L11. Indefinite Integrals

Summer Math Course
for Direct Entry Students,
CSE Department, HKUST.

Indefinite Integral

- Intuitively, the **indefinite integration** is the reverse process of differentiation.
- For a continuous function $f(x)$, $F(x)$ is called a **primitive function** of $f(x)$ if

$$\frac{d}{dx}F(x) = f(x).$$

- The **indefinite integral** of a function $f(x)$ denoted by $\int f(x)dx$, is defined to be the *collection* (i.e. it's not unique) of all primitive functions of $f(x)$.

Indefinite Integral

- Mathematically,

  if $\dfrac{d}{dx}F(x) = f(x)$, then

  $$\int f(x)dx = F(x) + C,$$

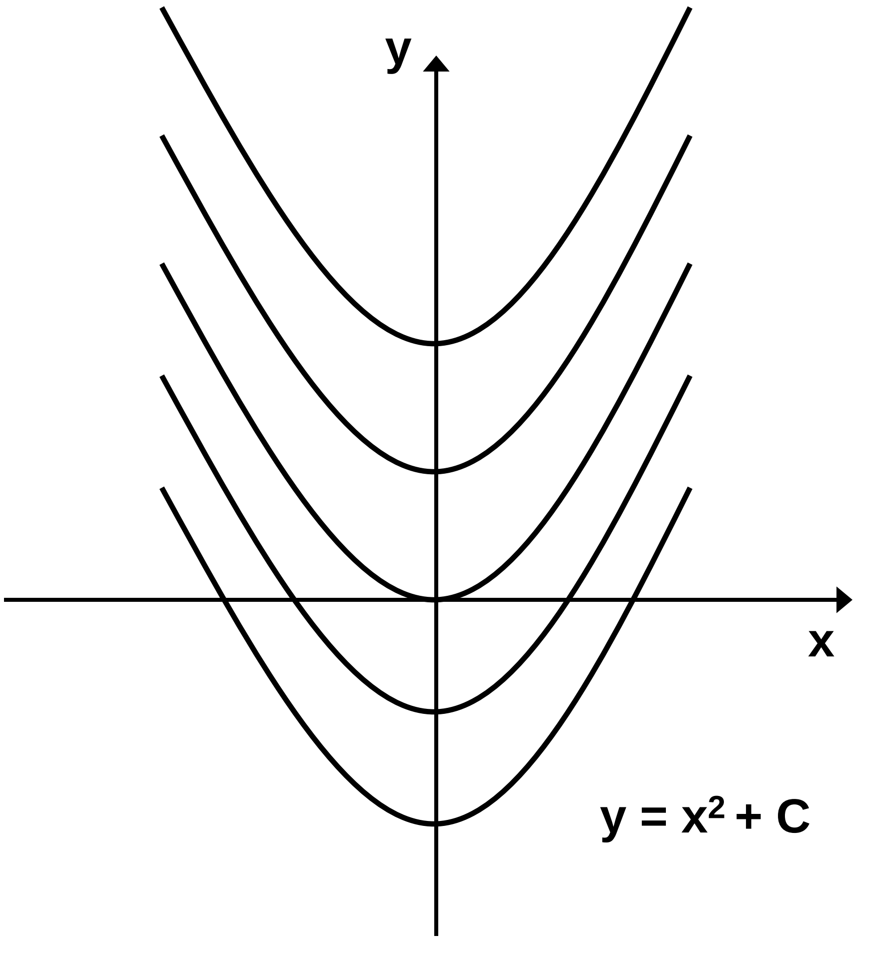
  where $f(x)$ is called the **integrand**, x the **variable** of integration, and C the **constant** of integration.
- Note: An indefinite integral is NOT a function, but a set of functions which differ by only an arbitrary constant C.

Example

- Let $f(x)=2x$.
- We know that: $\frac{d}{dx}x^2 = 2x.$
- Therefore, x^2 is a primitive function of $f(x)$.
- Moreover, for any constant C, $\frac{d}{dx}(x^2 + C) = 2x.$
- Therefore, for any constant C, x^2+C is a primitive function of $f(x)$.

Example

- A "family" of primitive functions of $f(x)=2x$.
- All gives the *same slope*, hence same $\frac{dy}{dx}$, at any given point.



Examples

- Using the notation of **indefinite integration**:

  $$\int 2x\,dx = x^2 + C.$$

- Another example:

  $$\frac{d}{dx}\left(\frac{x^4}{4}\right) = \frac{4x^3}{4} = x^3 \Rightarrow \int x^3\,dx = \frac{x^4}{4} + C$$

  where C is a constant.

Rules of indefinite integration

- Two rules of indefinite integration:
  - $\int k\ f(x)\,dx = k\int f(x)\,dx$ where k is a constant.
  - $\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx.$
- Proof of the first rule:
  - Let $F(x)$ be a primitive function of $f(x)$.

$$\frac{d}{dx}[k\,F(x)] = k\frac{d}{dx}F(x) = k\ f(x)$$

$$\Rightarrow \int k\ f(x)\,dx = k\,F(x) + C_1.$$

Rules of indefinite integration

And $k\int f(x)\,dx = k(F(x)+C_2) = k\,F(x)+C_3.$

- Note that C_1, C_2 and C_3 are constants.
- Neglecting the arbitrary constants C_1 and C_3, we have, $\int k\,f(x)\,dx = k\int f(x)\,dx.$

- Exercise: Prove the second rule, that is: $\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx.$

Some standard integration formulae

- Here are some standard integration formulae (where *C* is an constant):
  - (1) $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad \text{for all } n \neq \text{-1}.$
  - (2) $\int \frac{1}{x} \, dx = \ln(|x|) + C.$
  - (3) $\int e^x \, dx = e^x + C.$
  - (4) $\int a^x \, dx = \frac{a^x}{\ln a} + C.$

Proof of formulae 1

- We now give proofs for formulae 1 and formulae 2. The proofs of formulae 3 and 4 are left as exercises.

- Proof of formulae 1:

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^n}{n+1} = x^n \text{ if } n \neq \text{-}1.$$

$$\Rightarrow \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad \text{for all } n \neq \text{-}1.$$

Proof of formulae 2

- Proof of formulae 2:
  - For $x>0$,
    $$\frac{d}{dx}\ln(|x|) = \frac{d}{dx}\ln(x) = \frac{1}{x}.$$
  - For $x<0$,
    $$\frac{d}{dx}\ln(|x|) = \frac{d}{dx}\ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$
  - Therefore, $\int \frac{1}{x}dx = \ln(|x|) + C.$

Exercises

- Exercise: Evaluate the following integrals:

$$\int (x+1)^2 \, dx,$$

$$\int \frac{1-x^2}{1-x} \, dx,$$

$$\int \left[2e^x - \frac{3}{x} \right] dx.$$

Method of substitution

- The **method** of **substitution**:
  If $u=g(x)$ then $\int f(g(x))g'(x)\,dx = \int f(u)\,du.$
- Proof:
  - Let $F(u)$ be the primitive function of $f(u)$.
  - Then, $\int f(u)\,du = F(u) + C_1.$
  - By chain rule, $\dfrac{d}{dx}F(u) = \dfrac{d}{du}F(u) \times \dfrac{du}{dx} = f(u)g'(x).$
  - So, $\int f(u)g'(x)dx = F(u) + C_2 = \int f(u)du - C_1 + C_2.$
  - Neglecting the arbitrary constants gives the proof.

Method of substitution

- In the intermediate step, we can write:
  If $u=g(x)$ then $du=g'(x)dx$.
- Then, we have:
  $$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$
  with substitution $g'(x)dx$ by du.
- The variable of integration is changed form x to u. So, this method is also known as *integration by **change** of **variable***.

Example

- Example: Suppose we want to find $\int e^{2x+3}\,dx$.
- Let $u=2x+3$, then $du=2dx$.

$$\therefore \int e^{2x+3}\,dx = \int e^{u}\,\frac{1}{2}\,du$$

$$= \frac{e^{u}}{2} + C = \frac{e^{2x+3}}{2} + C.$$

Example

- Example: Suppose we want to find $\int \frac{x^3}{3+x^4}\,dx$.
- Let $u=3+x^4$, then $du=4x^3dx$.

$$\therefore \int \frac{x^3}{3+x^4}\,dx = \frac{1}{4}\int \frac{1}{3+x^4}\,4x^3dx = \frac{1}{4}\int \frac{1}{u}\,du$$

$$= \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln\left|3+x^4\right| + C.$$

Exercise

- Exercise: Find the following integrals:

$$\int \frac{x-1}{(3x^2-6x+5)^2}\,dx \quad \text{and}$$

$$\int \frac{\ln x}{x}\,dx.$$

Integration of rational function

- The technique of **partial fraction rule** to resolve the rational function $P(x)/Q(x)$ into fractions is useful for integration.
- It states that we can always decompose such a rational function into partial fractions if:
  - P(x) and Q(x) are polynomials of x only, and
  - Degree of P(x) is smaller than Q(x) (if not, we can achieve that by polynomial division).

Integration of rational function

- Example:

$$\text{let } \frac{4}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1},$$

$$4 = A(x+1) + B(x-1)$$

- Substituting x=1,-1 we get A=2 and B=-2. So,

$$\frac{4}{x^2-1} = \frac{2}{x-1} - \frac{2}{x+1}.$$

Integration of rational function

- The technique of **partial fraction rule** to resolve the rational function $P(x)/Q(x)$ into fractions is useful for integration.
- Example: Suppose we want to find $\int \frac{x^2-1}{x+2}\,dx.$
- Let $\frac{x^2-1}{x+2} = Ax + B + \frac{C}{x+2}.$
- Solving for A, B and C, we can easily obtain A=1, B=-2 and C=3. Thus
$$\frac{x^2-1}{x+2} = x - 2 + \frac{3}{x+2}.$$

Integration of rational function

$$\therefore \int \frac{x^2-1}{x+2}\,dx = \int \left[\mathrm{x} - 2 + \frac{3}{x+2} \right] dx$$

$$= \frac{x^2}{2} - 2x + 3\ln|x+2| + C.$$

- Example: To find $\int \frac{18-x}{12x^2-7x-12}\,dx.$
- Let $\frac{18-x}{12x^2-7x-12} = \frac{18-x}{(3x-4)(4x+3)} = \frac{A}{3x-4} + \frac{B}{4x+3}.$

Integration of rational function

- Solving for *A* and *B* we can easily obtain *A*=2, *B*=-3. Therefore

$$\frac{18-x}{12x^2-7x-12}=\frac{18-x}{(3x-4)(4x+3)}=\frac{2}{3x-4}-\frac{3}{4x+3}.$$

$$\therefore \int \frac{18-x}{12x^2-7x-12}dx=\int\left[\frac{2}{3x-4}-\frac{3}{4x+3}\right]dx$$

$$=\frac{2}{3}\ln|3x-4|-\frac{3}{4}\ln|4x+3|+C.$$