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A Lower Bound on the Distortion of Embedding Planar Metrics into Euclidean Space*

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Abstract. We exhibit a simple infinite family of series-parallel graphs that cannot be metrically embedded into Euclidean space with distortion smaller than $\Omega(\sqrt{\log n})$. This matches Rao's [14] general upper bound for metric embedding of planar graphs into Euclidean space, thus resolving the question how well do planar metrics embed in Euclidean spaces?

1. Introduction

Some of the most interesting questions in the study of finite metric spaces are about the relations between the structural properties of the underlying graph and of its geodetic metric (i.e., its shortest-path distance). In this paper we address one such question, and show a tight lower bound on the distortion of embedding a metric coming from a planar graph into Euclidean space.

Here are some basic definitions. A finite (semi-) metric space (S, μ) is a finite set S and a symmetric nonnegative distance function μ on $S \times S$ satisfying the triangle inequality and $\mu(x, x) = 0$. In what follows, we sometimes refer to μ as *metric*, regardless of S. A metric is called *planar* if it can be obtained by restricting the geodetic (i.e., shortest-path) metric of some weighted planar graph to a subset of its vertices. The weights should, of course, be nonnegative. *Series-parallel*, *tree*, etc., metrics are defined similarly. For example, one can easily check that the geodetic metric of the unit-weighted K_5 is a tree metric, while that of unit-weighted $K_{3,3}$ is not a planar metric.

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Given two metric spaces (S, μ), (R, δ) and an embedding $f\colon S \longrightarrow R$, the *distortion* of f is defined as

$$\mathrm{distr}(f) = \max_{x,y\in S} \frac{\delta(f(x), f(y))}{\mu(x, y)} \cdot \max_{x,y\in S} \frac{\mu(x, y)}{\delta(f(x), f(y))},$$

i.e., the product of the maximum *expansion* and the maximum *contraction* of f. Observe that the distortion is never less than 1, and is equal to 1 exactly when f preserves μ up to scaling.

Two important parameters of a finite metric μ are $c_2(\mu)$ and $c_1(\mu)$, the smallest possible distortion of embedding μ into real Euclidean and ℓ_1 space, respectively. Although the fundamental structural properties of finite metrics have just begun to emerge, numerous new exciting conjectures, theorems and applications (see, e.g., [8], [4], [10], [2], [3], [12], [6], [14], and [7]) seem to indicate that $c_1(\mu)$ and $c_2(\mu)$ indeed do capture some nontrivial aspects of μ.

We summarize the relevant facts about these two parameters. It always holds that $c_2(\mu) \geq c_1(\mu)$, since any Euclidean metric is ℓ_1-embeddable [13]. Assuming μ is a metric on n points, both parameters are at most $O(\log n)$ [4], which is tight. The bound it is attained at the geodetic metric of any unit-weighted constant degree expander graph on n points [10], [1].

Tree metrics behave significantly better: $c_1(\mu)$ is 1, while $c_2(\mu) = O(\sqrt{\log\log n})$ [5], [12], which is attained at the geodetic metric of the unit-weighted full binary tree of depth $\log n$ [4], [12]. A natural next question is what happens in the case when μ is a planar metric, or, more generally, a metric coming from a graph excluding some fixed minor? An elegant result of Rao [14] using [9], shows that for such metrics $c_2(\mu)$, and consequently $c_1(\mu)$, are at most $O(\sqrt{\log n})$.

What are the true values of $c_1(\mu)$ and $c_2(\mu)$ for planar metrics? There are reasons to believe that $c_1(\mu)$ is bounded by a constant; see [7] for a related discussion, and a proof that this is indeed the case for series-parallel and outerplanar metrics. For the last two years it was not clear whether $c_2(\mu)$ is closer to $O(\sqrt{\log n})$, as in the upper bound of [14], or to $O(\sqrt{\log\log n})$, as for trees [12].

In this paper we settle this question, and establish a lower bound of $\Omega(\sqrt{\log n})$ already for series-parallel metrics, which are a special case of planar metrics.

In addition to the main result, we obtain as a by-product of our construction an infinite family of weighted planar (in fact, tree-width 3) graphs H_n, so that not only the entire geodetic metric of H_n, but even the lengths of its edges must be distorted by a factor of $\Omega(\sqrt{\log n})$ by any embedding of H_n into Euclidean space. This contrasts with a construction of [11] showing that any weighted graph of tree-width 2 can be embedded into a line (!) so that the edges neither expand nor contract by more than a factor of 3, and also with a theorem of Seymour [15] about the existence of edge-preserving metric embeddings of weighted planar graphs into ℓ_1 space.

2. The Lower Bound

Define a family $\{G_k\}$ of graphs in the following inductive manner, similar to the one used in [7]:

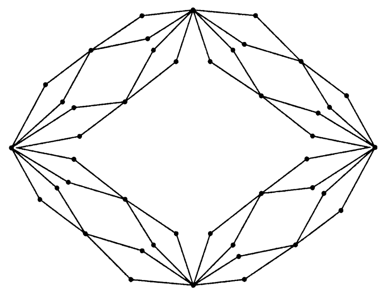

Fig. 2.1. The graph G_3.

G_0 consists of a single edge. G_i is a "refinement" of G_{i-1} obtained by replacing each edge of G_{i-1} by two parallel paths, each containing two edges. See Fig. 2.1. The length of every edge in G_i is defined as 2^{-i}, half that of G_{i-1}. It is convenient to identify $V(G_{i-1})$ with the "old" vertices of G_i. Observe that this natural identification is in fact an isometry, i.e., the restriction of the geodetic metric of G_i to the "old" vertices is identical with the geodetic metric of G_{i-1} on these vertices.

Thus, without risk of confusion, we can speak of edges of G_i, and of their length, while discussing any G_k, $k \geq i$. To simplify the presentation, we also introduce the notion of an anti-edge. Assume that the edge (a, b) of G_{i-1} was replaced in G_i by edges (a, x), (x, b) and (a, y), (y, b), respectively. The pair of vertices $\{x, y\} \subset V(G_i)$ will be called the *anti-edge* of (a, b). Observe that the distance between x and y is 2^{-i}, exactly as the length of the edge (a, b) of G_{i-1}.

It is easily checked that G_k is a series-parallel graph containing 4^k edges and $(2 \cdot 4^k + 4)/3$ vertices.

Theorem 2.1. *Let μ denote the geodetic metric of G_k. Then*

$$c_2(\mu) \geq \sqrt{k+1}.$$

Proof. Let $f\colon V(G) \longrightarrow \mathbb{R}^d$ be an embedding of μ into Euclidean space. Due to the scalability of Euclidean space, we may without loss of generality assume that f is nonexpanding, i.e., $\|f(v) - f(u)\|_2 \leq \mu(u, v)$ for any $u, v \in V(G_k)$. Let $\alpha = \min_{v,u \in V(G_k)}(\|f(v) - f(u)\|_2/\mu(v, u))$. Our goal will be to show that $\alpha \leq 1/\sqrt{k+1}$, i.e., some distance must contract under such f by at least $\sqrt{k+1}$.

First, we prove by downwards induction on i that the length of any edge of G_i, $i = k, k-1, \ldots, 0$, must contract under f by at least a factor of $[1 - (k-i)\alpha^2]^{-1/2}$. Formally, we claim that for any $(a, c) \in E(G_i)$,

$$\|f(a) - f(c)\|_2 \leq \sqrt{1 - (k-i)\alpha^2} \cdot \mu(a, c).$$

The claim is trivially true for $i = k$, since f is nonexpanding. Assume we have already demonstrated the claim for $i+1$; we demonstrate it for i. Recall the well known inequality that for any four points a, b, c, d in Euclidean space the sum of the squares of the diagonals never exceeds the sum of the squares of the sides:

$$\|a - b\|_2^2 + \|b - c\|_2^2 + \|c - d\|_2^2 + \|d - a\|_2^2 \geq \|a - c\|_2^2 + \|b - d\|_2^2. \tag{2.1}$$

The inequality holds since after subtracting the right-hand side from the left-hand side, one arrives at $\|a - b + c - d\|_2^2$.

Now, consider an edge (a, c) of G_i, its anti-edge (b, d) and the four surrounding edges of G_{i+1}: (a, b), (b, c), (c, d) and (d, a). By our inductive assumption, the images of each of the latter pairs are at most $\sqrt{1 - (k - i - 1)\alpha^2} \cdot 2^{-(i+1)}$ apart. By the definition of α, the images of b and d are at least $\alpha \cdot 2^{-i}$ apart. Combining this with (2.1), we get

$$4 \cdot 2^{-2(i+1)} \cdot [1 - (k - i - 1)\alpha^2] \geq \alpha^2 \cdot 2^{-2i} + \|f(a) - f(c)\|_2^2,$$

from which we conclude that

$$\|f(a) - f(c)\|_2^2 \leq [1 - (k - i)\alpha^2] \cdot 2^{-2i} = [1 - (k - i)\alpha^2] \cdot \mu^2(a, c),$$

as claimed.

Consider the edge (s, t) of G_0. The images of s and t are at least α apart by definition of α, and at most $\sqrt{1 - k \cdot \alpha^2}$ apart by the claim. Comparing the two terms we conclude that

$$\alpha \leq 1/\sqrt{k + 1}. \qquad \square$$

Theorem 2.1 can be slightly strengthened. Observe that in our proof we have used only the edges and the anti-edges of G_i's. Therefore, restoring all these pairs as edges, and assigning them weight equal to their distance in G_k, we arrive at the graph H_k, whose edges must suffer distortion $\sqrt{k + 1}$ in any embedding of H_k into Euclidean space. Formally:

Let H_0 consist of single edge of length 1, and let H_i be obtained by taking H_{i-1}, and in addition to existing vertices and edges, introducing for each edge $e = (a, c)$ of H_{i-1} of length $2^{-(i-1)}$ two new vertices b_e, d_e, a new edge (b_e, d_e) of length $2^{-(i-1)}$ and four new edges (a, b_e), (b_e, c), (c, d_e) and (d_e, a), each of length 2^{-i}. As before, H_{i-1} isometrically embeds into H_i under the natural identification of the vertices.

It turns out that H_k is still planar, and, moreover, has tree-width 3. It has $(5 \cdot 4^k - 2)/3$ edges and the same number of vertices as in G_k. By the preceding discussion we get

Theorem 2.2. *In any embedding f of H_k into Euclidean space which does not expand the edges, there exists an edge in $E(H_k)$ whose length is contracted by f by at least a factor of $\sqrt{k + 1}$.*

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