Graphical Models for Game Theory

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Multi-player General-sum Game

- n players
- Two actions for the players
- The payoff function of each player is a table of size 2^n
- No known algorithm to compute the Nash equilibria

Graphical Model

- Each player is a vertex of a graph
- There are edges linking between vertices (players)
- The payoff to a player is only determined by the action of him and his neighbors
- The payoff function of each player is then exponential to the size of his neighbors instead of the size of the graph

Graphical Model - Tree

- In this paper, only tree (graph without cycle) will be considered.
- A polynomial (in the size of presentation) time algorithm to compute the approximate Nash equilibria is presented

Preliminaries

- An n-player two-action game is defined by n matrices M_i $(1 \leq i \leq n)$
- The entry $M_i(x_1, \ldots, x_n) = M_i(\vec{x})$ specifies the payoff to player i
- The action 0 and 1 are the pure strategies
- $M_i(x_1, \ldots, x_n) = M_i(\vec{x})$ denotes the payoff of player i if player j plays x_j for $1 \leq j \leq n$
- The mixed strategy of player i is given the probability $p_i \in [0, 1]$ that he will play 0
- $M_i(p_1, \ldots, p_n) = M_i(\vec{p})$ denotes the expected payoff of player i if player j plays p_j for $1 \leq j \leq n$

Approximate Nash Equilibrium

- $\vec{p}[i : p'_i]$ denotes the vector $\vec{p}$ with the i-th component substituted by p'_i
- Nash equilibrium for the game is a mixed strategy $\vec{p}$ such that for all i and $p'_i \in [0, 1]$, $M_i(\vec{p}) \geq M_i(\vec{p}[i : p'_i])$
- ϵ-Nash equilibrium is a mixed strategy $\vec{p}$ such that for all i and $p'_i \in [0, 1]$, $M_i(\vec{p}) + \epsilon \geq M_i(\vec{p}[i : p'_i])$

Graphical Game

- n-player Graphical Game is a pair $(G, \mathcal{M})$ where G is an undirected graph of n vertices and $\mathcal{M}$ is a set of n matrices M_i
- $N_G(i) \subseteq \{1, \ldots, n\}$ denotes the neighbors of player i (including player i)
- If $|N_G(i)| = k$, M_i has k indices and has 2^k entries
- The game size is reduced substantially if $k \ll n$

Definitions

- Pick an arbitary vertex as root
- Any vertex on the path from a vertex v to the root is called downstream from v
- Each vertex (other than the root) has one downstream child
- Any vertex on the path from a vertex v to a leaf is called upstream from v
- Each vertex has zero or more upstream parents

Definitions

- Let G^U be the subgraph induced by the vertex U and all its upstream vertices
- $\mathcal{M}^U_{V=v}$ denotes the subset of $\mathcal{M}$ corresponding to vertices in G^U with a modification that M_U is collapsed by one index by fixing $V = v$
- A Nash equilibrium of the sub-graphical game $(G^U, \mathcal{M}^U_{V=v})$ is an equilibrium "upstream" from U, given V plays v

Definitions

- $T(v, u)$ denotes a binary-valued table with indices $v, u \in [0, 1]$
- $T(v, u) = 1$ iff there exists Nash equilibrium for $(G^U, \mathcal{M}^U_{V=v})$ in which U plays u
- Let V has parents $U_1, \ldots, U_k$ and a child W. If there exists $\vec{u} = (u_1, \ldots, u_k)$ such that
  - $T(v, u_i) = 1$ for all $1 \leq i \leq k$ and
  - $V = v$ is a best response to $\vec{U} = \vec{u}$ and $W = w$

  Then $T(w, v) = 1$ and $\vec{u}$ is said to be a witness of $T(w, v)$

TreeNash Algorithm - Downstream

- For each vertex V, let W be its child vertex
  - If V is a leaf, set $T(w, v)$ as 1 iff $V = v$ is a best response to $W = w$
  - Otherwise, let $\vec{U} = (U_1, \ldots, U_k)$ be the parents of V
    - For all $w, v \in [0, 1]$ and $\vec{u} \in [0, 1]^k$, if $V = v$ is a best response to $W = w$ and $\vec{U} = \vec{u}$, set $T(w, v) = 1$ and add $\vec{u}$ to the witness list of $T(w, v)$
- Pass $T(w, v)$ to W

TreeNash Algorithm - Upstream

- For each vertex V, let W and $\vec{U} = (U_1, \ldots, U_k)$ be its child and parents
- If V is the root
  - Pick an arbitary v suct that $T(v) = 1$
  - Label V as v
  - Pick an arbitary witness $\vec{u}$ from the witness list of $T(v)$
- Otherwise, let (w, v) be passed from W
  - Label V as v
  - Pick an arbitary witness $\vec{u}$ from the witness list of $T(w, v)$
- Pass (v, u_i) to U_i for $i = 1, \ldots, k$

Approximation Algorithm

- Players are constrained to play a discretized mixed strategy that is multiple of τ
- i.e. player i plays $p_i \in \{0, \tau, 2\tau, \ldots, 1\}$
- Joint strategy $\vec{p}$ will fall on the discretized τ-grid $p_i \in \{0, \tau, 2\tau, \ldots, 1\}^k$.

Expected Payoff

For any mixed strategy $\vec{p}$ for $(G, \mathcal{M})$ and any player i, Let $|N_G(i)| = k$, Then the expected payoff to i under $\vec{p}$ can be written as

$$M_i(\vec{p}) = \sum_{\vec{x} \in \{0,1\}^k} \left(\prod_{j=1}^{k} \alpha_j(x_j) \right) M_i(\vec{x})$$

where $\alpha_j(x_j) = (p_j)^{1-x_j}(1-p_j)^{x_j}$

Lemma 2

Let $\vec{p}, \vec{q} \in [0,1]^k$ satisfy $|p_i - q_i| \leq \tau$ for all $1 \leq i \leq k$ and $\tau \leq 2/(k \log^2(k/2))$. Then,

$$\left| \prod_{i=1}^{k} p_i - \prod_{i=1}^{k} q_i \right| \leq (2k \log k)\tau$$

Proof: By induction on k. Assume k is power of 2.

Lemma 2

Proof: (Continued)

$$
\begin{aligned}
\prod_{i=1}^{k} q_i &= \left(\prod_{i=1}^{k/2} q_i\right)\left(\prod_{i=k/2+1}^{k} q_i\right) \\
&\leq \left(\prod_{i=1}^{k/2} p_i + k(\log(k/2))\tau\right) \times \left(\prod_{i=k/2+1}^{k} p_i + k(\log(k/2))\tau\right) \\
&\leq \left(\prod_{i=1}^{k} p_i\right) + 2k(\log(k/2))\tau + (k(\log(k/2))\tau)^2 \\
&= \left(\prod_{i=1}^{k} p_i\right) + (2k\log k)\tau
\end{aligned}
$$

Lemma 3

Let the mixed strategies $\vec{p}$, $\vec{q}$ for $(G, \mathcal{M})$ satisfy $|p_i - q_i| \leq \tau$ for all i and $\tau \leq 2/(k \log^2(k/2))$. Then,

$$|M_i(\vec{p}) - M_i(\vec{q})| \leq 2^{k+1}(k \log k)\tau$$

Lemma 3

Proof:

$$
\begin{aligned}
|M_i(\vec{p}) - M_i(\vec{q})| &\leq \sum_{\vec{x}\in\{0,1\}^k} \left| \prod_{j=1}^{k} \alpha_j(x_j) - \prod_{j=1}^{k} \beta_j(x_j) \right| |M_i(\vec{x})| \\
&\leq \sum_{\vec{x}\in\{0,1\}^k} 2k\log(k)\tau \\
&\leq 2^{k+1}(k\log(k))\tau
\end{aligned}
$$

where $\alpha_j(x_j) = (p_j)^{1-x_j}(1-p_j)^{x_j}$, $\beta_j(x_j) = (q_j)^{1-x_j}(1-q_j)^{x_j}$

Lemma 4

Let $\vec{p}$ be a Nash equilibrium for $G, \mathcal{M}$ and $\vec{q}$ be the nearest mixed strategy on the τ-grid. If $\tau \leq 2/(k \log^2(k/2))$, $\vec{q}$ is a $2^{k+2}(k\log(k))\tau$-Nash equilibrium for $G, \mathcal{M}$.

Proof: Let r_i be a best response for player i to $\vec{q}$,

$$\begin{aligned}
|M_i(\vec{q}[i:r_i])| &\leq |M_i(\vec{p}[i:r_i])| + 2^{k+1}(k\log k)\tau \\
&\leq |M_i(\vec{p})| + 2^{k+1}(k\log k)\tau \\
&\leq |M_i(\vec{q})| + 2^{k+1}(k\log k)\tau + 2^{k+1}(k\log k)\tau \\
&\leq |M_i(\vec{q})| + 2^{k+2}(k\log k)\tau
\end{aligned}$$

ApproximateTreeNash

- Set $\tau = \min(\epsilon/(2^{k+2}(k\log(k))), 2/(k\log^2(k/2)))$
- TreeNash can compute a ϵ-Nash equilibrium
- Time complexity for processing a vertex is $(1/\tau)^{2k}$