L04: Simple Probability

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Probabilistic experiment

- Many experiments do not yield exactly the same results when performed repeatedly.
  - Coin tosses.
  - Dice tosses.
- Experiments of this type are called **probabilistic** experiments.
- When we talk about probabilities, we focus on probabilistic experiments.

Sample spaces

- A set A containing all the outcomes of an experiment is called a **sample space**.
- Suppose a one dollar coin and a two dollar coin are tossed.
  - When the sequence of heads (H) and tails (T) is considered, the sample space is $A = \{HH, HT, TH, TT\}$.
  - When the number of heads is considered, the sample space is $A = \{0, 1, 2\}$.

Events

- A statement about the outcome of an experiment, which for a particular outcome will be either true or false, is said to describe an **event**.
- The event described by a statement is taken to be the **set** of all outcomes for which the statement is true.
- With this interpretation, any event can be considered a **subset** of the sample space.

Events

- Suppose a coin is tossed 3 times in order.
  - Then the sample space is: {*HHH*, *HHT*, *HTH*, *HTT*, *THH*, *THT*, *TTH*, *TTT*}
  - The event described by "at least two tails are recorded" = {*HTT*, *THT*, *TTH*, *TTT*}.
  - The event described by "the first two tosses are heads" = {*HHH*, *HHT*}.

Event probability

- We denote the **probability** of any event E by $p(E)$.
- Intuitively, the number $p(E)$ reflects our assessment of the likelihood that the event E will occur.
- Denote n as the number of trials in the experiment and n_E as the number of occurrences of event E.
- When n becomes large, n_E / n tends to $p(E)$.

Event probability

- Formally speaking, an event should be a **set**.
- If event E is the sample space A of the experiment, E is called the **certain event** and $p(E)=1$.
  - Note that when considering the events as sets, sample space A is the universal set.
- If event E is an empty set, E is called the **impossible event**, and $p(E)=0$.

Mutually exclusive events

- Events X and Y are said to be **mutually exclusive** or **disjoint**, if $X \cap Y = \phi$.
  - For a particular outcome, it is impossible that both the events X and Y are true.
- When events X and Y are mutually exclusive, $p(X \cup Y) = p(X) + p(Y)$.
  - In the experiments, $n_{X \cup Y} = n_X + n_Y$.
  - Therefore, $n_{X \cup Y} / n = n_X / n + n_Y / n$.
  - When n becomes large, each fraction turns to probability, and we have $p(X \cup Y) = p(X) + p(Y)$.

Equally likely outcomes

- Given a finite sample space $A = \{x_1, x_2, \ldots, x_n\}$.
- In some situation each outcome in A is **equally likely** to occur.
- Then, the probability of an event E is
  $$p(E) = \frac{|E|}{|A|}.$$
- For example, if $E = \{x_1, x_2, \ldots, x_k\}$, then $p(E) = k/n$.

Example of mutually exclusive events

- Suppose a fair six-faced dice is tossed.
- Let X be the number that faces up.
  - $p(X$ is even or X is odd$)$

    $= p(\{X: X \text{ is even}\} \cup \{X: X \text{ is odd}\})$

    $= p(\{X: X \text{ is even}\}) + p(\{X: X \text{ is odd}\})$

    $= 3/6 + 3/6 = 1.$
  - $\{X: X \text{ is even}\}$ and $\{X: X \text{ is odd}\}$ are mutually exclusive, so we can go from step 2 to step 3.
  - There are 3 outcomes for each of these 2 events and 6 total possible outcomes, so the probability for these two events is 3/6.

Example of events that are not mutually exclusive

- Suppose a fair six-faced dice is tossed.
- Let X be the number that faces up.
  - $p(X \text{ is even or } X>3)$

$= p(\{X: X \text{ is even}\} \cup \{X: X>3\})$

$= p(\{X=2, X=4, X=6\} \cup \{X=4, X=5, X=6\})$

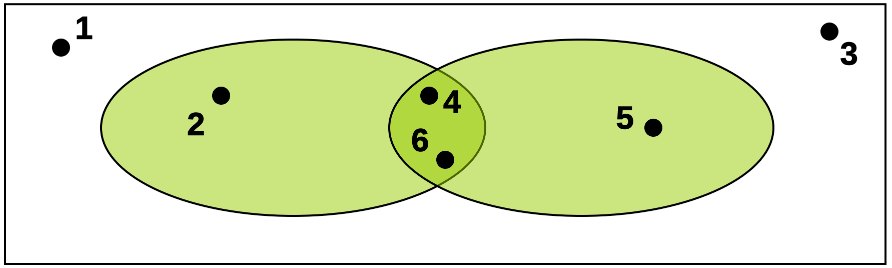
$= p(\{X=2, X=4, X=5, X=6\})=4/6=2/3.$

Example of events that are not mutually exclusive

- $\{X\text{: } X \text{ is even}\}$ and $\{X\text{: } X>3\}$ are **NOT** mutually exclusive.
- Note that:

  $p(\{X\text{: } X \text{ is even}\} \cup \{X\text{: } X>3\})=2/3$, while

  $p(X\text{: } X \text{ is even}) + p(\{X\text{: } X>3\})=1$.
- The problem occurs because we “count” some events twice (double-counting).



Handling events that are not mutually exclusive

- We have to "remove" the double-counted events by deducting the probability of the overlapping events once:
  - $p(X \text{ is even or } X>3)$

$= p(\{X\colon X \text{ is even}\} \cup \{X\colon X>3\})$

$= p(\{X\colon X \text{ is even}\}) + p(\{X\colon X>3\})$

$\quad - p(\{X\colon X \text{ is even}\} \cap \{X\colon X>3\})$

$= p(\{X=2, X=4, X=6\}) + p(\{X=4, X=5, X=6\})$

$\quad - p(\{X=4, X=6\})$

$= 3/6 + 3/6 - 2/6 = 4/6 = 2/3.$

Statistical independence

- Two events are **independent** intuitively means that knowing whether or not one of them occurs does not affect how likely it is that the other occurs.
- Assume a dice is tossed twice.
  - Let X_1 and X_2 be the numbers that face up in the two trials respectively.
  - The event described by "$X_1=1$" and the event described by "$X_2=1$" are independent.

Statistical independence

- Two events X and Y are **independent** if and only if $p(X \cap Y) = p(X)p(Y)$.

- Assume a dice is tossed twice. Let X_1 and X_2 be the numbers that face up in the two trials respectively.
- Suppose we want to compute $p(X_1 = 1$ and $X_2 = 1)$.
- Clearly, the events $\{X_1 = 1\}$ and $\{X_2 = 2\}$ are statistically independent.

Statistical independence

- Therefore,

  $p(X_1 = 1 \text{ and } X_2 = 1)$

  $= p(\{X_1 = 1\} \cap \{X_2 = 1\})$

  $= p(\{X_1 = 1\}) \times p(\{X_2 = 1\}$

  $= 1/6 \times 1/6 = 1/36.$

- Question: Are the events $\{X_1 + X_2 \text{ is even}\}$ and $\{X_2 \text{ is even}\}$ statistically independent? (hint: check to see if $p(X \cap Y) = p(X)p(Y)$)

Example for statistical independence

- Question: What is the probability of getting an odd number when tossing a 6-faced fair dice, and then getting an ace when selecting a card from a pack of 52 cards randomly?
  - Answer: $3/6 \times 4/52 = 1/26$.

- Exercise: A card C is selected from a pack of 52 cards randomly, find $p(\{\text{card } C \text{ is black}\} \cap \{\text{card } C \text{ is a diamond}\})$.

Conditional probability

- **Conditional probability** is the probability of some event *E*, assuming event *F*. Conditional probability is written $p(E|F)$, and is read "the probability of *E*, given *F*".
- Let *E* and *F* be events with $p(F)>0$. The conditional Probability of *E* given *F*, is given by

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}.$$

Examples for conditional probability

- Question: Suppose that we flip a coin three times, and all eight possibilities are equally likely.
  - Let *F* be the event that the first flip comes up tail.
  - Let *E* be the event that an odd number of tails appears.
  - Given *F*, what is the probability of event *E*?
- Answer:

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)} = \frac{2/8}{4/8} = 1/2.$$

Examples for conditional probability

- Exercise: There is a sequence of four binary digits, assume that the probability of each binary digit to be 0 or 1 are equally likely. What is the probability that the sequence contains at least two consecutive 0s, given that its first digit is 0?

Review for Discrete Math

Practical use of Probability

- Allows one to analyze and estimate running time of a particular algorithm with respect to input size.
  - E.g. Quicksort (a popular sorting algorithm) can be proved to have an average running time of O(nlogn) using probability