More on the correspondence between polytopes and LPs.

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Let $$F = \{x : Ax = b, \quad x \geq 0\}$$

be the feasible region of some LP, $x \in R^n$. Then the corresponding polytope P in R^{n-m} is the solution space to

$$\begin{array}{rclcrcl} b_i - \sum_{j=1}^{n-m} a_{i,j}\, x_j & \geq & 0, & & i & = & n-m+1, \ldots, n \\ x_j & \geq & 0, & & j & = & 1, \ldots, n-m \end{array}$$

The mapping from F to P is simply

$$\phi((x_1 \ldots, x_n)) = (x_1, \ldots, x_{n-m})$$

Now let P be a polytope in R^{n-m} defined by

$$h_{i,1}\hat{x}_1 + \cdots + h_{i,n-m}\hat{x}_{n-m} + g_i \leq 0, \quad i = 1, \ldots, n.$$

where the first $n-m$ equations are:

$$\hat{x}_i \geq 0, \quad i = 1, \ldots, n-m$$

Then the mapping from P to F is

$$\rho((\hat{x}_1 \ldots, \hat{x}_{n-m})) = (\hat{x}_1 \ldots, \hat{x}_{n-m}, x_{n-m+1}, \ldots, x_n)$$

where

$$x_i = -g_i - \sum_{j=1}^{n-m} h_{i,j}\, x_j, \quad i = n-m+1, \ldots, n.$$

Lemma: Using the notation of the previous page.

$$\rho(\phi(F) = F \quad \text{and} \quad \phi(\rho(P)) = P.$$

This just says that ϕ and ρ are 1-1 functions.

Proof: Next homework.

Lemma: Let

$$F = \{x : Ax = b, \quad x \geq 0\}$$

be the feasible region of a linear program and P the corresponding polytope in R^{n-m}.
Now let $c \in R^n$ be a cost vector.

Then there exists a cost vector $d \in R^{n-m}$ and $K \in R$ such that for every $\hat{x} \in P$

$$K + d'\hat{x} = c'\rho(\hat{x})$$

Proof: Next homework.

Note. This implies that solving the linear program is equivalent to minimizing $d'\hat{x}$ on P. This, in turn, is equivalent to sweeping in from infinity the hyperplanes corresponding to $d'\hat{x} = \text{const}$ until they hit P.