L17: Vector

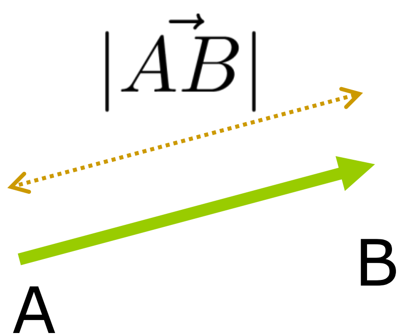
2012 Summer Math Course
for Direct Entry Students,
CSE Department, HKUST.

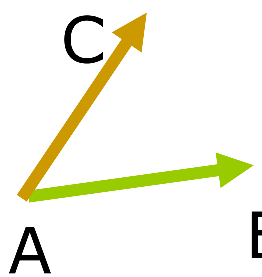
Definition of vector

- A quantity that can be completely specified by its numerical value is called a **scalar**.
- A scalar is merely a real number.
- A quantity that can be completely specified by both numerical value (**magnitude**) and **direction** is called a **vector**.
- For example, distance is a **scalar** while displacement is a **vector**.
- **Vector** is represented by $\vec{a}, \mathbf{a}, or, \underline{a}$

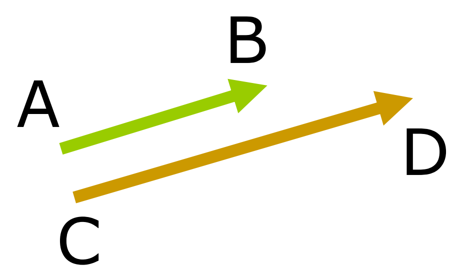
Representation of vector

- Pictorially, a geometric vector $\vec{AB}$ is represented by a directed line segment from A to B.
  - The length ($|\vec{AB}|$) stands for its **magnitude**.
  - The direction arrow stands for its **direction**.





Same magnitude, different direction



Same direction, different magnitude

Equality of vector

- Two vectors are said to be **equal** if and only if they have both the **same magnitude** and **direction**.
- $\vec{AB} = \vec{CD}$:

Same direction,
same magnitude

- Vector can be freely placed. (**Initial** and **end point** are not important)

Different kinds of vectors

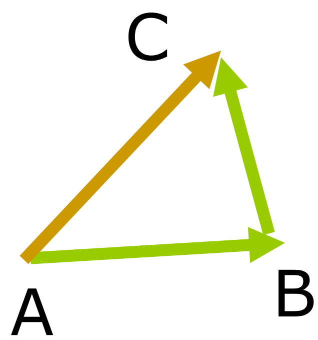
- A vector with zero magnitude is called a **zero vector** denoted by $\vec{0}/\mathbf{0}$.
- The zero vector has **no direction**.
- An **unit vector** is a vector with length equal to 1. (usually denoted by $\hat{a}$/hat(a))
- A vector can be **normalized** when divided by its length:

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

(length of normalized vector = 1)

Addition of vectors

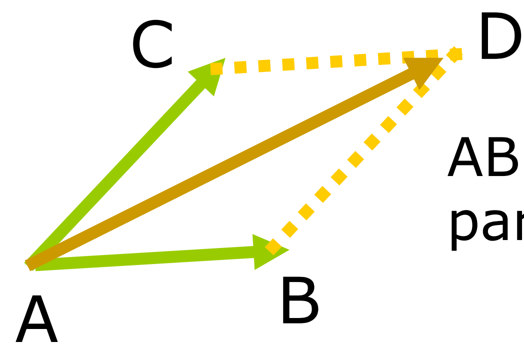
- **Triangle law:**



ABC is a triangle.

$$\vec{AB} + \vec{BC} = \vec{AC}$$

- **Parallelogram law:**

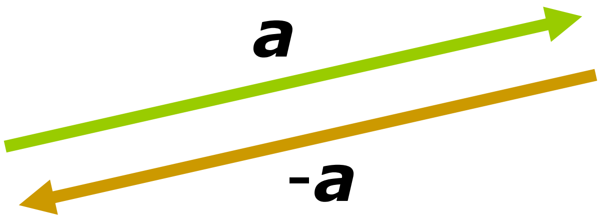


ABDC is a parallelogram.

$$\begin{aligned} & \vec{AB} + \vec{AC} \\ = \quad & \vec{AB} + \vec{BD} \quad \text{(Equality of vectors)} \\ = \quad & \vec{AD} \quad \text{(Triangle Law)} \end{aligned}$$

Subtraction of vectors

- The **negative** vector of ***a***, denoted by –***a***, is a vector having the same magnitude of ***a***, but having the opposite direction of ***a***.

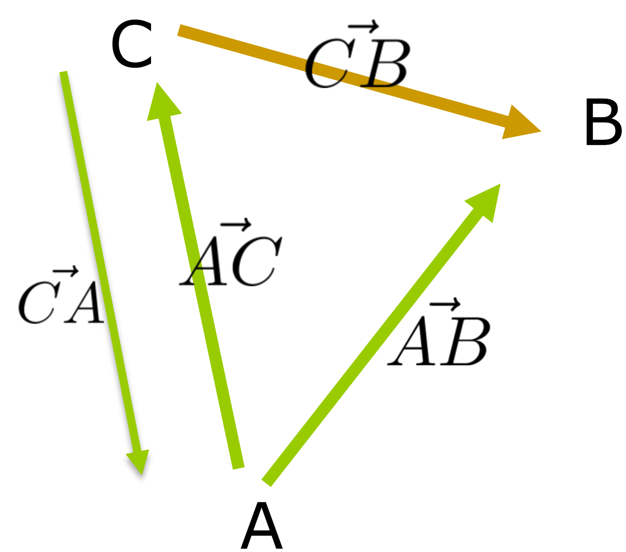


- The subtraction of two vectors is defined as follows: ***a***-***b***=***a***+(-***b***).

Subtraction of vectors

- Subtraction of vectors:

$$\begin{aligned} & \vec{AB} - \vec{AC} \\ = \; & \vec{AB} + \vec{CA} \quad \text{(negative of vector)} \\ = \; & \vec{CA} + \vec{AB} \\ = \; & \vec{CB} \quad \text{(Triangle Law)} \end{aligned}$$

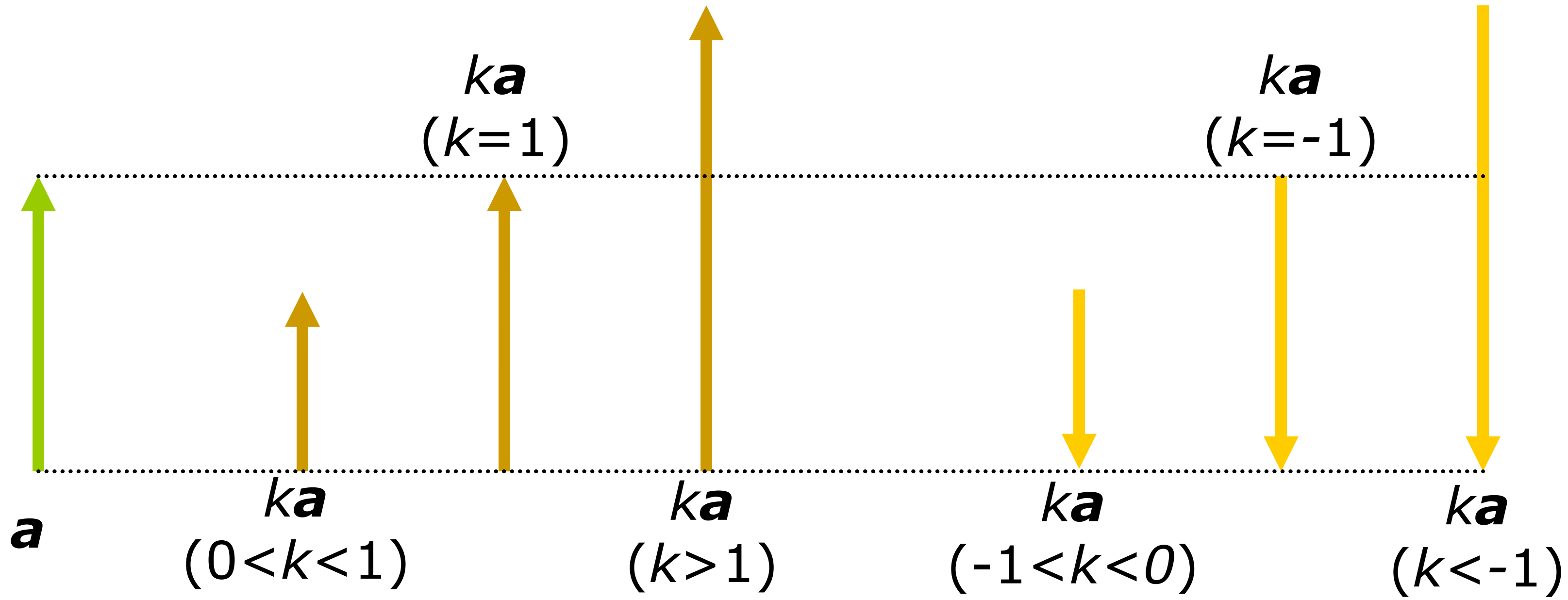


- It goes from the end of 2nd vector to the end of 1st vector.

Scalar multiplication of vector

- We use $k\mathbf{a}$ to denote the **scalar multiplication** of a vector $\mathbf{a}$ by scalar k.
- $k\mathbf{a}$ is a vector whose magnitude is $|k|$ times the magnitude of $\mathbf{a}$.
- The direction of $k\mathbf{a}$ is
  - the **same** as $\mathbf{a}$ when k is **positive** or
  - **opposite** to that of $\mathbf{a}$ when k is **negative**.

Scalar multiplication of vector



- Question: What happens when k=0?

Vector Properties

- Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ be vectors, and m, n be scalars.
- Properties:
  - $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$. (Commutative Law of Vector Addition)
  - $(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}=\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})$. (Associative Law of Vector Addition)
  - $m(\boldsymbol{a}+\boldsymbol{b})=m\boldsymbol{a}+m\boldsymbol{b}$. (Distributive law of Scalar Multiplication)
  - $m(n\boldsymbol{a})=mn(\boldsymbol{a})$. (Associative Law of Scalar Multiplication)

- Question: How to prove the above properties?

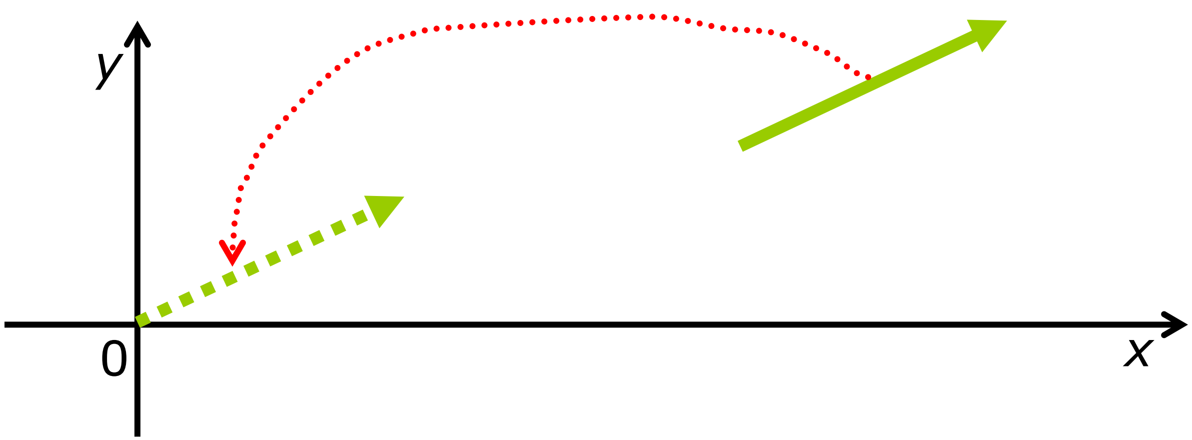
Parallel vectors

- Since vector carry both **magnitude** and **direction**, we can talk about **parallel** vectors by the direction.

- Definition:
  - Two nonzero vectors **v**, **w** are parallel (**v** // **w**)if and only if **v** = k **w**, for some nonzero scalar k.

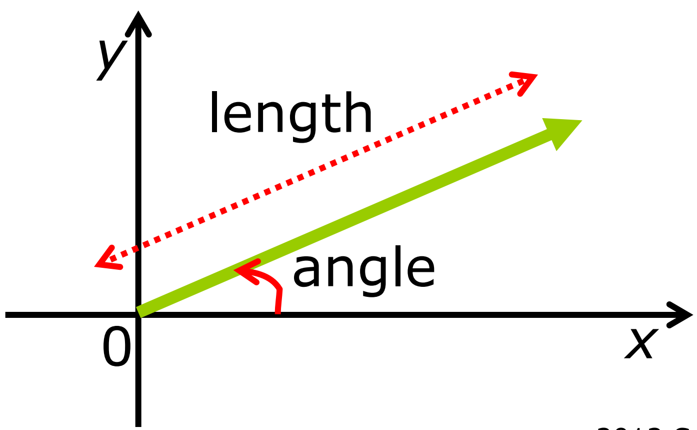
Vectors in $\mathbb{R}^2$

- When the vector is in $\mathbb{R}^2$, without affecting the magnitude and the direction, you can freely **fix** the **initial point** of a vector to the **origin**.
- This does not change the vector.



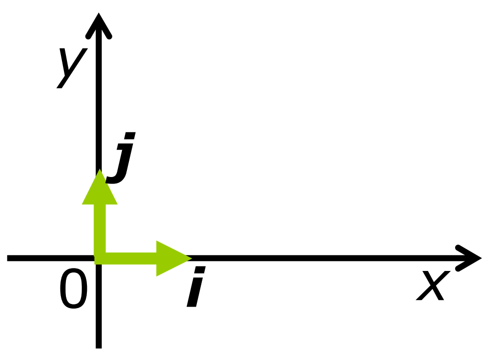
Vectors in $\mathbb{R}^2$

- The direction of a vector in $\mathbb{R}^2$ can be represented by the angle that the vector makes with the positive *x*-axis.
- Therefore, a vector in $\mathbb{R}^2$ can be **uniquely** represented by its **length** and this **angle**.



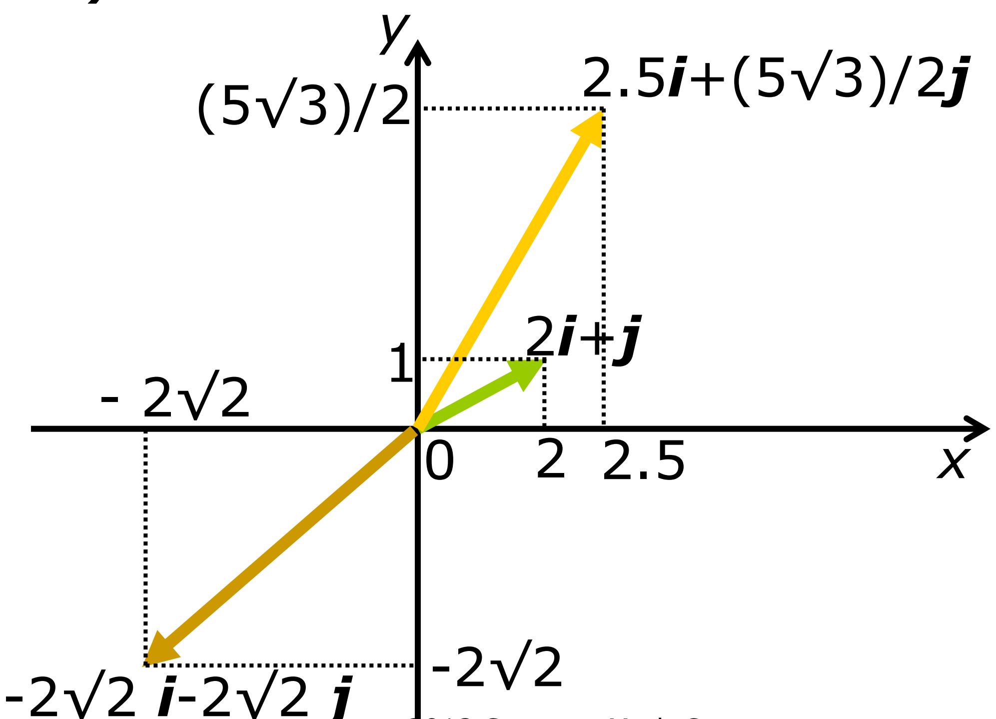
Vectors in $\mathbb{R}^2$

- Alternatively, instead of using the angle and the length, a vector in $\mathbb{R}^2$ can be uniquely represented as $a\boldsymbol{i}+b\boldsymbol{j}$,
  - for some real numbers *a* & *b*, where
  - $\boldsymbol{i}$ is the unit vector along the positive *x*-axis;
  - $\boldsymbol{j}$ is the unit vector along the positive *y*-axis.



Vectors in $\mathbb{R}^2$

- $a\boldsymbol{i}+b\boldsymbol{j}$ represents the vector whose initial point is the origin (0, 0) and the end point is (a, b).



Magnitude and Direction in $\mathbb{R}^2$

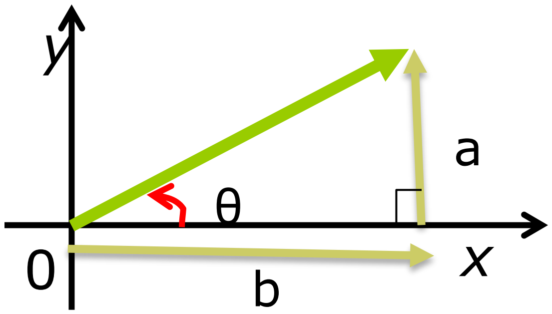
- Magnitude: $| a\vec{i} + b\vec{j} |$

$$= \text{distance} < (0,0), (a,b) >$$

$$= \sqrt{a^2 + b^2}.$$ (Pythagoras' theorem)

- Direction: $\tan(\theta) = \frac{a}{b}$

$$\theta = \tan^{-1}(\frac{a}{b})$$



Addition, subtraction and scalar multiplication of vectors in $\mathbb{R}^2$

- Addition:
  - $(a_1\boldsymbol{i}+b_1\boldsymbol{j})+(a_2\boldsymbol{i}+b_2\boldsymbol{j})=(a_1+a_2)\boldsymbol{i}+(b_1+b_2)\boldsymbol{j}$.
- Subtraction:
  - $(a_1\boldsymbol{i}+b_1\boldsymbol{j})-(a_2\boldsymbol{i}+b_2\boldsymbol{j})=(a_1-a_2)\boldsymbol{i}+(b_1-b_2)\boldsymbol{j}$.
- Scalar Multiplication:
  - $k(a\boldsymbol{i}+b\boldsymbol{j})=ka\boldsymbol{i}+kb\boldsymbol{j}$.

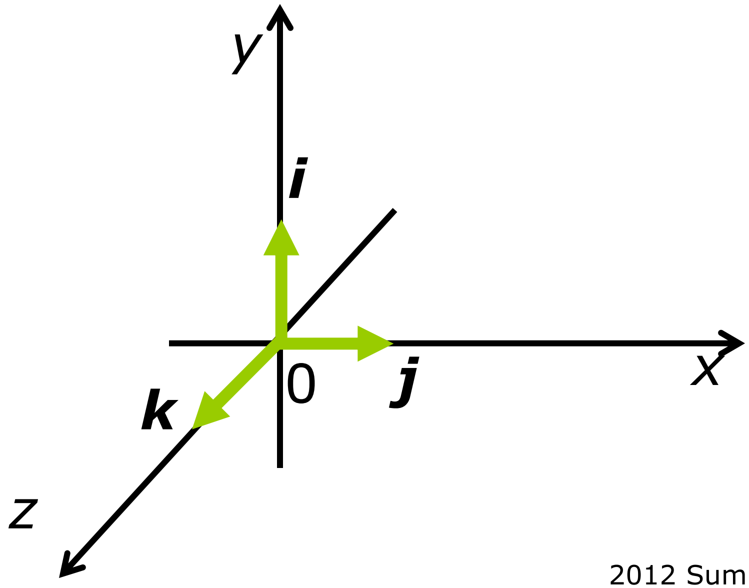
- Question: Find magnitude and angle of:
  a) $(3\boldsymbol{i}+4\boldsymbol{j})$, b) $(5\boldsymbol{i}+\boldsymbol{j})+5(-\boldsymbol{i}+\boldsymbol{j})$
  c) $2(2\boldsymbol{i}-\boldsymbol{j})-(-3\boldsymbol{i}+5\boldsymbol{j})$

Equality of vector in R^2

- For vectors $\mathbf{u}$ = a$\mathbf{i}$+b$\mathbf{j}$, $\mathbf{v}$ = c$\mathbf{i}$+d$\mathbf{j}$,
- Property:
  - If $\mathbf{u} = \mathbf{0}$
    => a$\mathbf{i}$+b$\mathbf{j}$ = 0$\mathbf{i}$+0$\mathbf{j}$
    => a=0 and b=0
  - If $\mathbf{u} = \mathbf{v}$
    => $\mathbf{u} - \mathbf{v} = \mathbf{0}$
    => (a-c)$\mathbf{i}$+(b-d)$\mathbf{j}$ = 0$\mathbf{i}$+0$\mathbf{j}$
    => (a-c)=0 and (b-d)=0
    => a = c and b = d
- The reason is: "**i** and **j** are **independent**".

Vectors in $\mathbb{R}^3$

- For any point *P*(*a*, *b*) in $\mathbb{R}^2$, its **position vector** $\overrightarrow{OP} = a\boldsymbol{i} + b\boldsymbol{j}$.
- Similarly, for any point *P*(*a*, *b*, *c*) in $\mathbb{R}^3$, its **position vector** $\overrightarrow{OP} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$.



- Note that ***k*** is the unit vector along the positive *z*-axis.

Magnitude of vectors in $\mathbb{R}^3$

- Magnitude: $|a\vec{i} + b\vec{j} + c\vec{k}|$

$$= \text{distance} \quad < (0,0,0), (a,b,c) >$$

$$= \sqrt{a^2 + b^2 + c^2}.$$

- Question:
  - a) Normalize ***i***+***j***-***k***.
  - b) Find |(5***i***+***j***+3***k***)+5(-***i+j***)|
  - c) Find |2(2***i***+***j***)-(3***i***+5***k***)|.

Addition, subtraction and scalar multiplication of vectors in $\mathbb{R}^3$

- Addition:
  - $(a_1\boldsymbol{i}+b_1\boldsymbol{j}+c_1\boldsymbol{k})+(a_2\boldsymbol{i}+b_2\boldsymbol{j}+c_2\boldsymbol{k})$
  $=(a_1+a_2)\boldsymbol{i}+(b_1+b_2)\boldsymbol{j}+(c_1+c_2)\boldsymbol{k}$.
- Subtraction:
  - $(a_1\boldsymbol{i}+b_1\boldsymbol{j}+c_1\boldsymbol{k})-(a_2\boldsymbol{i}+b_2\boldsymbol{j}+c_2\boldsymbol{k})$
  $=(a_1-a_2)\boldsymbol{i}+(b_1-b_2)\boldsymbol{j}+(c_1-c_2)\boldsymbol{k}$.
- Scalar Multiplication:
  - $k(a\boldsymbol{i}+b\boldsymbol{j}+c\boldsymbol{k})=ka\boldsymbol{i}+kb\boldsymbol{j}+kc\boldsymbol{k}$.

Equality of vector in R^3

- For vectors $\mathbf{u} = a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$, $\mathbf{v} = d\mathbf{i}+e\mathbf{j}+f\mathbf{k}$,

- Property:
  - If $\mathbf{u} = \mathbf{0}$
    => a=0 and b=0 and c=0
  - If $\mathbf{u} = \mathbf{v}$
    => a=d and b=e and c=f

- The reason is: "**i**,**j**,**k** are **independent**".