In this addendum we prove the Lemma given in the class notes on page 13.

Lemma: Let $S \subseteq X$ and $T = N_\ell(S) \neq Y$. Set

$$\alpha_\ell = \min_{x \in S, y \notin T} \{\ell(x) + \ell(y) - w(x,y)\}$$

and

$$\ell'(v) = \begin{cases} \ell(v) - \alpha_\ell & \text{if } v \in S \\ \ell(v) + \alpha_\ell & \text{if } v \in T \\ \ell(v) & \text{otherwise} \end{cases}$$

Then ℓ' is a feasible labeling and
(i) If $(x,y) \in E_\ell$ for $x \in S, y \in T$ then $(x,y) \in E_{\ell'}$.
(ii) If $(x,y) \in E_\ell$ for $x \notin S, y \notin T$ then $(x,y) \in E_{\ell'}$.
(iii) There is some edge $(x,y) \in E_{\ell'}$ for $x \in S, y \notin T$

Proof.: By definition $\forall x \in X, y \in Y$ we have $\ell(x) + \ell(y) \geq w(x,y)$. Furthermore, if $(x,y) \in E_\ell$ then $\ell(x) + \ell(y) = w(x,y)$.

There are four types of edges (x,y): For each type of edge we need to show that the feasible labelling condition $\ell'(x) + \ell'(y) \geq w(x,y)$ holds. We also need to show that (i) (ii) and (iii) are correct.

1. $x \in S, y \in T$:
   Then $\ell'(x)+\ell'(y) = \ell(x)-\alpha+\ell(y)+\alpha = \ell(x)+\ell(y)$. So $\ell'(x)+\ell'(y) \geq w(x,y)$ and if $(x,y) \in E_\ell$ then $(x,y) \in E_{\ell'}$.

2. $x \notin S, y \notin T$:
   Then $\ell'(x) + \ell'(y) = \ell(x) + \ell(y)$ so, as in the previous case, $\ell'(x) + \ell'(y) \geq w(x,y)$ and if $(x,y) \in E_\ell$ then $(x,y) \in E_{\ell'}$.

3. $x \notin S, y \in T$:
   $\ell'(x) + \ell'(y) = \ell(x) + \alpha + \ell(y)$ so $\ell'(x) + \ell'(y) \geq w(x,y)$

4. $x \in S, y \notin T$:
   First note that, by the definition of the problem $(x,y) \notin E_\ell$.
   Thus $\alpha_\ell = \min_{x \in S, y \notin T}\{\ell(x) + \ell(y) - w(x,y)\} > 0$.
   This implies $\ell'(x) + \ell'(y) - w(x,y) = \ell(x) + \ell(y) - \alpha - w(x,y) \geq 0$
   so $\ell'(x) + \ell'(y) \geq w(x,y)$.

   Let $x' \notin S, y' \in T$ such that $\ell(x') + \ell(y') - w(x,y) = \alpha$. Then (x', y') is in $E_{\ell'}$.

We point out that in the running of the Hungarian algorithm we always have the property that edges in M are either in (S,T) or $\bar{S}, \bar{T}$ (this can be proven by induction) so, when we upgrade E_ℓ to $E_{\ell'}$ in step 3, our matching M in E_ℓ remains a matching of $E_{\ell'}$.