MAX-SAT: Best of Two

We have so far seen two different approaches to approximating MAX-SAT:

- *Random MAX-SAT.*
  $E(W) \geq \frac{OPT}{2}$.
  This chose a random truth assignment using a fair coin. For clause C_j with length l_j

  $$Pr(C_j \text{ is satisfied}) = 1 - 2^{-l_j}.$$

- *Randomized Rounding.*
  $E(W) \geq \left(1 - \frac{1}{e}\right) OPT \approx 0.632 \cdot OPT$.
  Finds solution (y^*, x^*) to relaxed linear program. For clause C_j with length l_j

  $$Pr(C_j \text{ is satisfied}) \geq \left(1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right) z_j^*.$$

Notice that *Random MAX-SAT* is "good" for long clauses while *Randomized Rounding* is "good" for short clauses. We will now see how to combine the two to get an even better approximation.

The *Best of Two* algorithm is to run *Random MAX-SAT* to get assignment x^1 with weight W_1 and to also run *Randomized Rounding* to get assignment x^2 with weight W_2. Then compare W_1 and W_2. If $W_1 > W_2$ return x^1, else return x^2. Let W be the weight of the returned assignment.

Lemma: $E(W) \geq \frac{3}{4}OPT$.

Proof: We use the fact that

$$W = \max(W_1, W_2) \geq \frac{1}{2}W_1 + \frac{1}{2}W_2.$$

Therefore

$$\begin{aligned}
E(W) &\geq E\left(\frac{1}{2}W_1 + \frac{1}{2}W_2\right) \\
&= \frac{1}{2}E(W_1) + \frac{1}{2}E(W_2) \\
&= \frac{1}{2}\sum_j w_j Pr(C_j \text{ is satisfied by } x^1) \\
&\quad + \frac{1}{2}\sum_j w_j Pr(C_j \text{ is satisfied by } x^2) \\
&\geq \frac{1}{2}\sum_j w_j \left(1 - 2^{-l_j}\right) \\
&\quad + \frac{1}{2}\sum_j w_j \left(1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right) z_j^* \\
&= \sum_j w_j \left(\frac{1}{2}\left(1 - 2^{-l_j}\right) + \frac{1}{2}\left(1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right) z_j^*\right)
\end{aligned}$$

So far we have seen that

$$E(W) \geq \sum_j w_j \left(\frac{1}{2}\left(1 - 2^{-l_j}\right) + \frac{1}{2}\left(1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right) z_j^* \right).$$

We will now show that, for all j,

$$\frac{1}{2}\left(1 - 2^{-l_j}\right) + \frac{1}{2}\left(1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right) z_{l_j}^* \geq \frac{3}{4} z_j^*.$$

This will imply that

$$E(W) \ \geq \ \sum_j \frac{3}{4} w_j z_j^* \ \geq \ \frac{3}{4} OPT$$

and we will be done.

We prove this case by case.
If $l_j = 1$ then $\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} z_j^* \geq \frac{3}{4} z_j^*$.

If $l_j = 2$ then $\frac{1}{2} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{3}{4} z_j^* \geq \frac{3}{4} z_j^*$.

If $l_j \geq 3$ then

$$\frac{1}{2}\left(1 - 2^{-l_j}\right) + \frac{1}{2}\left(1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right) z_{l_j}^* \geq \frac{1}{2} \cdot \frac{7}{8} + \frac{1}{2}\left(1 - \frac{1}{e}\right) z_j^* \geq \frac{3}{4} z_j^*.$$