Additional Notes on Matrix Manipulation

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These notes are a quick review of the linear algebra and matrix facts needed for the study of linear programming.

In what follows A will be an $m \times n$ matrix.

1. If $m = n$ and A has rank m, then A^{-1} exists

    - A is called nonsingular.
    - Equivalent to $det(A) \neq 0$.
    - $A^{-1}A = I_m$. (I_m is the $m \times m$ *identity* matrix.)

    Example:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}.$$

2. If $n > m$ and A has rank m:

Assume w.l.o.g. that first m columns of A are linearly independent (otherwise just swap columns), i.e., A can be represented as $[B|J]$, where B is an $m \times m$ nonsingular matrix. Then

$$B^{-1}A = [I_m \,|\, B^{-1}J].$$

Example:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then

$$B^{-1}A = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & -2 & -3 \end{array}\right] = [I_m \,|\, B^{-1}J].$$

Given an $m \times n$ matrix A and an m-vector b, consider the solution of $Ax = b$,

1. If $m = n$ and A has rank m, then A^{-1} exists, so $x = A^{-1}b$

   Example:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

$$x = A^{-1}b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 3 \\ -6 \end{bmatrix}.$$

2. If $n > m$ and A has rank m, then WLOG, $A = [B|J]$, where B is an $m \times m$ nonsingular matrix. We then have,

$$B^{-1}b = B^{-1}Ax = [I_m|B^{-1}J]x.$$

Define $\beta = B^{-1}b$, and $\alpha = -B^{-1}J = [\alpha_{ji}]$, where $1 \leq j \leq m$ and $m + 1 \leq i \leq n$. We then have,

$$x_j = \beta_j + \sum_{i=m+1}^{n} \alpha_{ji} x_i, \qquad j = 1, \ldots, m$$

Example:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

$$B^{-1}b = \begin{bmatrix} 4 \\ 2 \\ 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & -2 & -3 \end{bmatrix} x.$$

In our example we solved

$$B^{-1}b = \begin{bmatrix} 4 \\ 2 \\ 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & -2 & -3 \end{bmatrix} x$$

This can be rewritten as,

$$\begin{aligned} x_1 &= 4 - x_5 - x_6 - x_7 \\ x_2 &= 2 - x_5 \\ x_3 &= 3 - x_6 \\ x_4 &= -6 + 3x_5 + 2x_6 + 3x_7 \end{aligned}$$