}.$$

Let p be an augmenting path in G_f and define

$$f_p(u,v) = \begin{cases} c_f(p) & \text{if } (u,v) \text{ is on } p \\ -c_f(p) & \text{if } (v,u) \text{ is on } p \\ 0 & \text{otherwise} \end{cases}$$

Lemma: If f is a flow and p an a.p.in G_f then:
f_p is a flow in G_f with $|f_p| = c_f(p) > 0$.

$f' = f + f_p$ is a flow in G with $|f'| = |f| + |f_p| > |f|$.



An initial flow f.

Its residual network G_f and an augmenting path f' in G_f.

The flow $f + f'$ and its residual network.

Optimality

Theorem: (Max-Flow Min-Cut Theorem)
Let f be a flow.
Then the following three conditions are equivalent:

1. f is a maximum flow in G.
2. G_f contains no augmenting paths
3. $|f| = C(S,T)$ for some (S,T) cut.

Proof:

- (1) $\Rightarrow$ (2): If G_f contained an augmenting path p then $|f + f_p| > |f|$ so f could not be maximal.

- (2) $\Rightarrow$ (3): Let $S = \{u \in V : \exists$ path from s to v in $G_f\}$. $T = V - S$. Then

  $$f(S,T) = f(S,V) - f(S,S) = f(S,V) = f(s,V) + f(S-s,V) = |f|.$$

  Now note that $\forall u \in S, v \in T$, $f(u,v) = c(u,v)$ since otherwise $c_f(u,v) > 0$ and $v \in S$.
  Thus $C(S,T) = f(S,T) = |f|$.

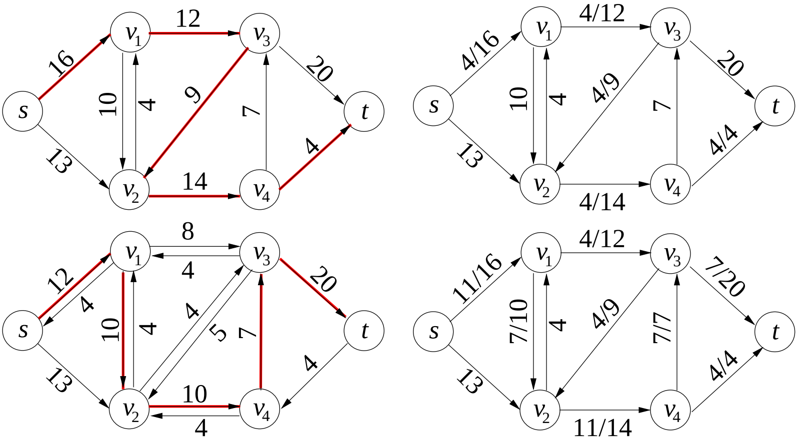
- (3) $\Rightarrow$ (1): We previously saw that every flow f' must satisfy $|f'| \le C(S,T)$ so if $|f| = C(S,T)$, f must be optimal.

Theorem Proof

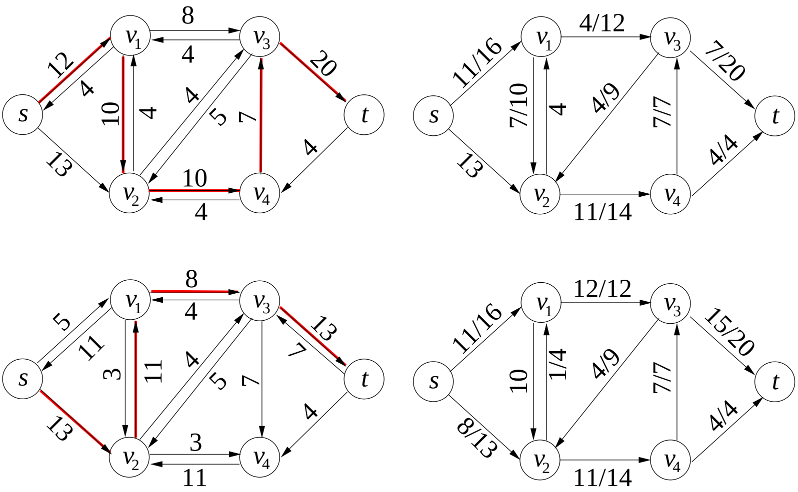
The Ford-Fulkerson Method

- Starts with flow $f \equiv 0$, $(\forall u, v, f(u,v) = 0)$
- Construct residual network G_f.
  If G_f contains no augmenting path, stop
  (f is optimal by MFMC theorem).
  Otherwise.
  1. Find an augmenting path $(s - t$ path) p in G_f
  2. Let f_p be the flow in G_f that pushes $c_f(p)$ units of flow along p.
  3. Let $f = f + f_p$ be new flow in G.

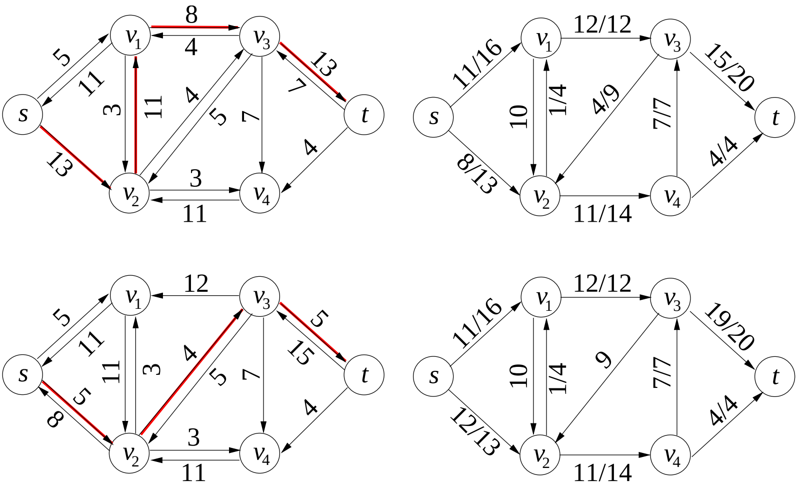
FF Example: Steps 1 & 2



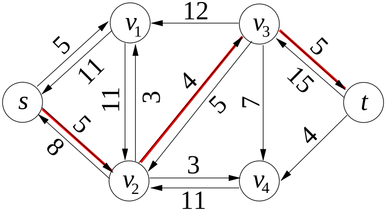
FF Example: Steps 2 & 3

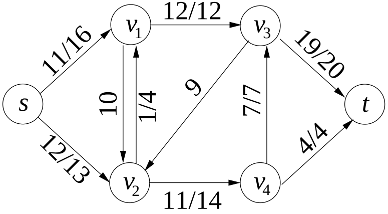


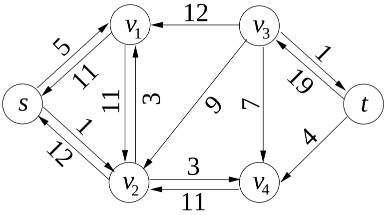
FF Example: Steps 3 & 4



FF Example: Steps 4 & 5 (End)







Running Time & Finiteness

The FF method is not a completely defined algorithm since it doesn't specify how to *choose* the augmenting paths.

In fact, if the capacities are irrational, it is possible that a "bad" way of choosing the a.p. will lead to a non-terminating algorithm that will never stop (it will keep on adding cheaper and cheaper augmenting paths).

If the capacities are all integers
$\Rightarrow$ then each c_p will be an integer ≥ 1
$\Rightarrow$ the algorithm must terminate after $|f^*|$ steps,
where f^* is a max-flow.

Maintaining the graphs G and G_f and the flow f using adjacency lists, while using DFS or BFS to find a s-t path, the algorithm can then be implemented to run in $O(|f^*||E|)$ time.

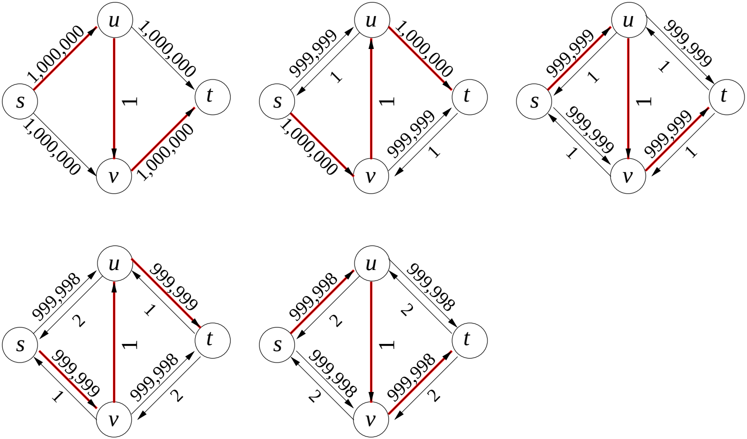
Note: This can be normalized to work if the capacities are rational.

Running Time

- Starts with flow $f \equiv 0$, $O(|E|)$
- Construct residual network G_f. $O(|E|)$
  If G_f contains no augmenting path, stop
  (f is optimal by MFMC theorem).
  Otherwise. Can be repeated $O(|f^*|)$ times.

  1. Find an augmenting $s - t$ path p in G_f $O(|E|)$
  2. Let f_p be the flow in G_f that pushes $c_f(p)$ units of flow along p.
  3. Let $f = f + f_p$ be new flow in G. $O(|E|)$

A pathological example in which each augmenting path only increases flow value by 1 unit.



The Edmonds-Karp Algorithm

Always choose an augmenting path of minimum-length in G_f (where each edge has unit length). This can be done **in $\mathbf{O(E)}$ time** using BFS.

Theorem: The EK alg performs at most $\mathbf{O(VE)}$ path-augmentations, so the E.K. alg runs in $\mathbf{O(VE^2)}$ time.

Let $\delta_f(u, v)$ denote shortest-path distance from u to v in G_f.

The proof of the Theorem is a consequence of the following two lemmas:

Lemma: $\forall v \in V - \{s,t\}$, $\delta_f(s,v)$ does not decrease after a flow augmentation.

Lemma:
Edge (u,v) is *critical* on a.p. p if $c_f(u,v) = c_f(p)$.
Suppose when running the E.K. algorithm that (u,v) is critical for a.p. p in G_f, and is later critical again for another a.p. p' in $G_{f'}$. Then

$$\delta_{f'}(s,u) \geq \delta_f(s,u) + 2.$$

Augmenting paths are simple and do not contain s,t internally, so $\delta_f(s,v)$ is always $\leq |V|-2$ (as long as v is reachable). Combining the two lemmas therefore shows that no specific edge can become critical more than $(|V|-2)/2 = O(|V|)$ times. *Some* edge is critical in each step, so there can be at most $O(|V||E|)$ steps.

Application: Max Bipartite Matching

A graph $G = (V, E)$ is *bipartite* if there exists partition $V = L \cup R$ with $L \cap R = \emptyset$ and $E \subseteq L \times R$.

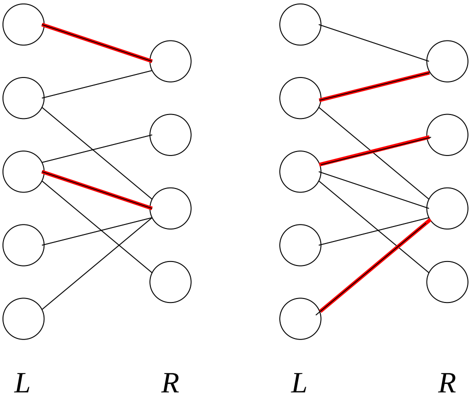
A *Matching* is a subset $M \subseteq E$ such that $\forall v \in V$ at most one edge in M is incident upon v.

The *size* of a matching is $|M|$, the number of edges in M.

A *Maximum Matching* is matching M such that every other matching M' satisfies $|M'| \leq M$.

Problem: Given bipartite graph G, find a maximum matching.

A bipartite graph with 2 matchings



Our approach will be to write the Max Bipartite Matching problem as a Max-Flow problem.

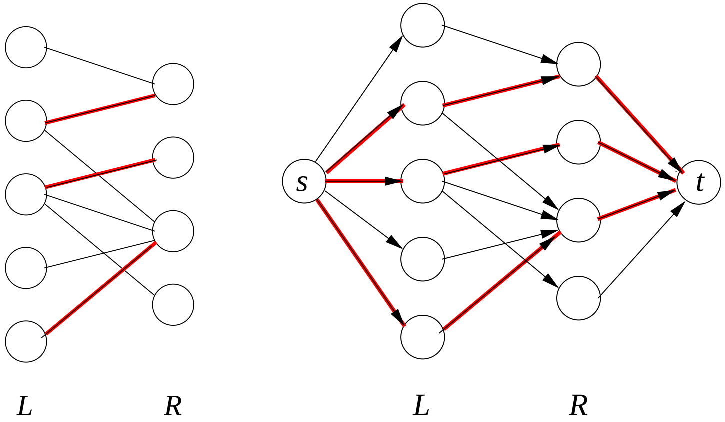
Our *flow network* will be $G' = (V', E')$ where
$V' = V \cup \{s, t\}$ and
$E' = \{(s, u) \,:\, u \in L\} \cup \{(u, v) \,:\, u \in L, v \in R \text{ and } (u, v) \in E\}$
$\cup \{(v, t) \,:\, t \in R\}$

We also assign
$\forall (u, v) \in E',\ c(u, v) = 1.$

Lemma: If f is an integer valued flow in G' then there is a matching M of G with $|f| = |M|$.
Similarly, if M is a matching of G then there is an integer valued flow f with $|f| = |M|$.

This *almost* tells us that Max-Flow solves our problem. The difficulty is that it's possible that the max-flow might not have integer value (it is possible that $|f|$ might be an integer but some $f(u, v)$ might not be integers).

A bipartite graph and its associated flow network.
A matching and associated flow are illustrated



Theorem:

Let $G' = (V', E')$ be a flow network in which c is integral. Then the max-flow f found by the F.F. method has the property that

$$\forall u, v, f(u, v) \text{is integer valued.}$$

The proof is by induction on the steps in the FF method.

At each step the current flow f is integer so the residual capacities are all integer.

This implies that the a.p. found has $c_f(p)$ integral, so the new flow $f + f'$ created is also integral.

The theorem guarantees that if G' is the flow network corresponding to a bipartite matching problem then max flow value $|f|$ is the value of a maximum matching.

The flow found by the FF algorithm can be modified to yield the max matching.

The FF algorithm run on this special graph will take $\mathbf{O(VE)}$ time (why?).

Odds and Ends

- A faster implementation of the FF method uses the idea of *blocking flows* developed by Dinic. This approach finds many augmenting paths at once.
- A totally different approach to the Max-Flow algorithm is the *push-relabel* method (see CLRS for details). This can run in $O(|V|^3)$ time as compared to the $O(|V||E|^2)$ of FF.
- General Culture: The max-flow problem can be written as a *linear program*. The FF method is essentially a special case of the *primal-dual* algorithm for solving combinatorial Linear Programs.