Addendum

This page proves the statement

$$\textit{if } y \in P_i \textit{ with } y \le t \textit{ then } \exists z \in L_i \textit{ such that}$$

$$\left(1-\frac{\epsilon}{n}\right)^i y \le z \le y \tag{1}$$

that appeared in the analysis of the subset-sum approximation scheme presented in COMP572. Please see the class notes for definitions of the terms.

The proof will be by induction on i. When $i = 1$ note that $P_1 =< 0, x_1 >$ while $L_1 =< 0, x_1 >$ or $L_1 =< 0 >$, depending upon whether or not $x_i \le t$. In both cases, if $y \in P_1$ with $y \le t$ then $y \in L_1$ so (1) is true.

Now assume that (1) is true for i. We will prove its correctness for $i + 1$. Recall that $P_{i+1} = Merge(P_i, P_i + x_{i+1})$ with all items $> t$ thrown out and L_{i+1} is the trimmed version of $Merge(L_i, L_i + x_{i+1})$ with all items $> t$ thrown out.

So now suppose that $y \in P_{i+1}$ with $y \le t$.
There are two cases: (i) $y \in P_i$ or (ii) $y \in P_i + x_{i+1}$.

If $y \in P_i$ then, by induction, $\exists z_i \in L_i$ such that $\left(1-\frac{\epsilon}{n}\right)^i y \le z_i \le y$. Since $z_i \in Merge(L_i, L_i + x_{i+1})$, $\exists z \in L_{i+1}$ such that $\left(1-\frac{\epsilon}{n}\right) z_i \le z \le z_i$. Combining the two sets of inequalities yields

$$\left(1-\frac{\epsilon}{n}\right)^{i+1} y \le \left(1-\frac{\epsilon}{n}\right) z_i \le z \le z_i \le y$$

which is what we wanted to show.

If $y \in P_i + x_{i+1}$ then $y = y_i + x_{i+1}$ for some $y_i \in P_i$. Again by induction $\exists z_i \in L_i$ such that $\left(1-\frac{\epsilon}{n}\right)^i y_i \le z_i \le y_i$. Therefore

$$\left(1-\frac{\epsilon}{n}\right)^i y = \left(1-\frac{\epsilon}{n}\right)^i (y_i+x_{i+1}) \le \left(1-\frac{\epsilon}{n}\right)^i y_i+x_{i+1} \le z_i+x_{i+1} \le y_i+x_{i+1} = y.$$

Since $z_i+x_{i+1} \in Merge(L_i, L_i+x_{i+1})$, $\exists z \in L_{i+1}$ such that $\left(1-\frac{\epsilon}{n}\right)(z_i+x_{i+1}) \le z \le z_i + x_{i+1}$. Combining the two sets of inequalities yields

$$\left(1-\frac{\epsilon}{n}\right)^{i+1} y \le \left(1-\frac{\epsilon}{n}\right)(z_i + x_{i+1}) \le z \le z_i + x_{i+1} \le y$$

which is what we wanted to show and the proof is now complete.