On embedding expanders into ℓ_p spaces

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Abstract

In this note we show that the minimum distortion required to embed all n-point metric spaces into the Banach space ℓ_p is between $(c_1/p)\log n$ and $(c_2/p)\log n$, where $c_2 > c_1 > 0$ are absolute constants and $1 \le p < \log n$. The lower bound is obtained by a generalization of a method of Linial et al. [LLR95], by showing that constant-degree expanders (considered as metric spaces) cannot be embedded any better.

1 Introduction

Let M be a metric space with a metric ρ, let X be a normed space (whose norm will be denoted by $\|.\|$), and let $f : M \to X$ be a mapping. We say that f is a *D-embedding* (or a mapping with *distortion* at most D), $D \ge 1$ a real number, if we have

$$\frac{1}{D}\,\rho(x,y) \le \|f(x) - f(y)\| \le \rho(x,y)$$

for any two points $x, y \in M$. We say that M D-embeds into X if there exists a D-embedding[1] $f : M \to X$.

The D-embeddability of finite metric spaces into various normed spaces was investigated in the context of the local Banach space theory ([Enf69a], [Enf69b], [JL84], [Bou85], [BMW86], [JLS87], [AR92], [Ma95]), and it seems that it can be of considerable interest also in more applied areas (see [LLR95]).

Let the symbol ℓ_p^n denote the n-dimensional real vector space equipped with the L_p-norm, given by $\|(x_1, x_2, \ldots, x_n)\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ (for $1 \le p < \infty$). Similarly ℓ_p denotes the space of countable sequences of real numbers with a finite L_p-norm. For a metric space M, let $D_p(M)$ be the minimum D such that M can be D-embedded into ℓ_p, and let $D_p(n)$ be the

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[1] A number of various terms besides the mentioned ones are used in the literature in this context; e.g., a D-embedding is also called a *D-isomorphism*, a *D-lipeomorphism*, etc.

supremum of $D_p(M)$ over all n-point metric spaces M. Since any n-point subset of ℓ_p can be isometrically embedded into $\ell_p^{n(n-1)/2}$ (see e.g., [Fic88]), we can restrict our attention to embeddings into finite-dimensional ℓ_p spaces.

The most well-studied case is that of $p = 2$, where we are dealing with embeddings into the usual Euclidean space. It is easy to find a 4-point metric space which cannot be isometrically embedded into any Euclidean space, but it is not so easy to prove that $D_2(n) \to \infty$ for $n \to \infty$; this was probably first done by Enflo [Enf69b], whose proof yields $D_2(n) = \Omega(\sqrt{\log n})$ (his example is the cube $\{0,1\}^k$ with the L_1 metric, $n = 2^k$; see also [Enf69a]). Bourgain [Bou85] proved an upper bound $D_2(n) = O(\log n)$ and a lower bound $D_2(n) = \Omega(\log n / \log\log n)$. The lower bound is non-constructive, using random graphs and a counting argument. Linial et al. [LLR95] discovered another lower bound technique, which allowed them to show the asymptotically tight lower bound $D_2(n) = \Omega(\log n)$; their proof, unlike Bourgain's, yields an explicit metric space exhibiting the lower bound (one can say that their method slightly resembles Enflo's, with expander graphs replacing the L_1-cube).

The situation for other values of p has been understood less satisfactorily. Concerning upper bounds, Bourgain's embedding technique in fact proves $D_p(n) = O(\log n)$ for any p (with the constant of proportionality independent of p). Concerning lower bounds, the argument of Linial et al. [LLR95] shows $D_p(n) = \Omega(\log n)$ for any $p \in [1, 2]$. For $p > 2$, however, the best known lower bound was apparently one following from the results of Bourgain et al. [BMW86], which is $c_\varepsilon (\log n)^{1/2-\varepsilon}$, with $\varepsilon > 0$ an arbitrarily small number and $c_\varepsilon > 0$ depending on ε. Here we obtain asymptotically tight bounds:

Theorem 1

(i) *There exist constants $c_1 > 0$ and n_0 such that for any $p \geq 1$ and any $n \geq n_0$ there exists an n-point metric space which D-embeds into ℓ_p only for $D \geq (c_1/p)\log n$.*

(ii) *Any n-point metric space can be embedded into ℓ_p with distortion at most $(c_2/p)\log n$, where c_2 is a constant and $1 \leq p < \log n$.*

Part (i) is proved in section 2 by generalizing the method of Linial et al. [LLR95]. Part (ii) is proved in section 4 by modifying Bourgain's embedding method very slightly.

2 Expanders

Let $G = (V, E)$ be a (simple, unoriented) graph on the vertex set $V = \{1, 2, \ldots, n\}$. We assume that G is d-regular (every vertex has exactly d neighbors) with d a constant (while n is a variable attaining arbitrarily large values). The graph G is called an *expander* if there exists a constant $\Phi > 0$ (independent of n; Φ is called the *conductance* of G) such that for any subset $A \subseteq V$ with $|A| \leq n/2$ we have

$$|\{\{i,j\} \in E;\ i \in A, j \in V \setminus A\}| \geq \Phi|A|. \tag{1}$$

It can be shown that a random d-regular graph is an expander (with a suitable $\Phi = \Phi(d)$) with a positive probability. Sophisticated explicit constructions of expanders are also known — see e.g., [AS92] for background information and references.

If $x_1, x_2, \ldots, x_n$ are real numbers, their *median* is defined as a real number m such that $|\{i;\, x_i \le m\}| \ge \lfloor n/2 \rfloor$ and $|\{i;\, x_i \ge m\}| \ge \lfloor n/2 \rfloor$ (thus, for n even, the median need not be determined uniquely). A basic property of expanders we use is the following (apparently due to Sinclair and Jerrum [JS88]; see e.g., Lovász [Lov93], Ex. 11.30):

Lemma 2 *Let G be an expander with conductance $\Phi > 0$, let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers, and let m be their median. Then*

$$\sum_{\{i,j\}\in E} |x_i - x_j| \ge \Phi \sum_{i \in V} |x_i - m| \,. \tag{2}$$

The proof of (2) has few lines. Inequality (1) can be viewed as a special case of (2), where the x_i's only attain values 0 and 1 (set $x_i = 1$ for $i \in A$ and $x_i = 0$ otherwise).

The key to the non-embeddability result of Linial et al. [LLR95] is a suitable generalization of (2) for the x_i's being elements of a Banach space instead of real numbers. Let $K = \binom{V}{2}$ denote the set of edges of the complete graph with vertex set V. A simple consequence of (2) is the following:

$$\sum_{\{i,j\}\in E} |x_i - x_j| \ge \frac{\Phi}{n-1} \sum_{\{i,j\}\in K} |x_i - x_j| \,. \tag{3}$$

Indeed, since (3) is clearly translation-invariant (it is not changed by adding the same real number to each x_i), we may assume that 0 is a median of the x_i's. Then we have $\sum_{\{i,j\}\in K} |x_i - x_j| \le \sum_{\{i,j\}\in K} (|x_i| + |x_j|) = (n-1)\sum_{i\in V} |x_i|$, so (3) follows from (2).

Linial et al. [LLR95] essentially observe (although they formulate it in a somewhat different manner) that (3) holds also for any $x_1, x_2, \ldots, x_n \in \ell_1$:

$$\sum_{\{i,j\}\in E} \|x_i - x_j\|_1 \ge \frac{\Phi}{n-1} \sum_{\{i,j\}\in K} \|x_i - x_j\|_1 \,. \tag{4}$$

This is an immediate consequence of (3): write each $x_i \in \ell_1$ in coordinates, use (3) for each coordinate separately, and sum the resulting inequalities. Since any finite subset of any ℓ_p with $1 \le p \le 2$ can be isometrically embedded into ℓ_1 [BDK66], an analog of (4) holds also in these ℓ_p's, in particular in a Euclidean space.

For $p > 2$, we need an "L_p-analog" of (3):

Proposition 3 *Let G be a d-regular expander with conductance Φ, let $x_1, x_2, \ldots, x_n$ be real numbers with median m, and let $p > 1$. Then we have*

$$\begin{aligned} \sum_{\{i,j\}\in E} |x_i - x_j|^p &\ge \frac{(\Phi/2p)^p}{d^{p-1}} \sum_{i\in V} |x_i - m|^p \\ &\ge \frac{(\Phi/4p)^p}{d^{p-1}} \frac{1}{n-1} \sum_{\{i,j\}\in K} |x_i - x_j|^p \,. \end{aligned} \tag{5}$$

The version of this result for $p = 2$ was essentially proved by Sinclair and Jerrum [JS88] (the method goes back to Alon [Alo86]); the result for a general p doesn't seem to be known.

We prove it in section 3 below by generalizing the known proof of the $p = 2$ case (following the presentation of Lovász [Lov93], Ex. 11.32).

As a consequence we get that for any $x_1, x_2, \ldots, x_n \in \ell_p$,

$$\sum_{\{i,j\}\in E} \|x_i - x_j\|_p^p \geq \left(\frac{c_0}{p}\right)^p \frac{1}{n-1} \sum_{\{i,j\}\in K} \|x_i - x_j\|_p^p \,, \tag{6}$$

with $c_0 = \Phi/4d^{1-1/p}$ a positive constant (this again follows by applying (5) to each coordinate and summing up).

To derive Theorem 1(i) from (6), we consider the expander G as an n-point metric space, with the metric ρ given by the usual graph-theoretic distance of vertices. Let us consider the ratio

$$R_\rho = \frac{\left(\frac{1}{|E|}\sum_{\{i,j\}\in E} \rho(i,j)^p\right)^{1/p}}{\left(\frac{1}{|K|}\sum_{\{i,j\}\in K} \rho(i,j)^p\right)^{1/p}} \,.$$