$\boxed{\text{Linear Programming Duality}}$

P&S Chapter 3

Last Revised – October 28, 2006

In this section we lean about duality, which is another way to approach linear programming. In particular, we will see:

- How to define the dual of a normal (primal) linear program.
  If the primal is a minimization problem, the dual is a maximization problem.

- That the optimum solution to the dual has cost equal to that of the optimum solution to the primal.

- Complementary Slackness:
  A combinatorial statement of the relationship between the primal and the dual

- A primal-dual interpretation of the shortest path problem.

Convert an LP in general form such as one on the right into a LP in standard form using method seen earlier. That is

$$\begin{aligned} &\min c'x \\ a_i'x &= b_i \quad i \in M \\ a_i'x &\geq b_i \quad i \in \bar{M} \\ x_j &\geq 0 \quad j \in N \\ x_j &\gtrless 0 \quad j \in \bar{N} \end{aligned}$$

$$\begin{aligned} &\min \hat{c}'\hat{x} \\ \hat{A}\hat{x} &= b \\ \hat{x} &\geq 0 \end{aligned}$$

where

$$\begin{aligned} \hat{A} &= \left[A_j, j \in N \middle| (A_j, -A_j), j \in \bar{N} \middle| \begin{array}{c} 0, i \in M \\ \hline -I, i \in \bar{M} \end{array} \right] \\ \hat{x} &= \operatorname{col}(x_j, j \in N | (x_j^+, x_j^-), j \in \bar{N} | x_i^s, i \in \bar{M}) \\ \hat{c} &= \operatorname{col}(c_j, j \in N | (c_j, -c_j), j \in \bar{N} | 0) \end{aligned}$$

$$\begin{aligned} \min\ & \widehat{c}'\widehat{x} \\ \widehat{A}\widehat{x} &= b \\ \widehat{x} &\geq 0 \end{aligned} \qquad \begin{aligned} \widehat{A} &= \left[A_j, j \in N \middle| (A_j, -A_j), j \in \bar{N} \middle| \frac{0, i \in M}{-I, i \in \bar{M}} \right] \\ \widehat{x} &= \text{col}(x_j, j \in N | (x_j^+, x_j^-), j \in \bar{N} | x_i^s, i \in \bar{M}) \\ \widehat{c} &= \text{col}(c_j, j \in N | (c_j, -c_j), j \in \bar{N} | 0) \end{aligned}$$

Recall that BFS $\widehat{x}_0$ is optimal iff corresponding $\bar{c} \geq 0$. If this occurs, there is a corresponding basis $\widehat{\mathcal{B}}$ s.t.

$$\widehat{c}' - \left(\widehat{c}'_B \widehat{B}^{-1}\right) \widehat{A} \geq 0$$

so $\pi' = \widehat{c}'_B \widehat{B}^{-1}$ is a *feasible* solution to

$$\pi' \widehat{A} \leq \widehat{c}'$$

where $\pi \in R^m$ and $m = |M| + |\bar{M}|$ is # of rows in original LP.

Note that there are *three* different sets of inequalities, one each for $j \in N$, $j \in \bar{N}$, and $i \in \bar{M}$.

$$\min \hat{c}'\hat{x} \qquad \hat{A} = \left[A_j, j \in N \middle| (A_j, -A_j), j \in \bar{N} \middle| \frac{0, i \in M}{-I, i \in \bar{M}} \right]$$

$$\hat{A}\hat{x} = b \qquad \hat{x} = \mathrm{col}(x_j, j \in N | (x_j^+, x_j^-), j \in \bar{N} | x_i^s, i \in \bar{M})$$

$$\hat{x} \geq 0 \qquad \hat{c} = \mathrm{col}(c_j, j \in N | (c_j, -c_j), j \in \bar{N} | 0)$$

which we saw leads to $\pi' \hat{\mathbf{A}} \leq \hat{\mathbf{c}}'$

1. If $j \in N$ then $\pi' \mathbf{A_j} \leq \mathbf{c_j}$

2. If $j \in \bar{N}$ then $\pi' A_j \leq c_j$ **and** $-\pi' A_j \leq -c_j$ so $\pi' \mathbf{A_j} = \mathbf{c_j}$.

3. If $i \in \bar{M}$ then $-\pi' \leq 0$ so $\pi' \geq \mathbf{0}$.

Using these equations, given a primal LP in general form, we can define a new LP in dual form.

Primal		**Dual**
$\min c'x$		$\max \pi' b$
$a_i' x = b_i$	$i \in M$	$\pi_i \gtrless 0$
$a_i' x \geq b_i$	$i \in \bar{M}$	$\pi_i \geq 0$
$x_j \geq 0$	$j \in N$	$\pi' A_j \leq c_j$
$x_j \gtrless 0$	$j \in \bar{N}$	$\pi' A_j = c_j$

Primal		**Dual**
$\min c'x$		$\max \pi' b$
$a_i' x = b_i$	$i \in M$	$\pi_i \gtrless 0$
$a_i' x \geq b_i$	$i \in \bar{M}$	$\pi_i \geq 0$
$x_j \geq 0$	$j \in N$	$\pi' A_j \leq c_j$
$x_j \gtrless 0$	$j \in \bar{N}$	$\pi' A_j = c_j$

Theorem: If an LP has an optimal solution, so does its dual and, at optimality, their costs are equal.

Proof: Let x, π be feasible solutions to the primal and dual. Then

$$c'x \geq \pi' Ax \geq \pi' b \tag{1}$$

Since primal has optimal solution, dual can not have *unbounded* feasible solutions. We saw before that, if $\hat{x}_0$ is optimal BFS in primal with basis B then $\pi' = \hat{c}_B' \hat{B}^{-1}$ is, by construction, *feasible* in dual. This means dual has a feasible solution so, by simplex algorithm, dual has an *optimal* (bounded) solution.

The cost of this particular π' is

$$\pi' b = \hat{c}_B' \hat{B}^{-1} b = \hat{c}_B' \hat{x}_0$$

Therefore, by (1), π' is optimal in dual.

Primal		**Dual**
$\min c'x$		$\max \pi' b$
$a_i' x = b_i$	$i \in M$	$\pi_i \gtrless 0$
$a_i' x \geq b_i$	$i \in \bar{M}$	$\pi_i \geq 0$
$x_j \geq 0$	$j \in N$	$\pi' A_j \leq c_j$
$x_j \gtrless 0$	$j \in \bar{N}$	$\pi' A_j = c_j$

Theorem: The dual of the dual is the primal.

Proof: Write dual as

$$
\begin{array}{rl}
\min \pi'(-b) & \\
(-A_j')\pi \geq -c_j & j \in N \\
(-A_j')\pi = -c_j & j \in \bar{N} \\
\pi_i \geq 0 & i \in \bar{M} \\
\pi_i \gtrless 0 & i \in M
\end{array}
$$

and consider it as primal. Then

$$
\begin{array}{rl}
\max x'(-c) & \\
x_j \geq 0 & j \in N \\
x_j \gtrless 0 & j \in \bar{N} \\
-a_i' x \leq -b & i \in \bar{M} \\
-a_i' x = -b & i \in M
\end{array}
$$

Theorem: Given a primal-dual pair, exactly one of the three situations occurring below occurs. The X's denote situations which can not occur.

Primal \ Dual	Finite optimum	Unbounded	Infeasible
Finite optimum	1	X	X
Unbounded	X	X	3
Infeasible	X	3	2

Proof: We already saw that (1) is possible. We also saw that if primal has finite optimum, then the dual *must* have a finite optimum so the X's in the first row are also correct. Since the dual of the dual is the primal we find, by symmetry, that the X's in the first column are correct.

We also saw that *any* feasible solution to the problem upper bounds the cost of *all* feasible solutions to the dual so it is impossible for both of them to simultaneously have unbounded solutions. We will see examples of (2) and (3) on the next slide.

PRIMAL

$$\begin{aligned} \min x_1 & \\ x_1 + x_2 &\geq 1 \\ -x_1 - x_2 &\geq 1 \\ x_1 &\gtreqless 0 \\ x_2 &\gtreqless 0 \end{aligned}$$

DUAL

$$\begin{aligned} \max \pi_1 &+ \pi_2 \\ \pi_1 - \pi_2 &= 1 \\ \pi_1 - \pi_2 &= 0 \\ \pi_1 &\geq 0 \\ \pi_2 &\geq 0 \end{aligned}$$

Note that both the Primal and Dual are infeasible, giving case 2. Now modify the primal so that $x_1, x_2 \geq 0$. Then

PRIMAL

$$\begin{aligned} \min x_1 & \\ x_1 + x_2 &\geq 1 \\ -x_1 - x_2 &\geq 1 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

DUAL

$$\begin{aligned} \max \pi_1 &+ \pi_2 \\ \pi_1 - \pi_2 &\leq 1 \\ \pi_1 - \pi_2 &\leq 0 \\ \pi_1 &\geq 0 \\ \pi_2 &\geq 0 \end{aligned}$$

and the primal is infeasible while the dual is unbounded (case 3). Primal unbounded and dual infeasible follows by flipping the primal and the dual.

Recall the **Diet Problem:**

$$\begin{aligned} \min\ & c'x \\ Ax &\geq r \\ x &\geq 0 \end{aligned}$$

$$\begin{aligned} a_{i,j} &= \text{amount of } i\text{th nutrient in a unit of the } j\text{th food} \\ & \quad i = 1, \ldots, m,\ j = 1, \ldots, n \\ r_i &= \text{yearly requirement of } i\text{th nutrient} \\ & \quad i = 1, \ldots, m \\ x_j &= \text{yearly consumption of the } j\text{th food (in units)} \\ & \quad j = 1, \ldots, n \\ c_j &= \text{cost per unit of the } j\text{th food} \\ & \quad j = 1, \ldots, n \end{aligned}$$

The Dual is

$$\begin{aligned} \max \pi' r \\ \pi' A \leq c' \\ \pi' \geq 0 \end{aligned}$$

This can be interpreted as saying that a pill-manufacturer wants to sell m nutrients; pill i containing one unit of nutrient i at price π_i. Manufacturer wants to maximize revenue. Constraint is that, for each food j, it must be cheaper for customer to satisfy nutritional requirements via pills rather than eating food.

Complementary Slackness Conditions

Theorem: A pair x, π respectively feasible in a primal-dual pair are optimal iff

$$\begin{aligned} u_i &= \pi_i(a_i'x - b_i) = 0 \quad \text{for all } i & (2) \\ v_j &= (c_j - \pi' A_j)x_j = 0 \quad \text{for all } j & (3) \end{aligned}$$

Proof: By duality definitions
$\forall i,\ u_i \geq 0$ and $\forall j,\ v_j \geq 0$. Now set

$$u = \sum_i u_i \geq 0 \quad \text{and} \quad v = \sum_j v_j \geq 0$$

Then $u = 0$ iff (2) and $v = 0$ iff (3).

Now note that

$$u + v = c'x - \pi' b.$$

So, (2) and (3) are true iff $c'x = \pi' b$ which is true iff both x and π are optimal.

Example

Primal

$$
\begin{array}{lrcrcrcrcrcr}
\min & x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & & \\
 & 3x_1 & + & 2x_2 & + & x_3 & & & & & = & 1 \\
 & 5x_1 & + & x_2 & + & x_3 & + & x_4 & & & = & 3 \\
 & 2x_1 & + & 5x_2 & + & x_3 & + & x_5 & & & = & 4 \\
 & & & & & & & x_i & & & \geq & 0
\end{array}
$$

Dual

$$
\begin{array}{lrcrcrcl}
\max & \pi_1 & + & 3\pi_2 & + & 4\pi_3 & & \\
 & 3\pi_1 & + & 5\pi_2 & + & 2\pi_3 & \leq & 1 \\
 & 2\pi_1 & + & \pi_2 & + & 5\pi_3 & \leq & 1 \\
 & \pi_1 & + & \pi_2 & + & \pi_3 & \leq & 1 \\
 & & & \pi_2 & & & \leq & 1 \\
 & & & & & \pi_3 & \leq & 1 \\
 & & & & & \pi_i & \gtrless & 0 \quad \text{for all } i
\end{array}
$$

Because primal is in standard form we have

$$\forall i,\ u_i = \pi_i(a_i'x - b_i) = 0.$$

We now need to satisfy

$$\forall j, v_j = (c_j - \pi' A_j)x_j = 0.$$

Primal

$$
\begin{array}{rrrrrrrrrrr}
\min & x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 \\
& 3x_1 & + & 2x_2 & + & x_3 & & & = & 1 \\
& 5x_1 & + & x_2 & + & x_3 & + & x_4 & = & 3 \\
& 2x_1 & + & 5x_2 & + & x_3 & + & x_5 & = & 4 \\
& & & & & & & x_i & \geq & 0
\end{array}
$$

Dual

$$
\begin{array}{rrrrrrl}
\max & \pi_1 + & 3\pi_2 + & 4\pi_3 & & & \\
& 3\pi_1 + & 5\pi_2 + & 2\pi_3 & \leq & 1 & \\
& 2\pi_1 + & \pi_2 + & 5\pi_3 & \leq & 1 & \\
& \pi_1 + & \pi_2 + & \pi_3 & \leq & 1 & \\
& & \pi_2 & & \leq & 1 & \\
& & & \pi_3 & \leq & 1 & \\
& & & \pi_i & \gtrless & 0 & \text{for all } i
\end{array}
$$

We now need to satisfy
$\forall j, v_j = (c_j - \pi' A_j)x_j = 0$.
Optimal primal is $(0, 1/2, 0, 5/2, 3/2)$ (with cost $9/2$) so need equality in 2nd, 4th and 5th equations, i.e.,

$$
\begin{array}{ccccccccccccc}
c_2 & - & \pi' A_2 & = & 0 & \text{or} & 2\pi_1 & + & \pi_2 & + & 5\pi_3 & = & 1 \\
c_4 & - & \pi' A_4 & = & 0 & \text{or} & & & \pi_2 & & & = & 1 \\
c_5 & - & \pi' A_5 & = & 0 & \text{or} & & & & & \pi_3 & = & 1
\end{array}
$$

which has solution

$$(\pi_1, \pi_2, \pi_3) = (-5/2,\ 1,\ 1).$$

Note that this has same cost $9/2$ as primal and is therefore optimal.

The Shortest Path problem & its dual

Let $G = (V, E)$ be a *directed* graph s.t. every edge $e_j \in E$ has cost $c_j \geq 0$. The shortest path problem is to find a directed path from source s to sink t with minimal cost. We will now see how to write this as an LP and then derive its dual.

The *feasible set* of this problem is

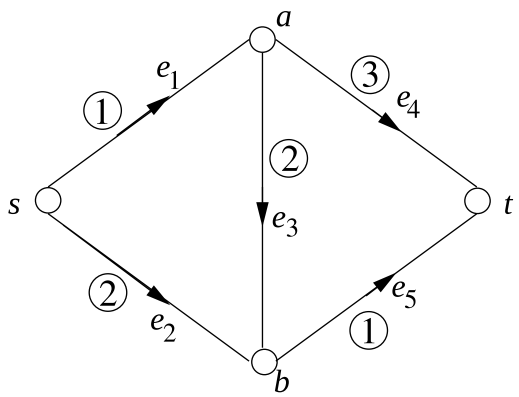
$$F = \{ \text{ sequences } P = (e_{j_1}, \ldots, e_{j_k}) : \text{this sequence is a directed path from } s \text{ to } t \text{ in } G\}$$

with cost $c(P) = \sum_{i=1}^{k} c_{j_i}$.

Now define the *node-incidence* matrix $A = [a_{i,j}]$,

$$a_{ij} = \begin{cases} +1 & \text{if arc } e_j \text{ leaves node } i \\ -1 & \text{if arc } e_j \text{ enters node } i \\ 0 & \text{otherwise} \end{cases} \quad \left(\begin{array}{l} i = 1, \ldots, |V| \text{ and} \\ j = 1, \ldots, |E| \end{array} \right)$$

$$a_{ij} = \begin{cases} +1 & \text{if arc } e_j \text{ leaves node } i \\ -1 & \text{if arc } e_j \text{ enters node } i \\ 0 & \text{otherwise} \end{cases} \quad \left(\begin{array}{l} i = 1, \ldots, |V| \text{ and} \\ j = 1, \ldots, |E| \end{array} \right)$$



$$A = \begin{array}{c} \\ s \\ t \\ a \\ b \end{array} \left[\begin{array}{rrrrr} e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline +1 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ -1 & 0 & +1 & +1 & 0 \\ 0 & -1 & -1 & 0 & +1 \end{array} \right]$$

To create a LP introduce f_j to denote *flow* through arc e_j. The intuition is that we would like to send one unit of flow from s to t. The cost of one unit of flow through e_j will be c_j so a shortest $s-t$ path "should" be one that minimizes cost.

We need *flow conservation* at every non s,t node v_i. This corresponds to $a_i' f = 0$. On the other hand, one net unit of flow leaves s and one net unit enters t so the problem that we want to solve is

$$\min c' f$$

$$Af = \begin{bmatrix} +1 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$f \geq 0$$

where the $+1$ corresponds to row s and the -1 corresponds to row t.

Note that it is possible that the f_j "could" take on non-integer values but it is easy to see that there is an optimal solution in which each $f_j = 1$ (in shortest path) or $f_j = 0$ (not in shortest path). We will prove this more formally later when discussing *unimodular* matrices.

$$\min c'f$$

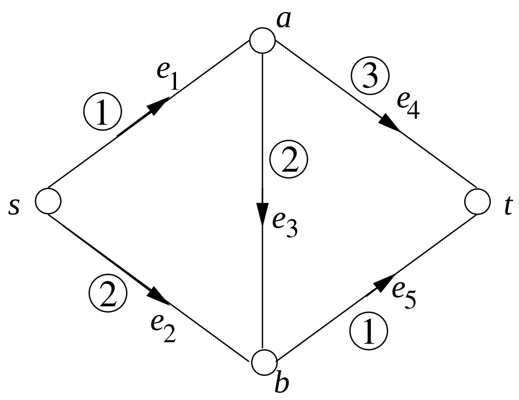
$$Af = \begin{bmatrix} +1 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$f \geq 0$$

where the $+1$ corresponds to row s and the -1 corresponds to row t.

Note that the $|V|$ equations are redundant and we can therefore leave out any **one** equation. It is most convenient to leave out the row t equation since this will leave a nonnegative cost column in our simplex tableau.

Example

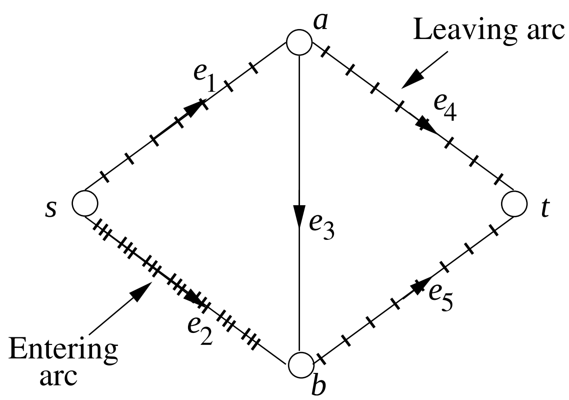


		f_1	f_2	f_3	f_4	f_5
$-z=$	0	1	2	2	3	1
	1	1	1	0	0	0
	0	-1	0	1	1	0
	0	0	-1	-1	0	1

Now create a basis with $\{f_1, f_4, f_5\}$.

		f_1	f_2	f_3	f_4	f_5
$-z=$	-4	0	-1	0	0	0
$f_1=$	1	1	1	0	0	0
$f_4=$	1	0	1	1	1	0
$f_5=$	0	0	-1	-1	0	1

A basis corresponds to a set of arcs containing a $s-t$ path. Degenerate elements in basis are non-path arcs.



		f_1	f_2	f_3	f_4	f_5
$-z=$	-4	0	-1	0	0	0
$f_1=$	1	1	1	0	0	0
$f_4=$	1	0	1	1	1	0
$f_5=$	0	0	-1	-1	0	1

Pivoting lets us remove f_4 from the basis and add f_2. Since $\bar{c} \geq 0$ we see that solution $(0, 1, 0, 0, 1)$ is feasible optimal so path $\{f_2, f_5\}$ is optimal.

		f_1	f_2	f_3	f_4	f_5
$-z=$	-3	0	0	1	1	0
$f_1=$	0	1	0	-1	-1	0
$f_2=$	1	0	1	1	1	0
$f_5=$	1	0	0	0	1	1

Primal

$$\min c' f$$

$$Af = \begin{bmatrix} +1 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$f \geq 0$$

Dual

$$\max \pi_s - \pi_t$$

$$\pi' A \leq c'$$

$$\pi \gtrless 0$$

Since the column corresponding to $e = (i, j)$ has a 1 in row i, a -1 in row j and a 0 elsewhere, the dual inequalities can be written as

$$\pi_i - \pi_j \leq c_{ij} \quad \text{for each } (i, j) \in E \qquad (4)$$

The complementary slackness conditions then say:

Path f and assignment π are jointly optimal iff
(i) each arc $e = (i, j)$ in shortest path, i.e.,
$f_e > 0$ corresponds to $\pi_i - \pi_j = c_{ij}$ and
(ii) $\pi_i - \pi_j < c_{ij}$ corresponds to $f_{(i,j)} = 0$, i.e.,
$e = (i, j)$ **not** in shortest path.

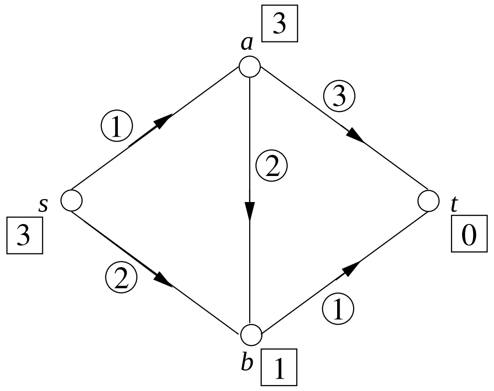
Path f and assignment π are jointly optimal iff
(i) each arc $e = (i, j)$ in shortest path, i.e.,
$f_e > 0$ corresponds to $\pi_i - \pi_j = c_{ij}$ and
(ii) $\pi_i - \pi_j < c_{ij}$ corresponds to $f_{(i,j)} = 0$, i.e.,
$e = (i, j)$ **not** in shortest path.

Let the shortest $s - t$ path found by simplex be

$$e_k\, e_{k-1}, \dots, e_1,$$

where $e_i = u_i, v_i$, $u_k = s$, $v_1 = t$ and $v_i = u_{i-1}$.

If v is some node on the path then, by definition, $\pi_v - \pi_t$ is the shortest distance from v to t so $\pi_s - \pi_t$, the objective maximized by the dual, is exactly the shortest distance from s to t.



The figure shows an optimal π corresponding to shortest path

$$(s, b),\ (b, t)$$

Note that when we calculate π we don't really have a value for π_t, but it can be calculated from our choices for π_s, π_b from the fact that $f_{b,t} = 1 > 0$ and the complementary slackness conditions which force

$$\pi_t = \pi_b - c_{b,t} = 1 - 1 = 0$$

Note: Any optimal π can be modified by adding the same constant to all of the π_i and still remain optimal. We may therefore start by assuming that $\pi_t = 0$ and construct the rest of the π_i under that assumption.

Initial Tableau

	c_j	
	I	

Final Tableau

	$c_j - \pi_j$	
	B^{-1}	

Assume, WLOG, that simplex starts with Identity matrix in left side, e.g., slack or artificial variables.

At end of algorithm, we have essentially multiplied matrix A from left by B^{-1} where B is the set of columns in original matrix corresponding to final optimal BFS.

At optimality, cost row is

$$0 \leq \bar{c} = c - \left(c'_B B^{-1}\right) A = c - \pi' A$$

where we have already seen that $\pi' = c'_B B^{-1}$ is an optimal solution to the dual. So,

$$\bar{c}_j = c_j - \pi_j \quad j = 1, \ldots, m$$

and

$$\pi_j = c_j - \bar{c}_j \quad j = 1, \ldots, m$$

and we can read off optimal dual solution from tableau.

Example

In the section on simplex we saw that the tableau on page 13 is equivalent to the *two-phase* tableau

		x_1^a	x_2^a	x_3^a	x_1	x_2	x_3	x_4	x_5
$-z=$	0	0	0	0	1	1	1	1	1
$-\xi=$	-8	0	0	0	-10	-8	-3	-1	-1
$x_1^a=$	1	1	0	0	3	2	1	0	0
$x_2^a=$	3	0	1	0	5	1	1	1	0
$x_3^a=$	4	0	0	1	2	5	1	0	1

This has $c_j = 0$ in real cost row. We also saw that at optimality this transforms to

		x_1^a	x_2^a	x_3^a	x_1	x_2	x_3	x_4	x_5
$-z=$	$-9/2$	$5/2$	-1	-1	$3/2$	0	$3/2$	0	0
$-\xi=$	0	1	1	1	0	0	0	0	0
$x_1^a=$	$1/2$	$1/2$	0	0	$3/2$	1	$1/2$	0	0
$x_2^a=$	$5/2$	$-1/2$	1	0	$7/2$	0	$1/2$	1	0
$x_3^a=$	$3/2$	$-5/2$	0	1	$-11/2$	0	$-3/2$	0	1

Thus, an optimal dual solution would be

$$\pi = (-5/2, 1, 1)$$

which is exactly what we derived on page 13.