Union Find

Version of October 11, 2016

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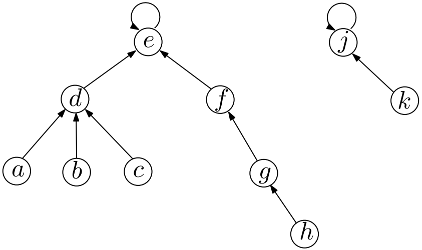
   - After this operation, we have Find-Set(x) = Find-Set(y).

Outline

- The Disjoint Set Union-Find data structure
  - The basic implementation
  - An improvement

Up-Tree Implementation



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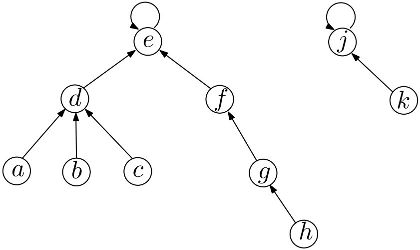

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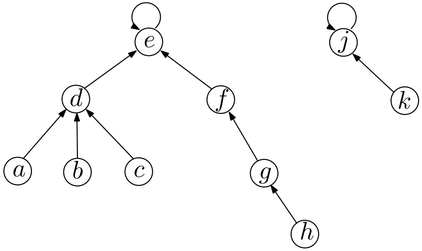

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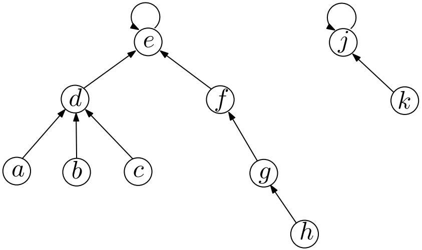

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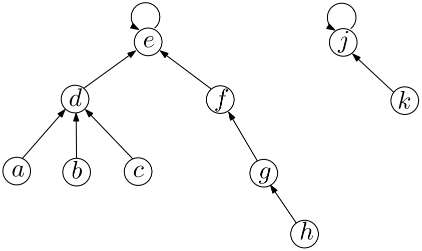

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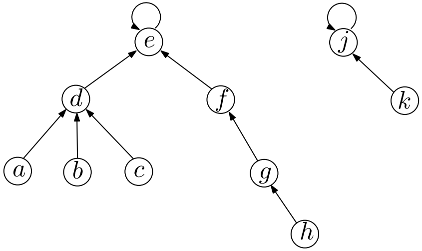

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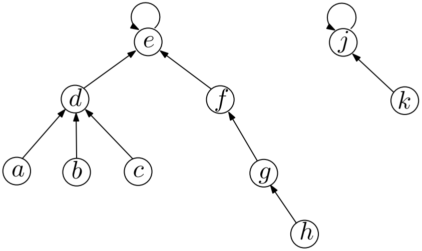

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  - The root element has a pointer pointing to itself.

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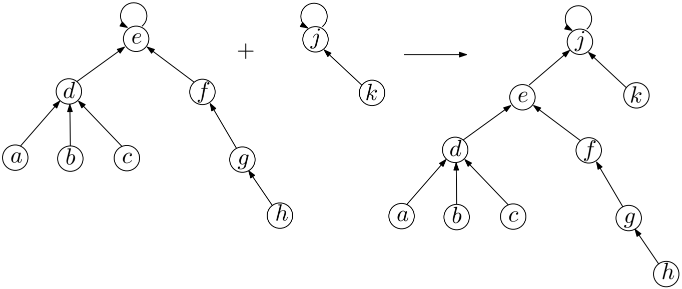

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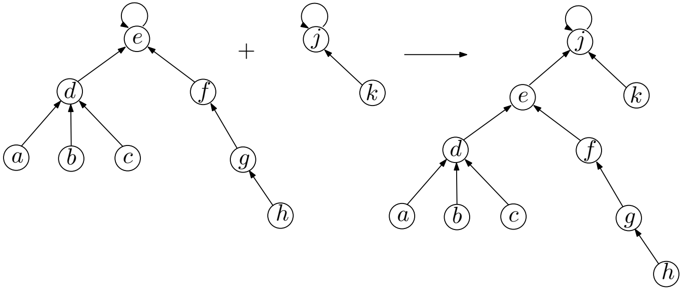

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Is this a good idea?

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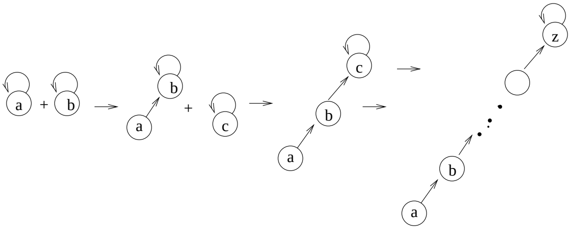

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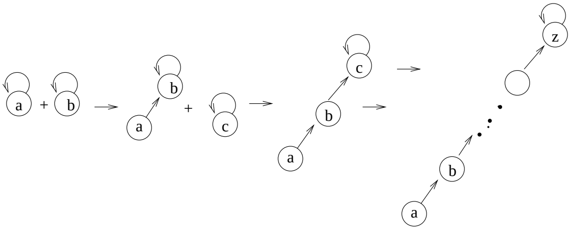

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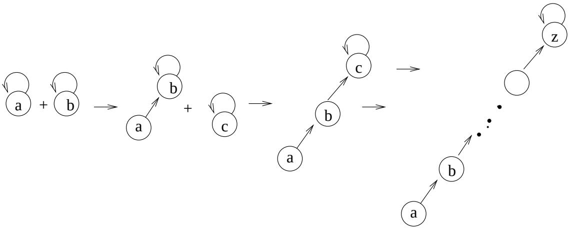

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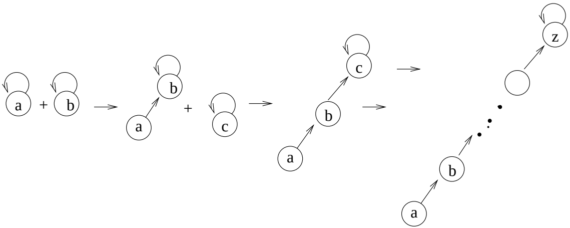

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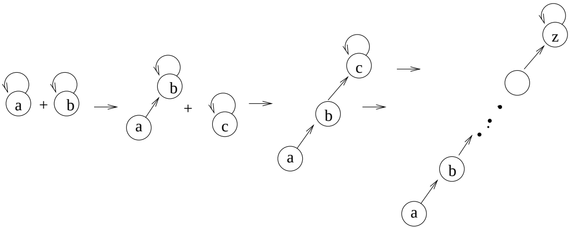

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- when we union two trees together, we always make the root of the taller tree the parent of shorter tree.

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Hence we have Find-Set(x) $= O(\log n)$. $\square$

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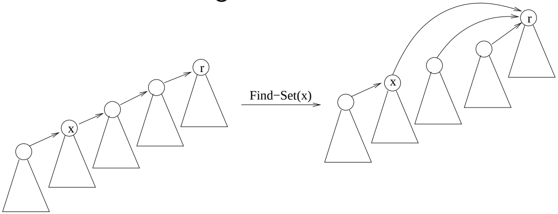

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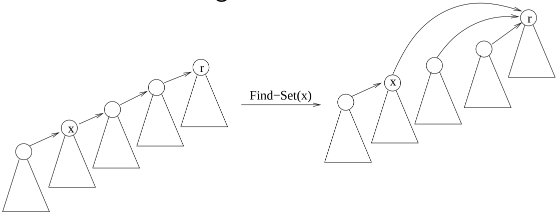

- This idea is called path compression.

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- e.g.,
  $\lg^* 2 = 1, \lg^* 4 = 2, \lg^* 16 = 3, \lg^* 65536 = 4, \lg^* 2^{65536} = 5.$

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What is the running time of Kruskal's algorithm if we employ this implementation of disjoint set Union-Find?