On the Average Complexity of 3D-Voronoi Diagrams of Random Points on Convex Polytopes*

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Abstract

It is well known that the complexity, i.e., the number of vertices, edges and faces, of the 3-dimensional Voronoi diagram of n points can be as bad as $\Theta(n^2)$. It is also known that if the points are chosen Independently Identically Distributed uniformly from a 3-dimensional region such as a cube or sphere, then the *expected* complexity falls to $O(n)$. In this paper we introduce the problem of analyzing what occurs if the points are chosen from a 2-dimensional region in 3-dimensional space. As an example, we examine the situation when the points are drawn from a Poisson distribution on the *surface* of a convex polytope.

1 Introduction

Given a set $S_n = \{p_1, p_2, \ldots, p_n\} \subseteq \mathbb{R}^k$ of n points in k-dimensional Euclidean space, the *Voronoi Diagram*, $\mathrm{VD}(S_n)$, of S_n is a very well understood subdivision of R^k. For each point $p_i \in S_n$ there is an associated (convex) cell

$$C_i = \{x \in \mathbb{R}^k \,:\, \forall j \neq i,\, d(x, p_i) \leq d(x, p_j)\}$$

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where $d(\cdot,\cdot)$ is the Euclidean distance function. By definition these cells partition R^k. The *complexity* of $VD(S_n)$ is the number of lower dimensional pieces that compose $\mathrm{VD}(S_n)$. For example, in the planar case, $k=2$, $\mathrm{VD}(S_n)$ contains edges $\mathcal{E}$ and vertices $\mathcal{V}$. The complexity of $\mathrm{VD}(S_n)$ will be $|\mathrm{VD}(S_n)| = |\mathcal{E}|+|\mathcal{V}|$. Since it is also known that the 2-dimensional Voronoi Diagram is a planar graph, Euler's theorem immediately implies that $|\mathrm{VD}(S_n)| = \Theta(n)$ [8].

If $k=3$ then $\mathrm{VD}(S_n)$ is composed not only of edges $\mathcal{E}$ and vertices $\mathcal{V}$ but also of the faces $\mathcal{F}$ of the convex cells. The complexity of $\mathrm{VD}(S_n)$ will then be $|\mathrm{VD}(S_n)| = |\mathcal{E}|+|\mathcal{V}|+|\mathcal{F}|$. In three dimensions it can be proven that $|\mathrm{VD}(S_n)| = O(n^2)$. For some cases, such as when all of the points in S_n are on the *moment curve* $\{(t,t^2,t^3) : t \in \mathbb{R}\}$, it can be easily proven that $\mathcal{V} = \Theta(n^2)$ so $|\mathrm{VD}(S_n)| = \Theta(n^2)$ [6]. Another well known example of this worst-case behavior is built around two line segments $L_1 = \left\{(x,0,0) : x \in [0,\frac{1}{2}]\right\}$ and $L_2 = \left\{(1,y,1) : y \in [\frac{1}{2},1]\right\}$; given any n_1 points on L_1 and any n_2 points on L_2, $|\mathcal{V}| = \Theta(n_1 n_2)$. In particular if $n_1 = n_2 = \frac{n}{2}$ then $|\mathcal{V}| = \Theta(n^2)$.

Moving away from worst-case behavior to average-case behavior it has been shown that if n points of S_n are independently identically distributed (IID) chosen from the uniform distribution over a "reasonably" smooth full dimensional bounded region $\mathcal{P}$ such as a cube or sphere then $E(|\mathrm{VD}(S_n)|) = \Theta(n)$ [2][4][5] [1].

Dropping the condition that $\mathcal{P}$ has full dimensionality dramatically changes the situation. For example, if we set $P = L_1 \cup L_2$ to be the union of the two 1-dimensional segments previously defined and choose n points S_n uniformly at 'random' from P then $n_1 = |S_n \cap L_1|$, the number of points on L_1, is a binomial random variable with parameters $n, \frac{1}{2}$, so $E(n_1) = \frac{n}{2}$ and $E(n_1^2) \sim \frac{n^2}{4}$. Since $n_2 = |S_n \cap L_2| = n - n_1$ and $\mathcal{V} = \Theta(n_1 n_2)$ we have that

$$E(|\mathcal{V}|) = \Theta(E(n_1(n-n_1))) = \Theta(n^2)$$

and $E(|\mathrm{VD}(S_n)|) = \Theta(n^2)$.

Combining the two previous paragraphs we see that, in 3-dimensional space, if n points S_n are chosen IID uniformly from $\mathcal{P}$ where $\mathcal{P}$ is a reasonably smooth 3-dimensional region then $E(|\mathrm{VD}(S_n)|) = \Theta(n)$ while for some 1-dimensional $\mathcal{P}$s, $E(|\mathrm{VD}(S_n)|) = \Theta(n^2)$. The obvious question then is what happens if $\mathcal{P}$ is a *2-dimensional* surface in 3-dimensional space and n points S_n are chosen IID uniformly from it. What will be the expected complexity $E(|\mathrm{VD}(S_n)|)$ of the 3-dimensional Voronoi diagram of those points? $\Theta(n^2)$? $\Theta(n)$? Something in between?

The problem of understanding the structure of the 3-dimensional Voronoi diagram of point sets from 2-dimensional surfaces has started to be of interest in

[1] These references don't exactly state this fact but it can be inferred from the general techniques developed there. The idea behind the proof is that points p_i *inside* $\mathcal{P}$ only have a constant number of Voronoi *neighbors* so their Voronoi cells C_i will have constant complexity. Points near the boundary of $\mathcal{P}$ might have Voronoi cells with high complexity but there are only a small number of such boundary cells.

recent years. This is because, as described in [1] and [7], Voronoi diagrams and their duals, the Delaunay triangulation, are of use in several geometric problems, e.g., surface reconstruction, mesh generation and surface modeling. In these problems a 2-dimensional surface is often sampled and then modeled, at least initially, by the Delaunay triangulation of the sample. Many parameters of such algorithms such as their running times and the complexity of their representations, then depend upon the complexity of the Delaunay triangulation (which is the same as that of the Voronoi diagram).

The two results [1] and [7] mentioned above seem to be the first to try and formally analyze the complexity of such Voronoi diagrams. In [1] Attali and Boissonnat prove that if n "well-sampled" points are chosen from a "smooth" closed surface then the complexity of their Voronoi diagram is $O(n^{7/4})$ where "well-sampled" is defined using the concept of local feature size. In [7] Erikson proves that there is a set of n "well-sampled" points from the cylinder with Voronoi diagram complexity $\Omega(n^{3/2})$.

There does not, though, seem to be any previous work on analyzing the *expected* complexity of the Voronoi Diagram when the points are chosen randomly from some 2-dimensional surface. In this note we make a first step towards answering this question by looking at random points chosen from the boundary of a convex polytope in $\mathbb{R}^3$. More specifically we prove

Theorem 1 *Let $\mathcal{P}$ be the boundary of a convex polytope in $\mathbb{R}^3$. Let S_n be a set of points drawn from the standard 2-dimensional Poisson distribution on $\mathcal{P}$ with rate n. Then $E(|\mathrm{VD}(S_n)|) = \Theta(n)$.*

The Poisson distribution on $\mathcal{P}$ with rate n is the one that has the properties

- If $M \subseteq \mathcal{P}$ is any measurable region let $N(M)$ be the random variable signifying the number of points the process generates in M (the dependence upon n is implicit). Then

$$\Pr(N(M) = k) = \frac{(n\mathrm{Area}(M))^k e^{-(n\mathrm{Area}(M))}}{k!} \tag{1}$$

  (so $E(N(M)) = n\mathrm{Area}(M)$).

- If M_1 and M_2 are non-overlapping regions, then $N(M_1)$ and $N(M_2)$ are independent random variables.

We note that we have restricted ourselves to proving Theorem 1 for a Poisson distribution because its mathematics are a bit cleaner (it allows us to assume that points in various regions are chosen independently of each other) but standard modifications allow the proof to also work for n points chosen IID from the uniform distribution over $\mathcal{P}$ and show, in this case as well, that $E(|\mathrm{VD}(S_n)|) = \Theta(n)$.

To get a feeling for the type of problem we are analyzing consider the box B with diagonal corners $(0,0,0)$ and $(3,3,1)$. In Figure 1 we see 9000 points chosen randomly IID from the uniform distribution over the surface of the box and the 24943 Voronoi vertices that correspond to them (we do not draw the full Voronoi diagram since such a large diagram would be impossible to view properly). Note that most of the Voronoi vertices are *inside* B with only a small fraction being outside the box. In our proof of Theorem 1 we will see why this happens.

In Section 2 we sketch the idea behind our proof and show how solving two smaller more specific subproblems would prove Theorem 1. In Section 3 we introduce definitions and utility lemmas that will be used throughout the rest of the paper. In Sections 4 and 5 we solve the two smaller subproblems introduced in Section 2. In Section 6 we review our work and discuss extensions and open problems.

2 A Sketch of the Proof

In what follows $\mathcal{P}$ will be the boundary of a given convex polytope and S_n will be a set of n points drawn from the 2-dimensional Poisson distribution on $\mathcal{P}$.

For a point $p \in \mathbb{R}^3$ and any closed or finite set set $X \subseteq \mathbb{R}^3$ extend the Euclidean distance function so that $d(p,X) = \min_{q \in X} d(p,q)$. Now define

Definition 1 *Let* $p \in \mathbb{R}^3$, $X \subseteq \mathbb{R}^3$ *and* $r \geq 0$:

- $S(p,r) = \{q \in \mathbb{R}^3 \ : \ d(q,p) \leq r\}$ *is the closed sphere of radius* r *around* p.
- *For point* $p \in \mathbb{R}^3$, $NN(p,X)$, *will denote a nearest neighbor* q *to* p *in* X, *i.e. a* $q \in X$ *such that*

$$\forall q' \in X, \ d(p,q) \leq d(p,q').$$

  In this paper all of the sets X *used will either be finite or closed. Thus such a* q *will always exist although it might not always be unique.*

- $r_p = d(p, S_n)$
- $S = S(p, r_p)$ *is the* defining sphere *around* p.

Note that S contains no points of S_n in its interior and at least one point of S_n, e.g., $NN(p, S_n)$, on its boundary. We will call the points of S_n on S's boundary the *defining* points of S. It is well known that every vertex/edge/face/region of $\mathrm{VD}(S_n)$ corresponds to at least one defining sphere $S = S(p, r_p)$ with 4/3/2/1 defining points of S_n on its boundary. Two spheres will correspond to the same vertex/edge/face/region of $\mathrm{VD}(S_n)$ if they have the exact same defining points. See Figure 2.

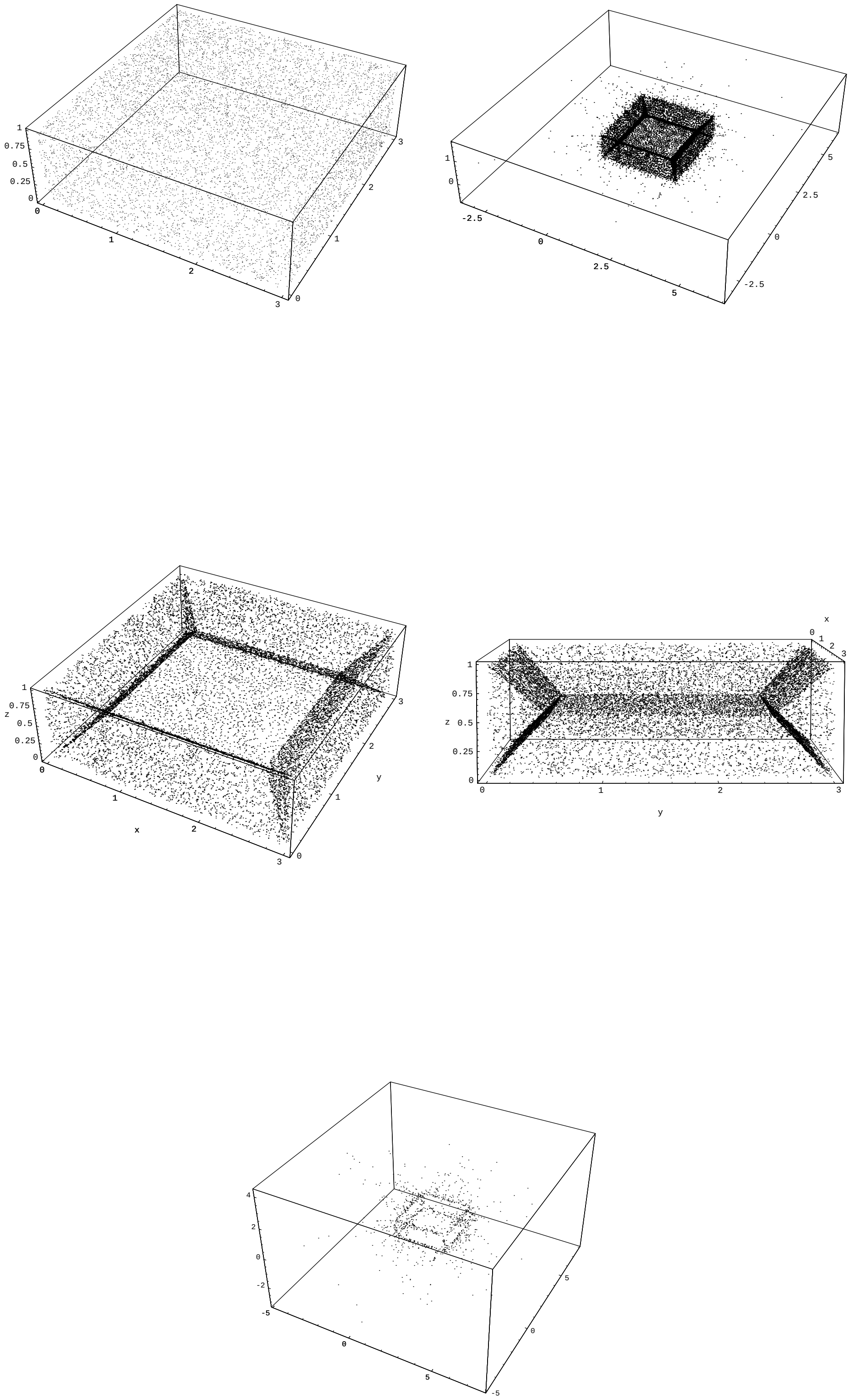


Figure 1: Top-Left: 9000 random points chosen from the surface of box B. Top-Right: the 24943 Voronoi vertices of the points. Middle: the 23455 Voronoi vertices inside B viewed from different view points; Bottom: the 1488 Voronoi vertices outside B. Note that the scales on the different figures are not the same.

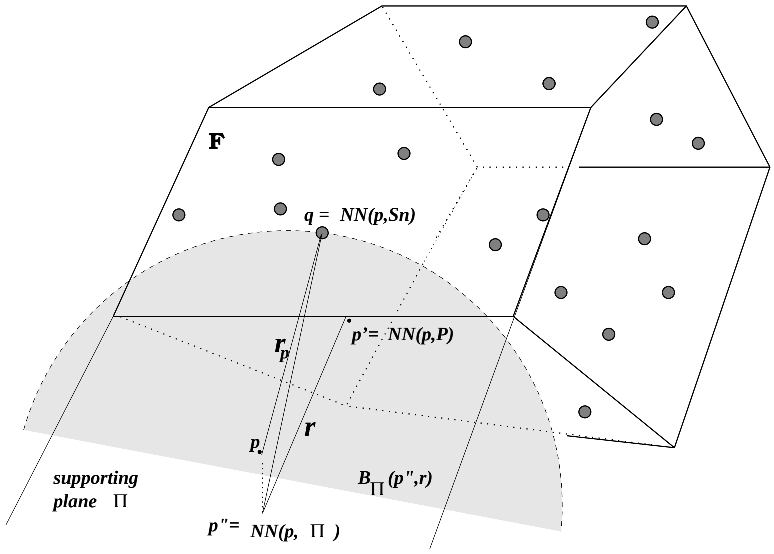


Figure 2: Notations: p is outside polytope $\mathcal{P}$. By definition, r_p is the distance from p to $q = NN(p, S_n)$. Note that q is different from $p' = NN(p, \mathcal{P})$ and from $p'' = NN(p, \Pi)$. q is a defining point of the sphere centered at p.

Our strategy for bounding the complexity of $\mathrm{VD}(S_n)$ will therefore be *to bound the number of different possible defining spheres S* where two spheres will be considered different only if they have different defining points. Note that we will use the terms *defining spheres* and *Voronoi spheres* interchangeably.

We will also need the following definition

Definition 2 *Let $\mathcal{P}$ be the boundary of a convex polytope. A sphere S in $\mathbb{R}^3$ is* x-bad *(with respect to $\mathcal{P}$) if*

$$Area(S \cap \mathcal{P}) \geq x^2.$$

A sphere S in $\mathbb{R}^3$ is x-good *(with respect to $\mathcal{P}$) if it is not* x-bad.

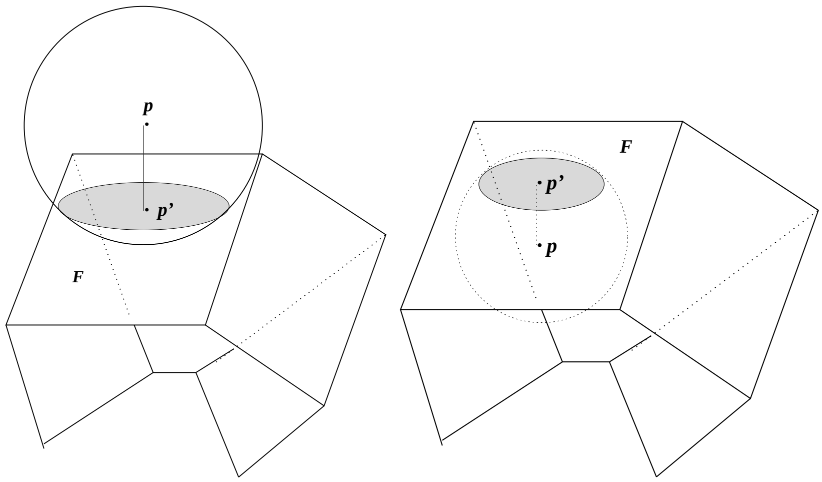
The reason for introducing this definition is the following lemma:

Lemma 1 *Let S_n be a set of points chosen from the standard 2-dimensional Poisson distribution on $\mathcal{P}$ with rate n. Then*

$$\Pr\left(\text{there exists a } \tfrac{\log n}{\sqrt{n}}\text{-}bad \text{ sphere } S \text{ such that } S_n \cap S = \emptyset\right) = n^{-\Omega(\log n)}.$$

The proof of this Lemma is deferred to the Appendix.

We need one more set of definitions before presenting our sketch proof of Theorem 1.



(a) p is outside $\mathcal{P}$ (b) p is inside $\mathcal{P}$

Figure 3: Type-I spheres

Definition 3 *Let $\mathcal{P}$ be a convex polytope and $S = S(p, r)$ a sphere. S will be a Type-I, Type-II or Type-III sphere if S contains at least one point of S_n on its boundary and:*

- *S is a* Type-I *sphere if $\exists$ a face F of $\mathcal{P}$ such that $S \cap \mathcal{P} \subseteq F$ (Figure 3).*
- *S is a* Type-II *sphere if (i) it is not a Type-I sphere, (ii) p is* inside $\mathcal{P}$ *and (iii) for every face F_i of $\mathcal{P}$ with corresponding supporting plane Π_i, if $S \cap F_i \neq \emptyset$ then $NN(p, \Pi_i) \in F_i$ (Figure 4).*
- *If S is not a Type-I or Type-II sphere then S is a* Type-III *sphere.*

(The *supporting plane* of a face F is the infinite plane Π that contains F.) Type-I spheres are easily understood. If p is inside $\mathcal{P}$ and S is not a Type-I sphere, then S is a Type-II sphere if, for every face F_i of $\mathcal{P}$ that S intersects, the center of the circle formed by $S \cap \Pi_i$ is in F_i. Type-III spheres are catch-alls that cover every other case.

We will now sketch the proof technique; it is to count Voronoi spheres, i.e., spheres of the form $S(p, r)$ where the boundary of $S(p, r)$ contains at least one of S_n such that the interior of $S(p, r)$ is empty. Recall that in our counting two spheres will be equivalent, if they have the same point(s) on their boundary. Then

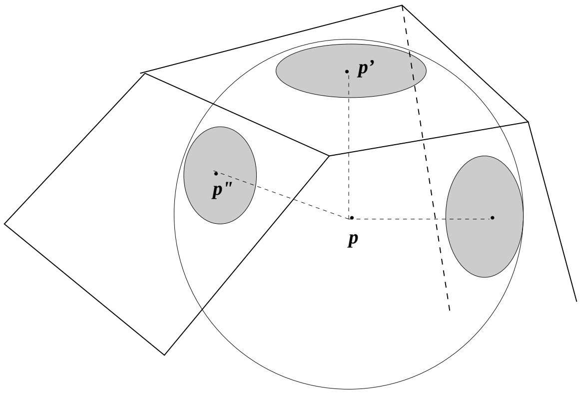


Figure 4: An example of a Type-II sphere. p, the center of the sphere, is inside polytope $\mathcal{P}$ (for clarity many of the faces of $\mathcal{P}$ have been left out of the diagram). For all the faces F_i of $\mathcal{P}$ that the sphere intersects, the center of the intersection circle with the supporting plane Π_i is on F_i.

$$\begin{aligned}
&\text{No. of Voronoi Spheres}\\
&= \text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-bad Voronoi Spheres} + \text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-good Voronoi Spheres}\\
&= \text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-bad V. Spheres} + \text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-good Type-I V. Spheres}\\
&\quad + \text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-good Type-II V. Spheres} + \text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-good Type-III V. Spheres}\\
&\leq \text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-bad V. Spheres} + \text{No. of Type-I V. Spheres}\\
&\quad + \text{No. of Type-II V. Spheres} + \text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-good Type-III Spheres}
\end{aligned} \tag{2}$$

The remaining sections of this paper are devoted to proving:

$$E\left(\text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-bad V. Spheres}\right) = o(1). \tag{3}$$

$$E\left(\text{No. of Type-I V. Spheres}\right) = O(n). \tag{4}$$

$$E\left(\text{No. of Type-II V. Spheres}\right) = O(n). \tag{5}$$

$$E\left(\text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-good Type-III Spheres}\right) = o(n). \tag{6}$$

Note: in our definitions of Type-I,II,III spheres we required that the spheres contained at least one point of S_n on their boundaries. This is obviously true for Voronoi Spheres. When counting $\frac{\log n}{\sqrt{n}}$-good Type-III Spheres this is also required so we will actually be counting the number of such spheres that have some point(s) of S_n on their boundaries. Two $\frac{\log n}{\sqrt{n}}$-good Type-III Spheres will be considered equivalent if they have the same points of S_n on their boundaries.

Now, taking expectations of equation (2) gives

$$
\begin{aligned}
&E(\text{Number of Voronoi Spheres})\\
&\leq E(\text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-bad V. Spheres}) + E(\text{No. of Type-I V. Spheres})\\
&\quad + E(\text{No. of Type-II V. Spheres}) + E(\text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-good Type-III Spheres}).
\end{aligned} \tag{7}
$$

Plugging in (3)-(6) will then prove Theorem 1, that the expected number of Voronoi spheres, which is the same as the expected complexity of the Voronoi Diagram, will be $O(n)$. The main reason that we use this decomposition into good and bad spheres is that it permits us to upperbound the number of $\frac{\log n}{\sqrt{n}}$-Good Type-III *Voronoi* Spheres by the number of $\frac{\log n}{\sqrt{n}}$-Good Type-III Spheres. Bounding the number of such *Voronoi* Spheres would be quite difficult since assuming that a sphere is Voronoi, i.e., empty, requires conditioning that skews the rest of the point distribution, making it very difficult to count the number of other Voronoi Spheres. Bounding the number of *Good* Spheres is much easier since it only requires calculating how many points could feasibly fall within a particular volume or area.

Proving (3) and (4) will be quite simple and we do that below; proving (5) and (6) will be more complicated and will require the remainder of this paper.

To prove (3) let $A = \sum_i Area(F_i)$ be the total surface area of $\mathcal{P}$. Then $\Pr(S_n = i) = \frac{2An^i}{i!}e^{-2An}$. Set X to be the number of $\frac{\log n}{\sqrt{n}}$-bad Voronoi Spheres. Recall that, in the worst case, point set S_n defines at most $O(|S_n|^2)$ Voronoi spheres. Thus

$$
E\left(XI_{\{}|S_n| > 2An\}\right) \leq \sum_{i>2An} O(i^2)\frac{2An^i}{i!}e^{-2An} = n^{-\Omega(\log n)} \tag{8}
$$

where I_B is the indicator random variable for event B, i.e, $I_B = 1$ if B occurs and 0 otherwise. On the other hand Lemma 1 states that the probability that there is a $\frac{\log n}{\sqrt{n}}$-bad Voronoi Sphere is $n^{-\Omega(\log n)}$ so

$$
E\left(XI_{\{|S_n| \leq 2An\}}\right) \leq O(|2An|^2)\Pr(X > 0) = n^{-\Omega(\log n)}. \tag{9}
$$

Combining (8) and (9) proves

$$
\begin{aligned}
E\left(\text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-bad V. Spheres}\right) &= E(X)\\
&= E\left(XI_{\{|S_n|>2An\}}\right) + E\left(XI_{\{|S_n|\leq 2An\}}\right)\\
&= n^{-\Omega(\log n)}
\end{aligned}
$$

and thus we have shown (3).

Proving (4) is also quite easy. From the definition of Type-I spheres we have that $S_n \cap S \subseteq F$ for some face F of $\mathcal{P}$. This means that S will be a defining sphere of the *planar* Voronoi diagram of the points $S_n \cap S$ on the supporting plane Π of F. Since planar Voronoi diagrams of m points have complexity $O(m)$ we immediately have that the total number of Type-I Voronoi Spheres S (we no longer have to restrict ourselves to good ones) intersecting face F_i is $O(|S_n \cap F_i|)$. Summing over all faces F_i gives that the total number of Type-I Voronoi Spheres is $O(\sum_i |S_n \cap F_i|) = O(|S_n|)$. The expected number of Type-I Voronoi Spheres is therefore $O(E(|S_n|)) = O(An) = O(n)$ proving (4).

The remainder of this paper is devoted to proving (5) (in Section 4) and (6) (in Section 5) which will both require tedious case by case analyses.

To prove (5) the intuition is that we essentially partition $\mathcal{P}$ up into small squares, each with area $\frac{1}{n}$. For each such square we then calculate the expected number of Voronoi spheres with center p inside $\mathcal{P}$ such that the nearest neighbor on $\mathcal{P}$ to p is on that square, and prove that this is $O(1)$. Since there are only $O(n)$ squares in the partition this will prove (5).

To prove (6) the intuition is that we show that for every p' on the edges of $\mathcal{P}$ there is associated a region, call it $A(p')$, with area $O\left(\frac{\log^3 n}{n}\right)$ that has the following property: $\forall p \in \mathbb{R}^3$, if $NN(p, \mathcal{P}) = p'$, then, with high probability, the points in any $\frac{\log n}{\sqrt{n}}$-good sphere centered at p must be in $A(p')$. Thus the number of such spheres can be bounded by the number of 4-tuples such that all four points must all be in $A(p')$ for the same p'. Summing over this calculation will prove (6).

3 Definitions and Utility Lemmas

In this subsection we introduce some basic definitions and utility lemmas that will be used in the rest of the paper.

We will often use the following basic properties of the Poisson distribution so we encapsulate them in two lemmas.

Lemma 2

- *Let M, M' be measurable regions with $M' \subseteq M$, X a set of points drawn from the Poisson distribution with rate n over M and X' a set drawn from the Poisson distribution with rate n over M'. Then X' has the same distribution as $X \cap M'$.*

- *Let F be a convex polygon and S_n a set of points drawn from the Poisson distribution with rate n over F. The probability that four points in S_n are cocircular is* 0.

- *Let $\mathcal{P}$ be the boundary of a convex polytope and S_n a set of n points drawn from the Poisson distribution over $\mathcal{P}$. The probability that five points in S_n are cospherical is* 0.

Lemma 3 *Let Z be any discrete Poisson distribution with any rate λ, i.e., $\forall i \geq 0$, $\Pr(Z = i) = \lambda^i \frac{e^{-\lambda}}{i!}$. Then $E(Z) = \lambda$ and $\forall j > 1$, $\exists d_j$ independent of λ such that $E(Z^j) \leq d_j \lambda^j$. That is $E(Z^j) \leq d_j (E(Z))^j$.*

We also strongly use the following geometric definitions:

Definition 4 *Let Π be a 2-dimensional plane in $\mathbb{R}^3$; C_Π, D_Π and $\overline{D_\Pi}$ will be the circle, open and closed disks*

$$\begin{aligned} C_\Pi(p,r) &= \{q \in \Pi : d(q,p) = r\}. \\ D_\Pi(p,r) &= \{q \in \Pi : d(q,p) < r\}. \\ \overline{D_\Pi}(p,r) &= \{q \in \Pi : d(q,p) \leq r\} = C_\Pi(p,r) \cup D_\Pi(p,r). \end{aligned}$$

Definition 5 *Let $F \subseteq \mathbb{R}^3$ be a 2-dimensional planar object in $\mathbb{R}^3$. Its* supporting plane *is the unique plane $\Pi \subset \mathbb{R}^3$ such that $F \subseteq \Pi$.*

Definition 6 *We also define the* Skeleton *and* r-boundary *of $\mathcal{P}$:*

$$\begin{aligned} Skel(\mathcal{P}) &= \{u \in \mathcal{P} : \ u \text{ is on some edge of } \mathcal{P}\} \\ Bd(r) &= \{u \in \mathcal{P} : \exists \text{point } v \in Skel(\mathcal{P}) \text{ such that } d(u,v) < r\}. \end{aligned}$$

Thus $Bd(r)$ is the set of points on $\mathcal{P}$ within distance r of an edge or vertex of $\mathcal{P}$.

Finally, we will need the following basic geometric lemmas in various places in the paper so we state them here at the beginning:

Lemma 4 *Let F be a convex polygon and Π its supporting plane. Then there exist $\sigma \geq 0$ (σ is a function of the angles of F) and $K \geq 0$ such that*

- $\forall r \leq K, \forall p \in F, \quad Area(F \cap D_\Pi(p,r)) \geq \sigma r^2.$
- $\forall r \geq K, \forall p \in F, \quad Area(F \cap D_\Pi(p,r)) \geq \sigma.$

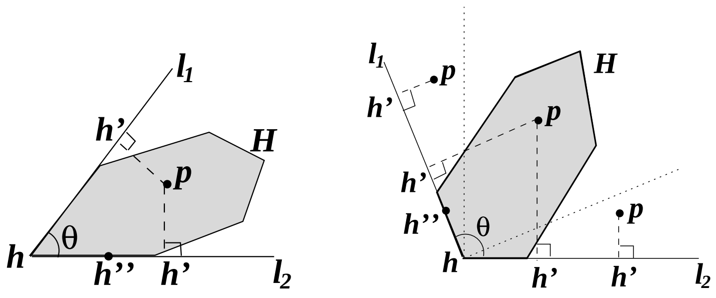
The proof of this lemma is straightforward so we omit it. The lemma permits us to introduce the following definition:

Definition 7 *Let $\mathcal{P}$ be a convex polytope, F_i, $i = 1, \ldots, k$, its faces and Π_i, $i = 1, \ldots, k$,, their respective supporting planes. Let σ_i and K_i be the σ and K associated with F_i in Lemma 4. Set*

$$c_0 = \frac{1}{\sqrt{\max_i \sigma_i}}$$

and $K_0 = \min_i K_i$. We note that this directly implies that if $p \in F_i$ for some F_i then

$$\forall r \leq K_0, \quad Area(F_i \cap D_{\Pi_i}(p, c_0 r)) \geq r^2.$$



(a) $\theta \leq 90°$ (b) $\theta > 90°$

Figure 5: A vertex h of H cannot be $NN(p, H)$.

Lemma 5 *Let H be the boundary of a (possibly unbounded) 2-dimensional convex polygon. Let p be inside H and $h = NN(p, H)$. Then h is not a vertex of H.*

Proof. Suppose that $h = NN(p, H)$ is a vertex of H. Let e_1 and e_2 be the edges of H incident upon h and θ the angle between them. Let l_1 and l_2 be the half lines defined by extending e_1 and e_2, respectively. There exists two cases: $\theta \leq 90°$ and $\theta > 90°$. In each case at least one, and possibly both, of the perpendiculars dropped from p onto the line supporting e_1, e_2 must be on l_1, l_2. Let h' be a point such that the perpendicular dropped from p onto e_1 or e_2 is on l_1 or l_2. See Figure 5. For every point $h'' \in hh'$ we have $length(ph'') < length(ph)$. Since there must exist some such point $h'' \in H$, this contradicts $h = NN(p, H)$ and we are done. $\square$

Lemma 6

1. *Let $\mathcal{P}$ be the boundary of a convex polytope. There exists c', with $0 < c' < 1$, depending only upon $\mathcal{P}$ such that the following property holds for all $p' \in \mathcal{P} \setminus Skel(\mathcal{P})$: Let F be the face of $\mathcal{P}$ such that $p' \in F$, Π its supporting plane and $r' = d(p', Skel(\mathcal{P}))$. Then*

   $$S(p', c'r') \cap \mathcal{P} \subseteq F.$$

   Equivalently, the distance from p' to any other face of $\mathcal{P}$ is greater than $c'r'$.

2. *Let $p' \in \mathcal{P} \setminus Skel(\mathcal{P})$, F, Π and r' as defined above. Let p be any point outside of $\mathcal{P}$ such that $NN(p, \mathcal{P}) = p'$. Also let $r \geq d(p, p')$. Then the following is true: If $S(p, r) \cap \mathcal{P} \not\subseteq F$ then $D_\Pi\left(p', \frac{c'r'}{2}\right) \subseteq S(p, r) \cap F$.*

Proof. The proof of (1) is straightforward from the convexity of $\mathcal{P}$. To prove (2) suppose that $S(p,r) \cap \mathcal{P} \not\subseteq F$ but, in contradiction to the lemma, $D_\Pi\left(p', \frac{c'r'}{2}\right) \not\subseteq S(p,r) \cap \Pi$.

Since $p' \notin Skel(\mathcal{P})$ and $p' = NN(p,\mathcal{P})$ we have that $p' = NN(p,\Pi)$ so $S(p,r) \cap \Pi = D_\Pi(p',\beta)$ for some β. Thus, $D_\Pi\left(p', \frac{c'r'}{2}\right) \not\subseteq S(p,r) \cap \Pi$ means that $\frac{c'r'}{2} > \beta$.

Also, since $S(p,r) \cap \mathcal{P} \not\subseteq F$, $\exists q \in S(p,r) \cap \mathcal{P}$ such that $q \notin F$. By the convexity of $\mathcal{P}$ and the fact that $p' = NN(p,\mathcal{P})$ we have that line segment pq intersects Π at some point that we will call q'.

Now $d(p,q') < d(p,q)$ so $q' \in S(p,r) \cap \Pi = D_\Pi(p',\beta)$. In particular, this means that $d(p',q') \le \beta < \frac{c'r'}{2}$.

We will now also see that $d(q,q') \le \frac{c'r'}{2}$. Suppose in contradiction that $d(q,q') > \frac{c'r'}{2}$. First note that because p, q and q' are collinear and $p' = NN(p,\Pi)$,

$$r \ge d(p,q) = d(p,q') + d(q',q) > d(p,p') + \frac{c'r'}{2}.$$

This then implies that

$$\forall p'' \in D_\Pi\left(p', \frac{c'r'}{2}\right), \quad d(p,p'') \le d(p,p') + d(p',p'') \le d(p,p') + \frac{c'r'}{2} < r,$$

so

$$D_\Pi\left(p', \frac{c'r'}{2}\right) \subseteq S(p,r) \cap \Pi = D_\Pi(p',\beta)$$

contradicting $\frac{c'r'}{2} > \beta$. This is impossible so $d(q,q') \le \frac{c'r'}{2}$.

Combining this with the previously proven $d(p',q') \le \frac{c'r'}{2}$ yields that

$$d(p',q) \le d(p',q') + d(q',q) \le 2\frac{c'r'}{2} = c'r'.$$

But now part (1) of the lemma tells us that for such q, if $q \in \mathcal{P}$ then q must be on F, contradicting our assumption that $q \notin F$. Thus our original assumption must be incorrect and

$$D_\Pi\left(p', \frac{c'r'}{2}\right) \subseteq S(p,r) \cap \Pi.$$

Since part (1) also tells us that $D_\Pi(p', c'r') \subseteq S(p,r) \cap F$ we have

$$D_\Pi\left(p', \frac{c'r'}{2}\right) \subseteq D_\Pi(p', c'r') \subseteq S(p,r) \cap F$$

and are done.

4 Bounding the Number of Type-II Voronoi Spheres

In this section we will investigate the expected number of Type-II Voronoi Spheres for points on polytope $\mathcal{P}$. Our goal will be to prove (5):

$$E(\text{Number of Type-II Voronoi Spheres}) = O(n).$$

Recall that a Voronoi Sphere $S = S(p, r)$ of point set S_n is Type-II if (i) p is inside $\mathcal{P}$ and, (ii) for all faces F_i of $\mathcal{P}$, if $S \cap F_i \neq \emptyset$ then $NN(p, \Pi_i) \in F_i$ where Π_i is the supporting plane of F_i. (See Figure 4.)

Our proof will require flipping back and forth between different related distributions. To do this we will need to introduce some new definitions that generalize our old ones:

Definition 8 *Let $F_1, F_2, \ldots, F_k \subset \mathbb{R}^3$ be a collection of 2-dimensional planar measurable surfaces and $\Pi_1, \Pi_2, \ldots, \Pi_k$ their respective supporting planes.*

1. *$S_{F_1,F_2,\ldots,F_k,n}$ is a set of points drawn from the 2-dimensional Poisson distribution with rate n on $\bigcup_{i=1}^{k} F_i$.*
2. *Let $k \geq 2$. A Voronoi sphere $S(p, r)$ for a point set $Z \subset \bigcup_{i=1}^{k} F_i$ is* Type-II *over $F_1, F_2, \ldots, F_k$ if, $\forall i$, if $S(p, r) \cap F_i \neq \emptyset$ then $NN(p, \Pi_i) \in F_i$.*
3. *$X_{F_1,F_2,\ldots,F_k,n}$ is the set of Type-II Voronoi Spheres for the point set $S_{F_1,F_2,\ldots,F_k,n}$ over $F_1, F_2, \ldots, F_k$.*
4. *$Y_{F_1,F_2,\ldots,F_k,n} = \{S(p, r) \in X_{F_1,F_2,\ldots,F_k,n} : \forall i \leq k,\ S(p, r) \cap F_i \neq \emptyset\}$*

The reason that this definition restricts itself to $k > 2$ is that, in our original definitions, if all of the defining points were located on one face then the sphere was Type-I. This means that Type-II spheres must have defining points on $k \geq 2$ faces. Note that $Y_{F_1,F_2,\ldots,F_k,n}$ refines the definition of $X_{F_1,F_2,\ldots,F_k,n}$ to only include Type-II Voronoi spheres that actually intersect *all* of the defining surfaces.

Recall that S_n was defined to be a set of points drawn from the standard 2-dimensional Poisson distribution on $\mathcal{P}$ with rate n. Thus S_n has the exact same distribution as $S_{F_1,F_2,\ldots,F_k,n}$ where $F_1, F_2, \ldots, F_k$ are now taken to be the faces of $\mathcal{P}$. By the standard properties of the Poisson distribution this implies that, for any subset $\{F_{i_1}, F_{i_2}, \ldots, F_{i_t}\}$ of $\{F_1, F_2, \ldots, F_k\}$, $S_{F_{i_1},F_{i_2},\ldots,F_{i_t},n}$ *has the same distribution* as $S_n \cap \left(\bigcup_{j=1}^{t} F_{i_j}\right)$.

If $S(p, r)$ is a Voronoi Sphere for S_n then $S(p, r)$ is defined by t points, $2 \leq t \leq 4$, $p_1, \ldots, p_t \in S_n$ on its boundary and $S(p, r) \cap (S_n \setminus \{p_1, \ldots, p_t\}) = \emptyset$. Thus, for every subset $S' \subset S_n$ such that $p_1, \ldots, p_t \in S'$, $S(p, r)$ is also a Voronoi sphere for S'. In particular, let F_{i_j} be the face of $\mathcal{P}$ such that $p_j \in F_{i_j}$. Then $S(p, r)$ is a Voronoi Sphere for $S_n \cap \left(\cup_{j=1}^{t} F_{i_j}\right)$. (Note that the converse is not necessarily true; $S(p, r)$ being a Voronoi Sphere for $S_n \cap \left(\cup_{j=1}^{t} F_{i_j}\right)$ does not necessarily imply that $S(p,r)$ is a Voronoi Sphere for S_n.) Furthermore, $S(p,r)$ has the property that, $\forall j \le t$, $S(p,r) \cap F_{i_j} \ne \emptyset$. Since the set of points $S_n \cap \left(\cup_{j=1}^{t} F_{i_j}\right)$ has the same distribution as $S_{F_{i_1},\dots,F_{i_t},n}$, the expected number of Voronoi Spheres that have this property is $E\left(Y_{F_{1_1},\dots,F_{i_t},n}\right)$.

Combining the above observations and using linearity of expectation we have just shown that if $F_1, F_2, \dots, F_k$ is the set of faces of $\mathcal{P}$ then

$$E\left(|X_{F_1,F_2,\dots,F_k,n}|\right) \le \sum_{1\le a<b\le k} E\left(|Y_{F_a,F_b,n}|\right) + \sum_{1\le a<b<c\le k} E\left(|Y_{F_a,F_b,F_c,n}|\right) + \sum_{1\le a<b<c<d\le k} E\left(|Y_{F_a,F_b,F_c,F_d,n}|\right) \quad (10)$$

The remainder of this section will be devoted to proving the following two lemmas

Lemma 7 *Let F_1 and F_2 be two convex polygons in $\mathbb{R}^3$, $\Pi_1 \ne \Pi_2$ their respective supporting planes. Furthermore, assume that $\Pi_1 \parallel \Pi_2$. Then $E\left(|Y_{F_1,F_2,n}|\right) = O(n)$.*

Lemma 8 *For t, $2 \le t \le 4$, let $F_1, F_2, \dots, F_t$ be t convex polygons, in $\mathbb{R}^3$ and $\Pi_1, \Pi_2, \dots, \Pi_t$ their respective supporting planes. Furthermore, assume that $\Pi_1 \nparallel \Pi_2$ and that $\forall i \ne j$, F_i is totally contained in one of the two closed halfspaces bounded by Π_j. Then $E\left(|Y_{F_1,\cdots,F_t,n}|\right) = O(n)$.*

Since no three faces of convex polytope $\mathcal{P}$ are parallel and all faces $F_i \ne F_j$ of $\mathcal{P}$ satisfy the condition "F_i is totally contained in one of the two closed halfspaces bounded by supporting plane Π_j" we can apply these two lemmas to equation (10) to prove

$$E\left(|X_{F_1,F_2,\dots,F_k,n}|\right) = O(n),$$

i.e., that the expected number of Type-II Voronoi Spheres in S_n is $O(n)$. Thus we will have proven (5).

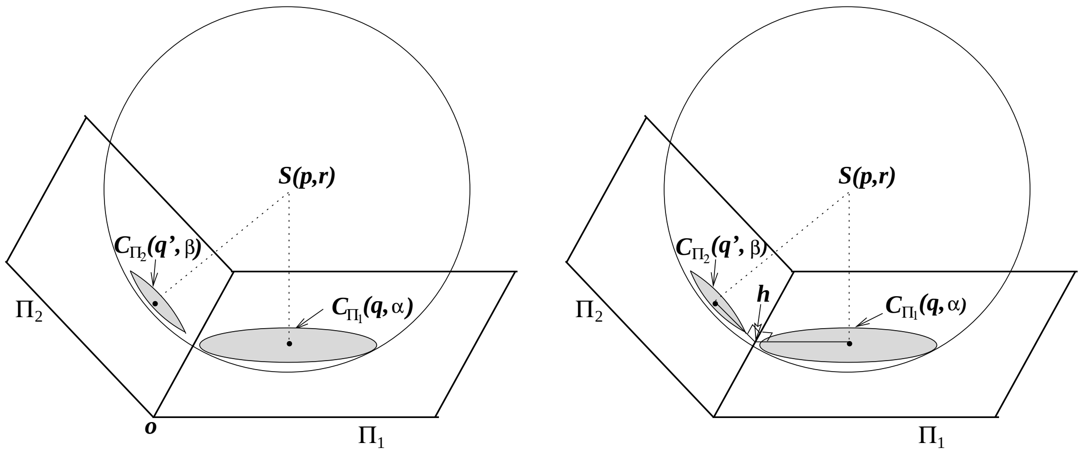
In the next subsection we introduce some properties and prove a utility lemma. In the one following we return and prove Lemmas 7 and 8.

4.1 Useful properties and a Utility Lemma

We start with some definitions:

Definition 9 *Let F be a planar surface in $\mathbb{R}^3$, Π its supporting plane and $R \subset \Pi$:*

- *Set $\phi_F : \mathbb{R}^3 \to \Pi$ to be the orthogonal projection onto Π.*
- *Set $c_R^F = \{p \in \mathbb{R}^3 \mid \phi_F(p) \in R\}$.*



(a) $\rho(o, S(p,r)) = \rho(o, C_{\Pi_1}(q,\alpha)) = \rho(o, C_{\Pi_2}(q',\beta))$

(b) For the perpendicular h dropped from q onto $\Pi_1 \cap \Pi_2$, $q'h \perp (\Pi_1 \cap \Pi_2)$.

Figure 6: Properties of the circles $C_{\Pi_1}(q,\alpha)$ and $C_{\Pi_2}(q',\beta)$, having a sphere $S = S(p,r)$ such that $S \cap \Pi_1 = C_{\Pi_1}(q,\alpha)$ and $S \cap \Pi_2 = C_{\Pi_2}(q',\beta)$.

For the special case that $F = F_k$ is one of the faces of $\mathcal{P}$ we will write ϕ_k for ϕ_{F_k} and c_R^k for $c_R^{F_k}$.

c_R^F can be thought of as the *cylinder* perpendicular to Π whose intersection with Π is R. Note too that the nearest neighbor of a point p to a plane is its orthogonal projection onto the plane so $\phi_F(p) = NN(p, \Pi)$.

Next we introduce some properties:

Property 1 *The power of a point $o(x,y,z)$ with respect to a sphere $S = S(p,r)$ is defined as the quantity $\rho(o,S) = d(o,p)^2 - r^2$. As its 2-dimensional analog, the power of a point $o(x,y)$ with respect to a circle $C = C(p,r)$ is defined by $\rho(o,C) = d(o,p)^2 - r^2$. The power of o with respect to a sphere $S(p,r)$ is equal to the power of o with respect to any circle obtained by intersecting the sphere with a plane containing o.*

Let Π_1 and Π_2 be planes such that $\Pi_1 \not\parallel \Pi_2$. Given a sphere $S = S(p,r)$ with $S \cap \Pi_1 \neq \emptyset$ and $S \cap \Pi_2 \neq \emptyset$, let q, α, q', β be such that $C_{\Pi_1}(q,\alpha) = S \cap \Pi_1$ and $C_{\Pi_2}(q',\beta) = S \cap \Pi_2$. See Figure 6(a). By Property 1, we have, $\forall o \in \Pi_1 \cap \Pi_2$,

$$\begin{aligned} \rho(o, S(p,r)) &= \rho(o, C_{\Pi_1}(q,\alpha)) = \rho(o, C_{\Pi_2}(q',\beta)) \\ &= d(o,q)^2 - \alpha^2 = d(o,q')^2 - \beta^2. \end{aligned}$$

Now let $C_{\Pi_1}(q,\alpha)$ be given. Let $S = S(p,r)$ be any sphere with $S \cap \Pi_1 = C_{\Pi_1}(q,\alpha)$ and q', β such that $S \cap \Pi_2 = C_{\Pi_2}(q',\beta)$. Let h be the perpendicular from q onto the line $\Pi_1 \cap \Pi_2$. Since $pq \perp \Pi_1$ and $qh \perp (\Pi_1 \cap \Pi_2)$, $ph \perp (\Pi_1 \cap \Pi_2)$. Also $pq' \perp \Pi_2$, so $ph \perp (\Pi_1 \cap \Pi_2)$ yielding $q'h \perp (\Pi_1 \cap \Pi_2)$, which means that q' must lie on the line of Π_2, passing through h and perpendicular to $\Pi_1 \cap \Pi_2$. See Figure 6(b). Moreover, Property 1 yields the following condition for the radius, denoted β, of the circle $S \cap \Pi_2$:

$$\beta^2 = d(h, q')^2 - d(h, q)^2 + \alpha^2. \tag{11}$$

As a consequence, we have

Property 2 *Given a circle $C_{\Pi_1}(q, \alpha)$ and a sphere S with $S \cap \Pi_1 = C_{\Pi_1}(q, \alpha)$, let h be the perpendicular from q onto the line $\Pi_1 \cap \Pi_2$. If $S \cap \Pi_2 \neq \emptyset$, then the center q' of the circle $S \cap \Pi_2$ must be on the line of Π_2 which passes through h and is perpendicular to $\Pi_1 \cap \Pi_2$, and the radius β of the circle $S \cap \Pi_2$ must satisfy $\beta^2 = d(h, q')^2 - d(h, q)^2 + \alpha^2$.*

Finally, we prove a utility lemma that will be useful in the proof of Lemma 7 and Lemma 8.

Definition 10 $\forall m \geq 0$, *set* $I_m = \left\{ r \in \mathbb{R} \mid \frac{m}{\sqrt{n}} \leq r \leq \frac{m+1}{\sqrt{n}} \right\}$.

Lemma 9 *Let F be a convex polygon in $\mathbb{R}^3$ and Π its supporting plane. Let $S_{F,n}$ be a set of points drawn from a Poisson distribution of rate n over F. Let B be a square with center o and sides of length $\frac{1}{\sqrt{n}}$. Let*

$$U_{B,m} = \left\{ u \in \Pi \mid \begin{array}{l} u \text{ is on some circle } C_\Pi(q, \alpha) \text{ with } q \in B, \alpha \in I_m \\ \text{and } D_\Pi(q, \alpha) \cap S_{F,n} = \emptyset. \end{array} \right\}.$$

Then (all constants implicit in the $O()$ notation depend only upon F)

(a) $\forall m \geq 0$, $E\left(|U_{B,m} \cap S_{F,n}|^3\right) = O(m^6)$.

(b) $\forall m : 2 \leq m \leq \sqrt{\frac{n}{\log n}}$, $\quad E\left(|U_{B,m} \cap S_{F,n}|^3\right) = O\left(m^3 e^{-\sigma m^2}\right)$.

(c) *If* $m > \sqrt{n}(\mathit{Diameter}(F) + 1)$ *then* $|U_{B,m} \cap S_{F,n}| = 0$.

(d) $\sum_{m=0}^{\infty} E\left(|U_{B,m} \cap S_{F,n}|^3\right) = O(1)$.

Proof. Since this is the first time that we do such a calculation we explicitly recall two facts about Poisson processes that we are using. The first is that if $A \subseteq \Pi$ is some region, then $|A \cap S_{F,n}|$, the number of points of $S_{F,n}$ in A, satisfies a Poisson distribution with rate $n\text{Area}(A \cap F)$. The second fact is that there exists some universal constant c, such that if Z is *any* discrete Poisson distribution then $E(|Z^3|) \leq c\,(E(|Z|))^3$ ($c = d_3$ from Lemma 3).

To compute $E\left(|U_{B,m} \cap S_{F_n}|^3\right)$ exactly would be quite complicated because by definition, $S_{F,n}$ and $U_{B,m}$ are not independent of each other so $|S_{F,n} \cap U_{B,m}|$ is not Poisson distributed with rate $nArea(U_{B,m} \cap F)$. Instead, we will upperbound $E\left(|U_{B,m} \cap S_{F,n}|^3\right)$ with something whose expectation is easier to calculate.

To do this first notice that, $\forall u \in U_{B,m}$,

$$\begin{aligned} d(o,u) &\leq d(o,q) + d(q,u) \text{ for } q \in B, \\ &\leq \tfrac{1}{\sqrt{n}} + \tfrac{m+1}{\sqrt{n}} = \tfrac{m+2}{\sqrt{n}}. \end{aligned} \tag{12}$$

Thus, we immediately have that $U_{B,m} \subseteq D_\Pi\left(o, \frac{m+2}{\sqrt{n}}\right)$ so

$$|S_{F,n} \cap U_{B,m}| \leq \left| S_{F,n} \cap D_\Pi\left(o, \frac{m+2}{\sqrt{n}}\right) \right|.$$

Since $D_\Pi\left(o, \frac{m+2}{\sqrt{n}}\right)$ *is* independent of $S_{F,n}$ we have that $\left| S_{F,n} \cap D_\Pi\left(o, \frac{m+2}{\sqrt{n}}\right) \right|$ *is* a Poisson Distributed random variable with rate

$$n\text{Area}\left(D_\Pi\left(o, \frac{m+2}{\sqrt{n}}\right) \cap F \right) \leq n\pi \frac{(m+2)^2}{n} = O(m^2).$$

Part **(a)** then follows directly from the standard properties of the Poisson distribution that were reviewed at the beginning of the proof.

If $m \geq 2$, then we can also lower bound $d(o,u)$ for all $u \in U_{B,m}$:

$$\begin{aligned} d(o,u) &\geq |d(o,q) - d(q,u)| = d(q,u) - d(o,q) \\ &\geq \tfrac{m}{\sqrt{n}} - \tfrac{1}{\sqrt{n}} = \tfrac{m-1}{\sqrt{n}}. \end{aligned} \tag{13}$$

Thus $\forall m \geq 2$,

$$U_{B,m} \subset D_\Pi\left(o, \frac{m+2}{\sqrt{n}}\right) \setminus D_\Pi\left(o, \frac{m-1}{\sqrt{n}}\right), \tag{14}$$

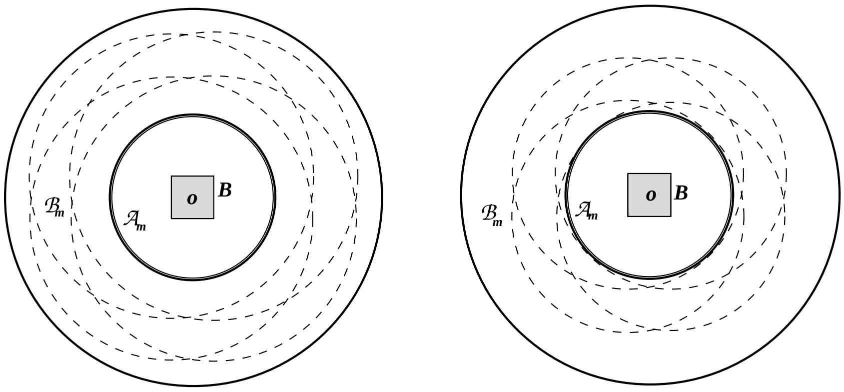
i.e. $U_{B,m}$ is contained in an annulus with center o and width $\frac{3}{\sqrt{n}}$. See Figure 7(a). Similarly it can be shown that

$$\forall q \in B, \forall \alpha \in I_m, \quad D_\Pi\left(o, \frac{m-1}{\sqrt{n}}\right) \subseteq D_\Pi(q,\alpha). \tag{15}$$

See Figure 7(b).

We can now prove part **(b)** of the lemma. Let $S_{F,n}$ be given, and for notational simplicity, set

$$\mathcal{A}_m = D_\Pi\left(o, \frac{m-1}{\sqrt{n}}\right) \text{ and } \mathcal{B}_m = D_\Pi\left(o, \frac{m+2}{\sqrt{n}}\right) \setminus D_\Pi\left(o, \frac{m-1}{\sqrt{n}}\right).$$



(a) Dotted circles are $C_\Pi\left(q, \frac{m+1}{\sqrt{n}}\right)$ for q the four corner points of C_m. Note that all circles are contained in $\mathcal{B}_m$.

(b) Dotted circles are $C_\Pi\left(q, \frac{m}{\sqrt{n}}\right)$ for q the four corner points of C_m. Note that all circles contain $\mathcal{A}_m$.

Figure 7: Let $\mathcal{A}_m = D_\Pi\left(o, \frac{m-1}{\sqrt{n}}\right)$ and $\mathcal{B}_m = D_\Pi\left(o, \frac{m+2}{\sqrt{n}}\right) \setminus D_\Pi\left(o, \frac{m-1}{\sqrt{n}}\right)$. If $m \geq 2$, then $\forall q \in B, \forall \alpha \in I_m, \mathcal{A}_m \subseteq D_\Pi(q, \alpha)$ and $U_{B,m} \subset \mathcal{B}_m$.

If $S_{F,n} \cap \mathcal{A}_m \neq \emptyset$, then by (15),

$$\forall q \in B,\ \forall \alpha \in I_m, \quad S_{F,n} \cap D_\Pi(q, \alpha) \neq \emptyset.$$

Thus by definition, $U_{B,m} = \emptyset$, so $|S_{F,n} \cap U_{B,m}| = 0$. If $S_{F,n} \cap \mathcal{A}_m = \emptyset$, then $U_{B,m}$ might not be empty. However using (14), we know that $|S_{F,n} \cap U_{B,m}| \leq |S_{F,n} \cap \mathcal{B}_m|$. Therefore for $\forall m \geq 2$

$$|S_{F,n} \cap U_{B,m}| \leq \begin{cases} 0, & \text{if } S_{F,n} \cap \mathcal{A}_m = \emptyset \\ |S_{F,n} \cap \mathcal{B}_m|, & \text{if } S_{F,n} \cap \mathcal{A}_m \neq \emptyset \end{cases} \tag{16}$$

which yields

$$\begin{aligned} E\left(|S_{F,n} \cap U_{B,m}|^3\right) &\leq E\left(|S_{F,n} \cap \mathcal{B}_m|^3 \;\; | S_{F,n} \cap \mathcal{A}_m = \emptyset\right) \times Pr\left(S_{F,n} \cap \mathcal{A}_m = \emptyset\right) \\ &= E\left(|S_{F,n} \cap \mathcal{B}_m|^3\right) \times Pr\left(S_{F,n} \cap \mathcal{A}_m = \emptyset\right) && (17) \\ &\leq c\left(E\left(|S_{F,n} \cap \mathcal{B}_m|\right)\right)^3 \times Pr\left(S_{F,n} \cap \mathcal{A}_m = \emptyset\right) && (18) \\ &\leq c\left(n\text{Area}(\mathcal{B}_m \cap F)\right)^3 \times e^{-n \times \text{Area}(\mathcal{A}_m \cap F)} && (19) \\ &= O\left(m^3 e^{-\sigma m^2}\right) && (20) \end{aligned}$$

where c is a universal constant and σ is a constant depending upon F.

Equality (17) follows from the fact that $\mathcal{A}_m$ and $\mathcal{B}_m$ are disjoint so $S_{F,n} \cap \mathcal{A}_m$ and $S_{F,n} \cap \mathcal{B}_m$ are independent. (18) comes from the previously noted fact that there is a universal constant c such that if Z is any Poisson random variable then $E(Z^3) \leq c(E(Z))^3$. (19) from the definition of Poisson random variables. (20) from the fact that $Area(\mathcal{B}_m \cap F) = O\left(\frac{m}{n}\right)$ and Lemma 4; the reason for restricting $m \leq \sqrt{\frac{n}{\log n}}$ is to guarantee that $\frac{m-1}{\sqrt{n}} < K$ so that for large enough n Lemma 4 can be applied to lower bound Area$(\mathcal{A}_m \cap F)$. We have thus proven part **(b)**.

To prove part **(c)** let $D = \text{diameter}(F)$. For $m > \sqrt{n}(D+1)$ we then have $F \subseteq \mathcal{A}_m$ so $S_{F,n} \cap \mathcal{A}_m = S_{F,n}$. Equation (16) and the definition of $\mathcal{B}_m$ then give $|U_{B,m} \cap S_{F_n}| = 0$.

Combining **(a)** and **(b)** proves that $\sum_{m=0}^{\sqrt{\frac{n}{\log n}}} E\left(|U_{B,m} \cap S_{F_n}|^3\right) = O(1)$. Part **(c)** gives $\sum_{m>\sqrt{n}(D+1)} E\left(|U_{B,m} \cap S_{F_n}|^3\right) = \sum_{m>\sqrt{n}(D+1)} 0 = 0$.

Thus, to prove part **(d)** it only remains to show that

$$\sum_{m=\sqrt{\frac{n}{\log n}}}^{\sqrt{n}(D+1)} E\left(|U_{B,m} \cap S_{F_n}|^3\right) = O(1).$$

Returning to (19) we see that $\forall m > \sqrt{\frac{n}{\log n}}$

$$\begin{aligned} E\left(|S_{F,n} \cap U_{B,m}|^3\right) &\leq c\left(n\text{Area}(\mathcal{B}_m \cap F)\right)^3 \times e^{-n\times\text{Area}(\mathcal{A}_m \cap F)} \\ &\leq O(n^3) \times e^{-n\times\text{Area}\left(\mathcal{A}_{\sqrt{\frac{n}{\log n}}} \cap F\right)} \end{aligned}$$

since $\forall m$, Area$(\mathcal{B}_m \cap F) \leq$ Area(F) and

$$\forall m > \sqrt{\frac{n}{\log n}}, \quad \text{Area}(\mathcal{A}_m \cap F) > \text{Area}\left(\mathcal{A}_{\sqrt{\frac{n}{\log n}}} \cap F\right).$$

Lemma 4 tells us that Area $\left(\mathcal{A}_{\sqrt{\frac{n}{\log n}}} \cap F\right) \geq \frac{\sigma n}{\log n}$ Thus

$$\sum_{m=\sqrt{\frac{n}{\log n}}}^{\sqrt{n}(D+1)} E\left(|U_{B,m} \cap S_{F_n}|^3\right) \leq \sqrt{n}(D+1)O(n)O\left(e^{-\frac{\sigma n}{\log n}}\right) = O(1)$$

and we are done. □

4.2 Proofs of Lemmas 7 and 8

We now have the tools to prove Lemma 7.

Proof. [Proof of Lemma 7]

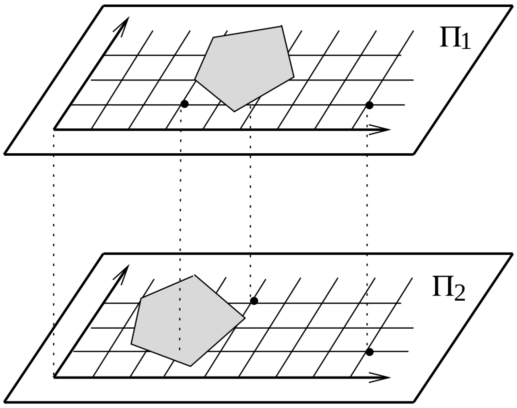


Figure 8: Coordinate system of Π_1 and Π_2 when $\Pi_1 \parallel \Pi_2$.

In this lemma we assume that we are using a coordinate system of Π_1 and its associated orthogonal projection onto Π_2 such that that every point of F_1 and F_2 have positive x- and y-coordinates in this system. See Figure 8.

Now let $k = 1, 2$. We partition plane Π_k into axis parallel squares

$$B^k_{s,i} = \left\{ (x, y) \in \Pi_k \mid \frac{s}{\sqrt{n}} \le x \le \frac{s+1}{\sqrt{n}}, \frac{i}{\sqrt{n}} \le y \le \frac{i+1}{\sqrt{n}} \right\}.$$

Let L be such that for $k = 1, 2$, $F_k \subset \{(x, y) \in \Pi_k \mid 0 \le x \le L, 0 \le y \le L\}$. L is a constant that is a function of F_1 and F_2. By the definition of L,

$$\text{For } k = 1, 2, \quad F_k \subseteq \bigcup_{s,i=0}^{\lfloor L\sqrt{n} \rfloor} B^k_{s,i}.$$

Let $Y_{B^1_{s,i}, B^2_{t,j}}$ be the set of Voronoi spheres $S(p, r) \in Y_{F_1,F_2,n}$ such that

$$NN(p, \Pi_1) \in B^1_{s,i} \text{ and } NN(p, \Pi_2) \in B^2_{t,j}.$$

Recall that by the definition of Type-II Voronoi spheres, for any sphere $S = S(p, r) \in Y_{F_1,F_2,n}$, $NN(p, \Pi_1) \in F_1$ and $NN(p, \Pi_2) \in F_2$. This means that

$$Y_{F_1,F_2,n} = \bigcup_{s,i=0}^{\lfloor L\sqrt{n} \rfloor} \bigcup_{t,j=0}^{\lfloor L\sqrt{n} \rfloor} Y_{B^1_{s,i}, B^2_{t,j}}.$$

However from the construction of $B^1_{s,i}$ and $B^2_{t,j}$, and the fact that $\Pi_1 \parallel \Pi_2$ we have that if $NN(p, \Pi_1) \in B^1_{s,i}$ then $NN(p, \Pi_2) \in B^2_{s,i}$ and vice-versa. Thus

$$Y_{F_1,F_2,n} = \bigcup_{s,i=0}^{\lfloor L\sqrt{n} \rfloor} Y_{B^1_{s,i}, B^2_{s,i}}$$

and

$$E\left(|Y_{F_1,F_2,n}|\right) \leq \sum_{s=0}^{\lfloor L\sqrt{n}\rfloor} \sum_{i=0}^{\lfloor L\sqrt{n}\rfloor} E\left(|Y_{B^1_{s,i},B^2_{s,i}}|\right). \tag{21}$$

For $s,i \in \{0,1,...,\lfloor L\sqrt{n}\rfloor\}$, $m \geq 0$, let

$$U^k_{s,i,m} = \left\{ u \in \Pi_k \mid \begin{array}{l} u \text{ is on some circle } C_{\Pi_k}(q,\alpha) \text{ with } q \in B^k_{s,i}, \alpha \in I_m \\ \text{and } D_{\Pi_k}(q,\alpha) \cap S_{F_1,F_2,n} = \emptyset. \end{array} \right\}.$$

Let $S = S(p,r)$ be a Voronoi sphere in $Y_{B^1_{s,i},B^2_{s,i}}$. Then by the definition of $Y_{B^1_{s,i},B^2_{s,i}}$, $S \cap \Pi_1 = \overline{D_{\Pi_1}}(q,\alpha)$ for $q = NN(p,\Pi_1) \in B^1_{s,i}$ and $\alpha \geq 0$; furthermore $D_{\Pi_1}(q,\alpha) \cap S_{F_1,F_2,n} = \emptyset$ and some defining point of S is on $C_{\Pi_1}(q,\alpha)$. Letting m be such that $\alpha \in I_m$ we see that $S_{F_1,F_2,n} \cap S \cap \Pi_1 = S_{F_1,F_2,n} \cap C_{\Pi_1}(q,\alpha) \subseteq U^1_{s,i,m}$. Similarly there exists m' such that $S \cap \Pi_2 \subseteq U^2_{s,i,m'}$. This means that $\forall w \in S_{F_1,F_2,n} \cap S$,

$$\begin{cases} \text{if } w \in \Pi_1, \text{ then } w \in S_{F_1,F_2,n} \cap U^1_{s,i,m} \text{ and} \\ \text{if } w \in \Pi_2, \text{ then } w \in S_{F_1,F_2,n} \cap U^2_{s,i,m'}. \end{cases}$$

Recall that a defining Voronoi sphere for $S_{F_1,F_2,n}$ is determined by four defining points in general position. By the discussion above these four defining points are in $U^1_{s,i,m} \cup U^2_{s,i,m'}$ for some m, m'. Furthermore, there is probability zero of all four of these points being on one plane (since the Poisson process has probability zero of generating 4 cocircular points on a plane). Hence the number of these spheres, i.e. $|Y_{B^1_{s,i},B^2_{s,i}}|$, is bounded by the number of all possible 6-tuples of points of $S_{F_1,F_2,n}$ such that for arbitrary m, m', 3 points are from $S_{F_1,F_2,n} \cap U^1_{s,i,m}$ and 3 points are from $S_{F_1,F_2,n} \cap U^2_{s,i,m'}$. This leads automatically to

$$|Y_{B^1_{s,i},B^2_{s,i}}| \leq \sum_{m,m'=0}^{\infty} \left(\begin{array}{l} \text{No. of 6 tuples } (w^1_1, w^1_2, w^1_3, w^2_1, w^2_2, w^2_3) \\ \text{such that } \left(\begin{array}{l} w^1_1, w^1_2, w^1_3, \in S_{F_1,F_2,n} \cap U^1_{s,i,m} \\ w^2_1, w^2_2, w^2_3, \in S_{F_1,F_2,n} \cap U^2_{s,i,m'} \end{array} \right) \end{array} \right). \tag{22}$$

For *fixed* m, m', the number of such 6 tuples is equal to

$$\left|S_{F_1,F_2,n} \cap U^1_{s,i,m}\right|^3 \times \left|S_{F_1,F_2,n} \cap U^2_{s,i,m}\right|^3.$$

Also, since $S_{F_1,F_2,n}$ is distributed by a Poisson process, the distribution of $|S_{F_1,F_2,n} \cap F_1|$ is independent of that of $|S_{F_1,F_2,n} \cap F_2|$. Hence taking expectations over (22) yields

$$\begin{aligned} E\left(|Y_{B^1_{s,i},B^2_{s,i}}|\right) &\leq \textstyle\sum_{m,m'=0}^{\infty} E\left(\left|S_{F_1,F_2,n} \cap U^1_{s,i,m}\right|^3 \times \left|S_{F_1,F_2,n} \cap U^2_{s,i,m'}\right|^3\right) \\ &\leq \textstyle\sum_{m,m'=0}^{\infty} E\left(\left|S_{F_1,F_2,n} \cap U^1_{s,i,m}\right|^3\right) \times E\left(\left|S_{F_1,F_2,n} \cap U^2_{s,i,m'}\right|^3\right) \\ &\leq \left[\textstyle\sum_{m=0}^{\infty} E\left(\left|S_{F_1,F_2,n} \cap U^1_{s,i,m}\right|^3\right)\right] \times \left[\textstyle\sum_{m'=0}^{\infty} E\left(\left|S_{F_1,F_2,n} \cap U^2_{s,i,m'}\right|^3\right)\right]. \end{aligned} \tag{23}$$

By Lemma 9, we have that both $\sum_{m'=0}^{\infty} E\left(\left|S_{F_1,F_2,n} \cap U^1_{s,i,m}\right|^3\right)$ and $\sum_{m'=0}^{\infty} E\left(\left|S_{F_1,F_2,n} \cap U^2_{s,i,m'}\right|^3\right)$ are $O(1)$, automatically yielding

$$E\left(\left|Y_{B^1_{s,i},B^2_{s,j}}\right|\right) = O(1). \tag{24}$$

Plugging this into (21) we get

$$E\left(\left|Y_{F_1,F_2,n}\right|\right) = O(n), \tag{25}$$

and are done. □

We now prove Lemma 8.

Proof. [Proof of Lemma 8]

1. We first bound $E\left(\left|Y_{F_1,F_2,n}\right|\right)$.

We assume that we are given coordinate systems for Π_k, $k = 1, 2$, that satisfy the following conditions (see Figure 9(a)).:

(i) the x-axis of Π_k is $\Pi_1 \cap \Pi_2$. It is oriented so that F_1 lies on the right-hand side of the positive direction of $\Pi_1 \cap \Pi_2$. The origin is chosen so that every point of $F_1 \cup F_2$ has positive x-coordinate. (ii) Among the two half planes of Π_k separated by $\Pi_1 \cap \Pi_2$, the one containing F_k corresponds to positive y-coordinate. Such a pair of coordinate systems can be chosen because $\Pi_1 \not\parallel \Pi_2$ and the facts that F_1 is totally contained in one of the closed halfspaces bounded by Π_2 and F_2 is totally contained in one of the closed halfspaces bounded by Π_1.

We partition Π_k into axis parallel squares

$$B^k_{s,i} = \left\{(x, y) \in \Pi_k \mid \frac{s}{\sqrt{n}} \le x \le \frac{s+1}{\sqrt{n}}, \frac{i}{\sqrt{n}} \le y \le \frac{i+1}{\sqrt{n}}\right\}.$$

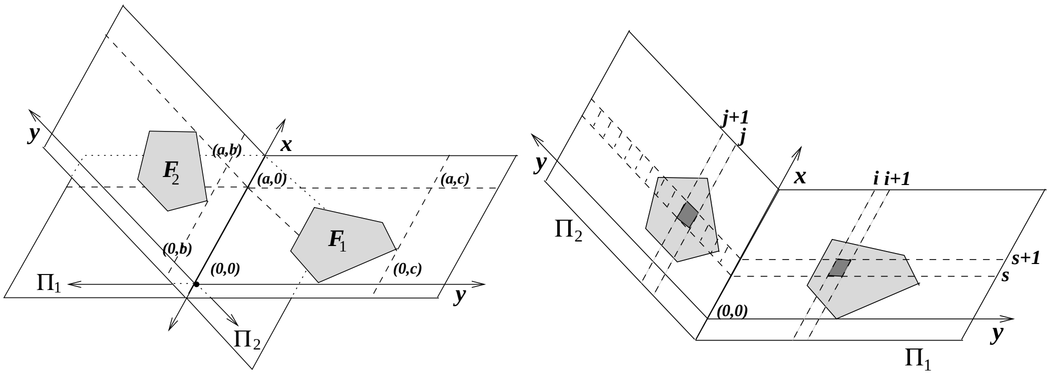
See Figure 9(b). Set $o^k_{s,i}$ to be the center of $B^k_{s,i}$: $o^k_{s,i} = \left\{\left(\frac{2s+1}{2\sqrt{n}}, \frac{2i+1}{2\sqrt{n}}\right) \in \Pi_k\right\}$. Let L be a constant large enough so that $L \ge \max(\text{diameter}(F_1), \text{diameter}(F_2))$ and for $k = 1, 2$, $F_k \subset \{(x, y) \in \Pi_k \mid 0 \le x \le L, 0 \le y \le L\}$. In particular, this last implies that

$$\text{For } k = 1, 2, \quad F_k \subseteq \bigcup_{s,i=0}^{\lfloor L\sqrt{n} \rfloor} B^k_{s,i}.$$

Note that by the definition of Type-II spheres, for any $S = S(p, r) \in Y_{F_1,F_2,n}$, $NN(p, \Pi_1) \in F_1$ and $NN(p, \Pi_2) \in F_2$. As in the proof of Lemma 7 we analyze $Y_{F_1,F_2,n}$ by partitioning it into into smaller sets indexed by the squares in which the $NN(p, \Pi_k)$ are located.

Let $Y_{B^1_{s,i},B^2_{t,j}}$ be the set of Voronoi spheres $S(p, r) \in Y_{F_1,F_2,n}$ such that

$$NN(p, \Pi_1) \in B^1_{s,i} \text{ and } NN(p, \Pi_2) \in B^2_{t,j}.$$



(a) Coordinate system of Π_1 and Π_2 (b) The boxes $B^1_{s,i_1} \subset \Pi_1$ and $B^2_{s,i_2} \subset \Pi_2$

Figure 9: The case that $\Pi_1 \not\parallel \Pi_2$.

In the proof of Lemma 7, where $\Pi_1 \parallel \Pi_2$, we strongly used the fact that $Y_{B^1_{s,i}, B^2_{t,j}} = \emptyset$ unless $s = t$ and $i = j$. Now that $\Pi_1 \not\parallel \Pi_2$ this is no longer true. Instead we use something weaker. Let $p_1 = NN(p, \Pi_1)$ and $p_2 = NN(p, \Pi_2)$. Express $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$ using the respective coordinate systems for Π_1 and Π_2 that were described above. In this notation, Property 2 states that $x = x'$. Thus, by the definition of the $B^k_{s,i}$, we have that $\forall i, j$, if $s \neq t$ then $Y_{B^1_{s,i}, B^2_{t,j}} = \emptyset$. This means that

$$E\left(|Y_{F_1,F_2,n}|\right) \leq \sum_{s=0}^{\lfloor L\sqrt{n}\rfloor} \sum_{i=0}^{\lfloor L\sqrt{n}\rfloor} \sum_{j=0}^{\lfloor L\sqrt{n}\rfloor} E\left(|Y_{B^1_{s,i}, B^2_{s,j}}|\right). \tag{26}$$

Hence we can compute $E\left(|Y_{F_1,F_2,n}|\right)$ if we can calculate $E\left(|Y_{B^1_{s,i}, B^2_{s,j}}|\right)$ for arbitrary s, i, and j. We start by showing that $\forall s, i, j, E\left(|Y_{B^1_{s,i}, B^2_{s,j}}|\right) = O(1)$ (this will be needed later) and then proceed to a more delicate analysis based on taking the relationship between the s, i and j values into account.

For $s, i \in \{0, 1, ..., \lfloor L\sqrt{n}\rfloor\}$, let

$$U^k_{s,i,m} = \left\{ u \in \Pi_k \mid \begin{array}{l} u \text{ is on some circle } C_{\Pi_k}(q, \alpha) \text{ with } q \in B^k_{s,i}, \alpha \in I_m \\ \text{and } D_{\Pi_k}(q, \alpha) \cap S_{F_1,F_2,n} = \emptyset. \end{array} \right\}.$$

Let S be a Voronoi sphere in $Y_{B^1_{s,i}, B^2_{s,j}}$. As in the proof of Lemma 7 we have that there exist m, m' such that the four defining points of S must belong to $U^1_{s,i,m} \cup U^2_{s,j,m'}$. Again as in Lemma 7 there is probability zero of all four of these points being on one plane so $|Y_{B^1_{s,i}, B^2_{s,j}}|$ is bounded by the number of all possible 6-tuples of points of $S_{F_1,F_2,n}$ such that for arbitrary m, m', 3 points are from $S_{F_1,F_2,n} \cap U^1_{s,i,m}$ and 3 points are from $S_{F_1,F_2,n} \cap U^2_{s,j,m'}$.

Thus

$$|Y_{B^1_{s,i},B^2_{s,j}}| \le \sum_{m,m'=0}^{\infty} \left(\begin{array}{l} \text{No. of 6 tuples } (w^1_1, w^1_2, w^1_3, w^2_1, w^2_2, w^2_3) \\ \text{such that } \left(\begin{array}{c} w^1_1, w^1_2, w^1_3 \in S_{F_1,F_2,n} \cap U^1_{s,i,m} \\ w^2_1, w^2_2, w^2_3 \in S_{F_1,F_2,n} \cap U^2_{s,j,m'} \end{array} \right) \end{array} \right). \tag{27}$$

For *fixed* m, m', the number of such 6 tuples is equal to

$$\left|S_{F_1,F_2,n} \cap U^1_{s,i,m}\right|^3 \times \left|S_{F_1,F_2,n} \cap U^2_{s,j,m}\right|^3.$$

Following equation (23) we have

$$\begin{aligned} E\left(|Y_{B^1_{s,i},B^2_{s,j}}|\right) &\le \textstyle\sum_{m,m'=0}^{\infty} E\left(\left|S_{F_1,F_2,n} \cap U^1_{s,i,m}\right|^3 \times \left|S_{F_1,F_2,n} \cap U^2_{s,j,m'}\right|^3\right) \\ &\le \textstyle\sum_{m,m'=0}^{\infty} E\left(\left|S_{F_1,F_2,n} \cap U^1_{s,i,m}\right|^3\right) \times E\left(\left|S_{F_1,F_2,n} \cap U^2_{s,j,m'}\right|^3\right) \\ &\le \left[\textstyle\sum_{m=0}^{\infty} E\left(\left|S_{F_1,F_2,n} \cap U^1_{s,i,m}\right|^3\right)\right] \times \left[\textstyle\sum_{m'=0}^{\infty} E\left(\left|S_{F_1,F_2,n} \cap U^2_{s,j,m'}\right|^3\right)\right]. \end{aligned} \tag{28}$$

Applying Lemma 9 to both factors of the right hand side then proves that

$$\forall s, i, j, \quad E\left(|Y_{B^1_{s,i},B^2_{s,j}}|\right) = O(1). \tag{29}$$

Plugging this into (26) yields that $E\left(|Y_{F_1,F_2,n}|\right) = O(n^{\frac{3}{2}})$. To prove that $E\left(|Y_{F_1,F_2,n}|\right) = O(n)$, requires a more delicate analysis. In (28) we fixed s, i, j and then summed over *all* possible values of m, m'. We now take advantage of the fact that Property 2 will enable us to restrict the range of m' that we have to sum over for fixed s, i, j, m.

Let

$$V^2_{s,i,j,m} = \left\{ u \in \Pi_2 \mid \begin{array}{l} u \text{ is on some circle } C_{\Pi_2}(q', \beta) \text{ s.t. } D_{\Pi_2}(q', \beta) \cap S_{F_1,F_2,n} = \emptyset \\ \quad \text{where } q', \beta \text{ satisfy that} \\ \quad \text{(i) } q' \in B^2_{s,j} \\ \quad \text{(ii) there exists Voronoi sphere } S = S(p, r) \text{ for } S_{F_1,F_2,n} \\ \qquad \text{with } S \cap \Pi_1 = \overline{D_{\Pi_1}}(q, \alpha) \text{ and } S \cap \Pi_2 = \overline{D_{\Pi_2}}(q', \beta) \\ \qquad \text{for some } q \in B^1_{s,i}, \alpha \in I_m \end{array} \right\}$$

Then, using the same type of argument as before,

$$|Y_{B^1_{s,i},B^2_{s,j}}| \le \sum_{m=0}^{\infty} \left(\begin{array}{l} \text{No. of 6 tuples } (w^1_1, w^1_2, w^1_3, w^2_1, w^2_2, w^2_3) \\ \text{such that } \left(\begin{array}{c} w^1_1, w^1_2, w^1_3 \in S_{F_1,F_2,n} \cap U^1_{s,i,m} \\ w^2_1, w^2_2, w^2_3 \in S_{F_1,F_2,n} \cap V^2_{s,i,j,m} \end{array} \right) \end{array} \right). \tag{30}$$

Note that we now sum over only one index m rather than two indices m, m'. This is at the expense of pushing the restricted range of m's into the definition of $V^2_{s,i,j,m}$.

$V^2_{s,i,j,m}$ only depends on the values s, i, j, m and the point set $S_{F_1,F_2,n} \cap \Pi_2$. It is therefore *independent* of $S_{F_1,F_2,n} \cap U^1_{s,i,m}$. Thus

$$\begin{aligned} E\left(|Y_{B^1_{s,i},B^2_{s,j}}|\right) &\le \sum_{m=0}^{\infty} E\left(\left|S_{F_1,F_2,n} \cap U^1_{s,i,m}\right|^3 \times \left|S_{F_1,F_2,n} \cap V^2_{s,i,j,m}\right|^3\right) \\ &\le \sum_{m=0}^{\infty} E\left(\left|S_{F_1,F_2,n} \cap U^1_{s,i,m}\right|^3\right) \times E\left(\left|S_{F_1,F_2,n} \cap V^2_{s,i,j,m}\right|^3\right). \end{aligned} \tag{31}$$

By definition $S_{F_1,F_2,n} \cap \Pi_1$ has the same distribution as $S_{F_1,n}$. Thus we can apply Lemma 9. Since $L >$ Diameter(F_1), part (c) of Lemma 9 says that for $m > \sqrt{n}(L+1)$, $E\left(\left|S_{F_1,F_2,n} \cap U^1_{s,i,m}\right|^3\right) = 0$. Part (a) of the lemma says that $\forall m$, $E\left(\left|S_{F_1,F_2,n} \cap U^1_{s,i,m}\right|^3\right) = O(m^6)$. Thus

$$E\left(|Y_{B^1_{s,i},B^2_{s,j}}|\right) = O\left(\sum_{m=0}^{\sqrt{n}(L+1)} m^6 E\left(\left|S_{F_1,F_2,n} \cap V^2_{s,i,j,m}\right|^3\right)\right). \tag{32}$$

It remains to calculate $E\left(\left|S_{F_1,F_2,n} \cap V^2_{s,i,j,m}\right|^3\right)$.

Let l be the line of intersection $l = \Pi_1 \cap \Pi_2$ and note that, by Property 2,

$$\beta^2 = d(q', l)^2 + \alpha^2 - d(q, l)^2.$$

Since $\frac{j}{\sqrt{n}} \le d(q', l) \le \frac{j+1}{\sqrt{n}}$, $\frac{i}{\sqrt{n}} \le d(q, l) \le \frac{i+1}{\sqrt{n}}$ and $\frac{m}{\sqrt{n}} \le \alpha \le \frac{m+1}{\sqrt{n}}$, it follows that

$$\sqrt{\frac{j^2 + m^2 - (i+1)^2}{n}} \le \beta \le \sqrt{\frac{(j+1)^2 + (m+1)^2 - i^2}{n}}. \tag{33}$$

For $\forall u \in V^2_{s,i,j,m}$,

$$\begin{aligned} d(o^2_{s,j}, u) &\le d(o^2_{s,j}, q') + d(q', u) \text{ for } q' \in B^2_{s,j}, \\ &\le \tfrac{1}{\sqrt{n}} + \sqrt{\tfrac{(j+1)^2+(m+1)^2-i^2}{n}} = \tfrac{\sqrt{(j+1)^2+(m+1)^2-i^2}+1}{\sqrt{n}} \text{ by (33)} \end{aligned}$$

and if $j \ge i+3$, then

$$\begin{aligned} d(o^2_{s,j}, u) &\ge |d(o^2_{s,j}, q') - d(q', u)| = d(q', u) - d(o^2_{s,j}, q') \text{ by } j \ge i+3, \\ &\ge \sqrt{\tfrac{j^2+m^2-(i+1)^2}{n}} - \tfrac{1}{\sqrt{n}} = \tfrac{\sqrt{j^2+m^2-(i+1)^2}-1}{\sqrt{n}} \text{ by (33).} \end{aligned}$$

Hence $\forall j \ge i+3$,

$$V^2_{s,i,j,m} \subset \overline{D_{\Pi_2}}\left(o^2_{s,j}, \frac{\sqrt{(j+1)^2 + (m+1)^2 - i^2} + 1}{\sqrt{n}}\right) \setminus D_{\Pi_2}\left(o^2_{s,j}, \frac{\sqrt{j^2 + m^2 - (i+1)^2} - 1}{\sqrt{n}}\right). \tag{34}$$

i.e. $V^2_{s,i,j,m}$ is contained in the given annulus. Also note that

$$\overline{D_{\Pi_2}}\left(o^2_{s,j}, \frac{\sqrt{j^2+m^2-(i+1)^2}-1}{\sqrt{n}}\right) \subseteq \overline{D_{\Pi_2}}\left(q', \beta\right), \forall q' \in B^2_{s,j}, \forall \beta \text{ satisfying (33).} \tag{35}$$

Let $\mathcal{A}'_m = D_{\Pi_2}\left(o^2_{s,j}, \frac{\sqrt{j^2+m^2-(i+1)^2}-1}{\sqrt{n}}\right)$ and

$$\mathcal{B}'_m = \overline{D_{\Pi_2}}\left(o^2_{s,j}, \frac{\sqrt{(j+1)^2+(m+1)^2-i^2}+1}{\sqrt{n}}\right) \setminus D_{\Pi_2}\left(o^2_{s,j}, \frac{\sqrt{j^2+m^2-(i+1)^2}-1}{\sqrt{n}}\right).$$

Using (34) and (35), we can write analogs of (17), (18) and (19).

$$\begin{aligned} E\left(\left|S_{F_1,F_2,n} \cap V^2_{s,i,m,j}\right|^3\right) &\leq E\left(\left|S_{F_1,F_2,n} \cap \mathcal{B}'_m\right|^3\right) \times Pr\left(S_{F_1,F_2,n} \cap \mathcal{A}'_m = \emptyset\right) \\ &\leq c\left(E\left(\left|S_{F_1,F_2,n} \cap \mathcal{B}'_m\right|\right)\right)^3 \times Pr\left(S_{F_1,F_2,n} \cap \mathcal{A}'_m = \emptyset\right) \\ &= c\left(n\text{Area}(\mathcal{B}'_m \cap F_2)\right)^3 \times e^{-n\times \text{Area}(\mathcal{A}'_m \cap F_2)} \end{aligned} \tag{36}$$

To evaluate the right hand side of the bottom term we will first need to calculate $n\text{Area}(\mathcal{B}'_m \cap F_2)$. Noting that

$$\forall a,b,c > 0, \quad \sqrt{a^2+b^2-c^2} \leq \sqrt{a^2+b^2+c^2} \leq (a+b+c),$$

we see that

$$\begin{aligned} \text{Area}(\mathcal{B}'_m \cap F_2) &= \tfrac{\pi}{n}\left(\left(\sqrt{(j+1)^2+(m+1)^2-i^2}+1\right)^2 - \left(\sqrt{j^2+m^2-(i+1)^2}-1\right)^2\right) \\ &= \tfrac{\pi}{n}\left(2i+2j+2m+3+2\sqrt{(j+1)^2+(m+1)^2-i^2}+2\sqrt{j^2+m^2-(i+1)^2}\right) \\ &\leq \tfrac{\pi}{n}\left(2i+2j+2m+3+2(j+1+m+1+i)+2(j+m+i+1)\right) \\ &\leq \tfrac{\pi}{n}6(i+j+m+2). \end{aligned}$$

Thus we have

$$n\text{Area}(\mathcal{B}'_m \cap F_2) \leq n\frac{6\pi(i+j+m+2)}{n} = O(i+j+m+2). \tag{37}$$

Next note that from Lemma 4 there exist σ, K dependent only upon F_2 such that if $\frac{\sqrt{j^2+m^2-(i+1)^2}-1}{\sqrt{n}} \leq K$ then

$$n \times \text{Area}(\mathcal{A}'_m \cap F_2) \geq n\sigma\left(\frac{\sqrt{j^2+m^2-(i+1)^2}-1}{\sqrt{n}}\right)^2$$

while if $\frac{\sqrt{j^2+m^2-(i+1)^2}-1}{\sqrt{n}} > K$ this gives

$$n \times \text{Area}(\mathcal{A'}_m \cap F_2) \geq \sigma n.$$

If we restrict ourselves to $j \geq i+3$ and take a slightly smaller σ then we have that

$$e^{-n \times \text{Area}(\mathcal{A'}_m \cap F_2)} = \begin{cases} O\left(e^{-\sigma\left(j^2+m^2-i^2\right)}\right), & \text{if } \frac{\sqrt{j^2+m^2-(i+1)^2}-1}{\sqrt{n}} \leq K \\ O\left(e^{-\sigma n}\right), & \text{if } \frac{\sqrt{j^2+m^2-(i+1)^2}-1}{\sqrt{n}} > K. \end{cases} \tag{38}$$

Assuming $j \geq i+3$ we can use Lemma 9, (36), (37) and (38) to evaluate (32). Set $T = \sqrt{K^2(\sqrt{n}+1)^2 - j^2 + (i+1)^2}$. Then

$$\begin{aligned} E\left(|Y_{B^1_{s,i},B^2_{s,j}}|\right) &= O\left(\sum_{m=0}^{\min(T,\sqrt{n}(L+1))} m^6(i+j+m+2)^3 e^{-\sigma\left(j^2+m^2-i^2\right)}\right) \\ &\quad +O\left(\sum_{m=T}^{\sqrt{n}(L+1)} m^6(i+j+m+2)^3 e^{-\sigma n}\right). \end{aligned}$$

Noting that $\forall i,j,m$, $(i+j+m+2) \leq (i+j)(m+2)$ and that $m^6(m+2)^3 e^{-\sigma m} = O(1)$ we find that for $j \geq i+3$,

$$E\left(|Y_{B^1_{s,i},B^2_{s,j}}|\right) = O\left(\sqrt{n}(i+j)^3 e^{-\sigma(j^2-i^2)}\right) + O\left(e^{-\sigma n/2}\right). \tag{39}$$

Now we will prove that

$$\forall s, \quad \sum_{i=0}^{\lfloor L\sqrt{n}\rfloor} \sum_{j=0}^{\lfloor L\sqrt{n}\rfloor} E\left(|Y_{B^1_{s,i},B^2_{s,j}}|\right) = O(\sqrt{n}):$$

From (29) and (39), we have

$$\begin{aligned} \sum_{i=0}^{\lfloor L\sqrt{n}\rfloor} \sum_{j\geq i}^{\lfloor L\sqrt{n}\rfloor} E\left(|Y_{B^1_{s,i},B^2_{s,j}}|\right) &= \sum_{i=0}^{\lfloor L\sqrt{n}\rfloor} \sum_{j=i}^{i+2} E\left(|Y_{B^1_{s,i},B^2_{s,j}}|\right) + \sum_{i=0}^{\lfloor L\sqrt{n}\rfloor} \sum_{j\geq i+3}^{\lfloor L\sqrt{n}\rfloor} E\left(|Y_{B^1_{s,i},B^2_{s,j}}|\right) \\ &= O(\sqrt{n}) + \sqrt{n}O\left(\sum_{i=0}^{\lfloor L\sqrt{n}\rfloor} \sum_{j\geq i+3}^{\lfloor L\sqrt{n}\rfloor} (i+j)^3 e^{-\sigma\left(j^2-i^2\right)}\right) \\ &= O(\sqrt{n}) \end{aligned} \tag{40}$$

The last equality comes from the fact that $\sum_{i=0}^{\infty}\sum_{j\geq i+3}^{\infty}(i+j)^3 e^{-\sigma\left(j^2-i^2\right)} = O(1)$. Now note that in the proof so far we have arbitrarily chosen which face is

F_1 and which face is F_2. If we swap F_1 and F_2 in the proof and also swap i and j we would derive

$$\sum_{j=0}^{\lfloor L\sqrt{n}\rfloor}\sum_{i\geq j}^{\lfloor L\sqrt{n}\rfloor} E\left(|Y_{B^2_{s,j},B^1_{s,i}}|\right) = O(\sqrt{n}). \tag{41}$$

But $Y_{B^1_{s,i},B^2_{s,j}} = Y_{B^2_{s,j},B^1_{s,i}}$ so this just says

$$\sum_{i=0}^{\lfloor L\sqrt{n}\rfloor}\sum_{j\leq i}^{\lfloor L\sqrt{n}\rfloor} E\left(|Y_{B^1_{s,i},B^2_{s,j}}|\right) = O(\sqrt{n}).$$

Thus

$$\begin{aligned}\sum_{i=0}^{\lfloor L\sqrt{n}\rfloor}\sum_{j=0}^{\lfloor L\sqrt{n}\rfloor} E\left(|Y_{B^1_{s,i},B^2_{s,j}}|\right) &\leq \sum_{i=0}^{\lfloor L\sqrt{n}\rfloor}\sum_{j\geq i}^{\lfloor L\sqrt{n}\rfloor} E\left(|Y_{B^1_{s,i},B^2_{s,j}}|\right) + \sum_{i=0}^{\lfloor L\sqrt{n}\rfloor}\sum_{j\leq i}^{\lfloor L\sqrt{n}\rfloor} E\left(|Y_{B^1_{s,i},B^2_{s,j}}|\right)\\ &= O(\sqrt{n}) + O(\sqrt{n}) = O(\sqrt{n})\end{aligned}$$

Therefore (26) can be rewritten as

$$E\left(|Y_{F_1,F_2,n}|\right) = \sum_{s=0}^{\lfloor L\sqrt{n}\rfloor}\sum_{i=0}^{\lfloor L\sqrt{n}\rfloor}\sum_{j=0}^{\lfloor L\sqrt{n}\rfloor} E\left(|Y_{B^1_{s,i},B^2_{s,j}}|\right) = \sum_{s=0}^{\lfloor L\sqrt{n}\rfloor} O(\sqrt{n}) = O(n), \tag{42}$$

and we are done.

2. We now bound $E\left(|Y_{F_1,F_2,F_3,n}|\right)$. Note that by the definition of Type-II Voronoi spheres, for any $S = S(p,r) \in Y_{F_1,F_2,F_3,n}$, $NN(p,\Pi_1) \in F_1$, $NN(p,\Pi_2) \in F_2$, and $NN(p,\Pi_3) \in F_3$.

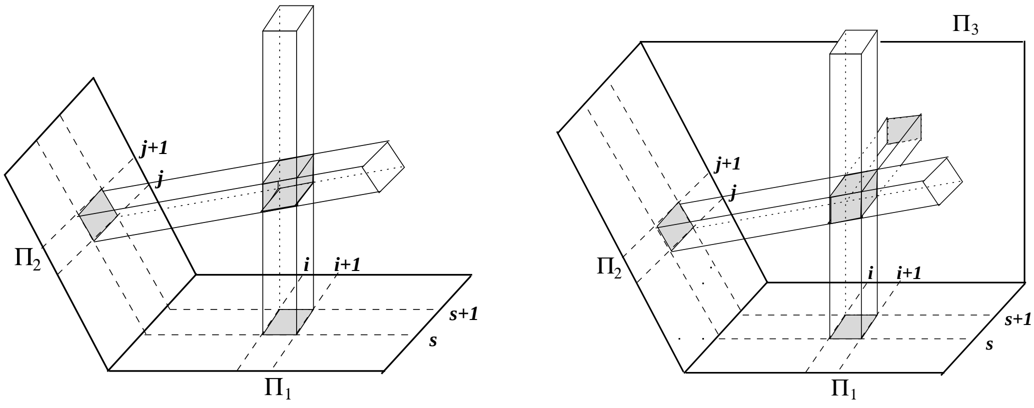
For $k = 1,2$, let $B^k_{s,i}$ be boxes as given in Part 1. Let $Y_{B^1_{s,i},B^2_{t,j},F_3}$ be the set of Voronoi spheres $S(p,r) \in Y_{F_1,F_2,F_3,n}$ such that

$$NN(p,\Pi_1) \in B^1_{s,i},\ NN(p,\Pi_2) \in B^2_{t,j},\ NN(p,\Pi_3) \in F_3.$$

As in the proof of Part 1, Property 2 implies that if $s \neq t$ then $Y_{B^1_{s,i},B^2_{t,j},F_3} = \emptyset$, so

$$E\left(|Y_{F_1,F_2,F_3,n}|\right) \leq \sum_{s=0}^{\lfloor L\sqrt{n}\rfloor}\sum_{i=0}^{\lfloor L\sqrt{n}\rfloor}\sum_{j=0}^{\lfloor L\sqrt{n}\rfloor} E\left(|Y_{B^1_{s,i},B^2_{s,j},F_3}|\right). \tag{43}$$

By the definition of Type-II spheres if $S(p,r) \in Y_{B^1_{s,i},B^2_{s,j},F_3}$, then, $NN(p,\Pi_1)$, the projection of p onto Π_1, is in $B^1_{s,i}$ and, $NN(p,\Pi_2)$, the projection of p onto Π_2, is in $B^2_{s,j}$, i.e., $\phi_1(p) \in B^1_{s,i}$ and $\phi_2(p) \in B^2_{s,j}$ Thus $p \in c^1_{B^1_{s,i}} \cap c^2_{B^2_{s,j}}$. This means



(a) (b)

Figure 10: In (a), $c^1_{B^1_{s,i}} \cap c^2_{B^2_{s,j}}$ is the shaded parallelepiped. In (b), $\phi_3\left(c^1_{B^1_{s,i}} \cap c^2_{B^2_{s,j}}\right)$ is the projection of the parallelepiped onto Π_3.

that if $S(p,r) \in Y_{B^1_{s,i},B^2_{s,j},F_3}$, $NN(p,\Pi_3) \in \phi_3\left(c^1_{B^1_{s,i}} \cap c^2_{B^2_{s,j}}\right)$. See Figure 10. We now show that

$$\forall s,i,j, \quad \text{Diameter}\left(\phi_3\left(c^1_{B^1_{s,i}} \cap c^2_{B^2_{s,j}}\right)\right) \leq \frac{C}{\sqrt{n}} \tag{44}$$

for some constant C dependent only upon F_1, F_2, F_3.

Since both of $c^1_{B^1_{s,i}}$ and $c^2_{B^2_{s,j}}$ are regular prisms having sides of length $\frac{1}{\sqrt{n}}$, their intersection, i.e. the region $c^1_{B^1_{s,i}} \cap c^2_{B^2_{s,j}}$ is a parallelepiped having sides of length $\frac{d}{\sqrt{n}}$ for some constant d dependent only upon the angle between Π_1 and Π_2. It follows that the diameter of the region $\phi_3\left(c^1_{B^1_{s,i}} \cap c^2_{B^2_{s,j}}\right)$ must be at most $\frac{C}{\sqrt{n}}$ for some constant C dependent only upon the angles between Π_1 and Π_2, Π_1 and Π_3, and Π_2 and Π_3, and we have proven (44).

Now suppose we are given any partition of Π_3 consisting of squares having sides of length $\frac{1}{\sqrt{n}}$. (44) implies that if $S(p,r) \in Y_{B^1_{s,i},B^2_{s,j},F_3}$ then $\phi_3\left(c^1_{B^1_{s,i}} \cap c^2_{B^2_{s,j}}\right)$, the region containing $NN(p,\Pi_3)$ can intersect at most $\delta = O(C^2)$ squares of that partition. Let $\mathcal{Q}^3_{s,i,j}$ be the set of squares B of the partition such that $B \cap \phi_3\left(c^1_{B^1_{s,i}} \cap c^2_{B^2_{s,j}}\right) \neq \emptyset$. As discussed we have that $|\mathcal{Q}^3_{s,i,j}| = O(C^2)$.

For given $B \in \mathcal{Q}^3_{s,i,j}$, let $Y_{B^1_{s,i},B^2_{s,j},B}$ be the set of Voronoi spheres $S(p,r) \in Y_{B^1_{s,i},B^2_{s,j},F_3}$ such that $NN(p,\Pi_3) \in B$. Then

$$E\left(|Y_{B^1_{s,i},B^2_{s,j},F_3}|\right) \leq \sum_{B \in \mathcal{Q}^3_{s,i,j}} E\left(|Y_{B^1_{s,i},B^2_{s,j},B}|\right). \tag{45}$$

Now set

$$U^3_{B,m'} = \left\{ u \in \Pi_3 \mid \begin{array}{l} u \text{ is on some circle } C_{\Pi_3}(q,\alpha) \text{ with } q \in B, \alpha \in I_{m'} \\ \text{and } D_{\Pi_3}(q,\alpha) \cap S_{F_1,F_2,F_3,n} = \emptyset. \end{array} \right\}.$$

Suppose that $S = S(p,r) \in Y_{B^1_{s,i},B^2_{s,j},B}$ is defined by t vertices, $t \leq 4$, $p_1, \ldots, p_t \in S_{F_1,F_2,F_3,n}$. Without loss of generality suppose that $p_1, \ldots p_{t'} \in F_1 \cup F_2$ and $p_{t'+1}, \ldots p_t \in F_3$. Then $S(p,r)$ is a Type-II sphere for $S_{F_1,F_2,F_3,n} \cap (F_1 \cup F_2)$ that has points $p_1, \ldots p_{t'}$ on its boundary. Let $\overline{Y_{B^1_{s,i},B^2_{s,j}}}$ be the number of spheres $S = S(p,r)$ that are Type-II over F_1, F_2 for $S_{F_1,F_2,F_3,n} \cap (F_1 \cup F_2)$ such that $S \cap F_1 \neq \emptyset$ and $S \cap F_2 \neq \emptyset$.

Now let q, α be such that $S \cap F_3 = \overline{D_{\Pi_3}}(q,\alpha)$ and m' such that $\alpha \in I_m$. By the definition of Voronoi spheres, $p_{t'+1}, \ldots p_t \in U^3_{B,m'}$.

Thus, for any fixed m',

$$\begin{aligned} \{|S(p,r) \in Y_{B^1_{s,i},B^2_{s,j},B} \quad &: \quad S(p,r) \cap F_3 = \overline{D_{\Pi_3}}(q,\alpha) \text{ for some } q \in B \text{ and } \alpha \in I_m\}| \\ &\leq \quad |\overline{Y_{B^1_{s,i},B^2_{s,j}}}| \quad |U^3_{B,m'} \cap S_{F_1,F_2,F_3,n}|^3 \end{aligned}$$

Summing over m' yields

$$|Y_{B^1_{s,i},B^2_{s,j},B}|| \leq \overline{|Y_{B^1_{s,i},B^2_{s,j}}|} \sum_{m=0}^{\infty} \quad |U^3_{B,m'} \cap S_{F_1,F_2,F_3,n}|^3$$

Now notice that the distribution of $S_{F_1,F_2,F_3,n} \cap (F_1 \cup F_2)$ is the same as that of $S_{F_1,F_2,n}$ so

$$E\left(|\overline{Y_{B^1_{s,i},B^2_{s,j}}}|\right) = E\left(|Y_{B^1_{s,i},B^2_{s,j}}|\right).$$

Also, the distribution of $S_{F_1,F_2,F_3,n} \cap (F_3)$ is the same as that of $S_{F_3,n}$ so

$$E\left(|U^3_{B,m'} \cap S_{F_1,F_2,F_3,n}|^3\right) = E\left(|U^3_{B,m'} \cap S_{F_3,n}|^3\right).$$

Using these two equalities and the fact that $\overline{Y_{B^1_{s,i},B^2_{s,j}}}$ is independent of $U^3_{B,m'}$ yields

$$E(|Y_{B^1_{s,i},B^2_{s,j},B}|) \leq E(|Y_{B^1_{s,i},B^2_{s,j}}|) \sum_{m=0}^{\infty} E\left(|U^3_{B,m'} \cap S_{F_3,n}|^3\right).$$

Lemma 9 says that $\sum_{m=0}^{\infty} E\left(|U^3_{B,m'} \cap S_{F_3,n}|^3\right) = O(1)$ so we have just shown that

$$E(|Y_{B^1_{s,i},B^2_{s,j},B}|) = O\left(E(|Y_{B^1_{s,i},B^2_{s,j}}|)\right).$$

Plugging this back into (45) and using the fact that $|\mathcal{Q}^3_{s,i,j}| = O(C^2)$ gives

$$E(|Y_{B^1_{s,i},B^2_{s,j},F_3}|) = O\left(E(|Y_{B^1_{s,i},B^2_{s,j}}|)\right). \tag{46}$$

Plugging this back into (43) we have

$$
\begin{aligned}
E\left(|Y_{F_1,F_2,F_3,n}|\right) &\le \sum_{s=0}^{\lfloor L\sqrt{n}\rfloor}\sum_{i=0}^{\lfloor L\sqrt{n}\rfloor}\sum_{j=0}^{\lfloor L\sqrt{n}\rfloor} E\left(|Y_{B^1_{s,i},B^2_{s,j},F_3}|\right)\\
&= \sum_{s=0}^{\lfloor L\sqrt{n}\rfloor}\sum_{i=0}^{\lfloor L\sqrt{n}\rfloor}\sum_{j=0}^{\lfloor L\sqrt{n}\rfloor} O\left(E\left(|Y_{B^1_{s,i},B^2_{s,j}}|\right)\right)\\
&= O\left(\sum_{s=0}^{\lfloor L\sqrt{n}\rfloor}\sum_{i=0}^{\lfloor L\sqrt{n}\rfloor}\sum_{j=0}^{\lfloor L\sqrt{n}\rfloor} E\left(|Y_{B^1_{s,i},B^2_{s,j}}|\right)\right)
\end{aligned}
$$

From (42) we have that

$$
\sum_{s=0}^{\lfloor L\sqrt{n}\rfloor}\sum_{i=0}^{\lfloor L\sqrt{n}\rfloor}\sum_{j=0}^{\lfloor L\sqrt{n}\rfloor} E\left(|Y_{B^1_{s,i},B^2_{s,j}}|\right) = O(n) \tag{47}
$$

so we have actually just shown that $E\left(|Y_{F_1,F_2,F_3,n}|\right) = O(n)$ and are done.

3. We now bound $E\left(|Y_{F_1,F_2,F_3,F_4,n}|\right)$. The proof will be very similar to the one in Part 2 so we only sketch the approach but leave out much of the justification.

Note that for any $S = S(p,r) \in Y_{F_1,F_2,F_3,F_4,n}$,

$$
NN(p,\Pi_1) \in F_1, \quad NN(p,\Pi_2) \in F_2, \quad NN(p,\Pi_3) \in F_3, \quad NN(p,\Pi_4) \in F_4.
$$

For $k = 1,2$, let $B^k_{s,i}$ be boxes as given in Part 1. Let $Y_{B^1_{s,i},B^2_{t,j},F_3,F_4}$ be the set of Voronoi spheres $S(p,r) \in Y_{F_1,F_2,F_3,F_4,n}$ such that

$$
NN(p,\Pi_1) \in B^1_{s,i},\ NN(p,\Pi_2) \in B^2_{t,j},\ NN(p,\Pi_3) \in F_3,\ NN(p,\Pi_4) \in F_4.
$$

Property 2 implies that if $s \ne t$ then $Y_{B^1_{s,i},B^2_{t,j},F_3,F_4} = \emptyset$, so

$$
E\left(|Y_{F_1,F_2,F_3,F_4,n}|\right) \le \sum_{s=0}^{\lfloor L\sqrt{n}\rfloor}\sum_{i=0}^{\lfloor L\sqrt{n}\rfloor}\sum_{j=0}^{\lfloor L\sqrt{n}\rfloor} E\left(|Y_{B^1_{s,i},B^2_{s,j},F_3,F_4}|\right). \tag{48}
$$

If $S(p,r) \in Y_{B^1_{s,i},B^2_{s,j},F_3,F_4}$, then $\phi_1(p) \in B^1_{s,i}$ and $\phi_2(p) \in B^2_{s,j}$. i.e. $p \in c^1_{B^1_{s,i}} \cap c^2_{B^2_{s,j}}$. As in Part 2 we have that

$$
\forall s,i,j, \quad \text{Diameter}\left(\phi_3\left(c^1_{B^1_{s,i}} \cap c^2_{B^2_{s,j}}\right)\right) \le \frac{C}{\sqrt{n}} \tag{49}
$$

for some constant C dependent only upon F_1, F_2 and F_4 and the relations between them.

Now suppose we are given any partition of Π_4 consisting of squares having sides of length $\frac{1}{\sqrt{n}}$. Let $\mathcal{Q}^4_{s,i,j}$ be the set of squares B of the partition such that $B \cap \phi_4\left(c^1_{B^1_{s,i}} \cap c^2_{B^2_{s,j}}\right) \neq \emptyset$. Then $|\mathcal{Q}^4_{s,i,j}| = O(C^2)$.

For given $B \in \mathcal{Q}^4_{s,i,j}$, let $Y_{B^1_{s,i},B^2_{s,j},F_3,B}$ be the set of Voronoi spheres $S(p,r) \in Y_{B^1_{s,i},B^2_{s,j},F_3,F_4}$ such that $NN(p,\Pi_4) \in B$. Then

$$E\left(|Y_{B^1_{s,i},B^2_{s,j},F_4,F_3}|\right) \leq \sum_{B \in \mathcal{Q}^3_{s,i,j}} E\left(|Y_{B^1_{s,i},B^2_{s,j},F_3,B}|\right). \tag{50}$$

Now set

$$U^4_{B,m'} = \left\{ u \in \Pi_4 \mid \begin{array}{l} u \text{ is on some circle } C_{\Pi_4}(q,\alpha) \text{ with } q \in B, \alpha \in I_{m'} \\ \text{and } D_{\Pi_4}(q,\alpha) \cap S_{F_1,F_2,F_3,F_4,n} = \emptyset. \end{array} \right\}.$$

Let $\overline{Y_{B^1_{s,i},B^2_{s,j},F_3}}$ be the number of spheres $S = S(p,r)$ that are Type-II over F_1, F_2, F_3 for $S_{F_1,F_2,F_3,F_4,n} \cap (F_1 \cup F_2 \cup F_3)$ such that $S \cap F_1 \neq \emptyset$, $S \cap F_2 \neq \emptyset$. and $S \cap F_3 \neq \emptyset$. Then

$$\begin{aligned} \{|S(p,r) \in Y_{B^1_{s,i},B^2_{s,j},F_4B} \quad &: \quad S(p,r) \cap F_3 = \overline{D_{\Pi_4}}(q,\alpha) \text{ for some } q \in B \text{ and } \alpha \in I_m\}| \\ &\leq \quad |\overline{Y_{B^1_{s,i},B^2_{s,j},F_3}}| \quad |U^4_{B,m'} \cap S_{F_1,F_2,F_3,F_4,n}|^3 \end{aligned}$$

Summing over m yields

$$|Y_{B^1_{s,i},B^2_{s,j},F_3,B}| \leq |\overline{Y_{B^1_{s,i},B^2_{s,j},F_3}}| \sum_{m=0}^{\infty} \quad |U^4_{B,m'} \cap S_{F_1,F_2,F_3,F_4,n}||^3$$

Taking expectations and using Lemma 9 yields

$$E(|Y_{B^1_{s,i},B^2_{s,j},F_3,B}|) \leq E(|Y_{B^1_{s,i},B^2_{s,j},F_3},|) \sum_{m=0}^{\infty} E\left(|U^4_{B,m'} \cap S_{F_4,n}|^3\right) = O\left(E(|Y_{B^1_{s,i},B^2_{s,j},F_3}|)\right).$$

Plugging this back into (50) using the fact that $|\mathcal{Q}^4_{s,i,j}| = O(C^2)$ and (46) gives

$$E(|Y_{B^1_{s,i},B^2_{s,j},F_3,F_4}|) = O\left(E(|Y_{B^1_{s,i},B^2_{s,j},F_3}|)\right) = O\left(E\left(|Y_{B^1_{s,i},B^2_{s,j}}|\right)\right)$$

To finish the proof we insert this into (48) to get

$$\begin{aligned} E\left(|Y_{F_1,F_2,F_3,F_4}|\right) &= \sum_{s=0}^{\lfloor L\sqrt{n}\rfloor} \sum_{i=0}^{\lfloor L\sqrt{n}\rfloor} \sum_{j=0}^{\lfloor L\sqrt{n}\rfloor} O\left(E\left(|Y_{B^1_{s,i},B^2_{s,j}}|\right)\right) \\ &= O(n) \end{aligned}$$

□

5 Bounding the Number of $\frac{\log n}{\sqrt{n}}$-good Type-III Spheres

In this section we prove (6), i.e.,

$$E\left(\text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-good Type-III Spheres}\right) = o(n).$$

We do this by splitting it into two cases. In subsection 5.1 we show that the expected number of of $\frac{\log n}{\sqrt{n}}$-good Type-III Spheres *whose center is outside or on* $\mathcal{P}$ is $O(n)$. In subsection 5.2 we show that the expected number of of $\frac{\log n}{\sqrt{n}}$-good Type-III Spheres *whose center is inside* $\mathcal{P}$ is $O(n)$. Combining these two results will prove (6).

Before starting we note again that we are only counting $\frac{\log n}{\sqrt{n}}$-good Type-III spheres in this section; *we are not assuming that the spheres are Voronoi spheres, i.e., we are not assuming that the interiors of the spheres are empty.* We are assuming though that the spheres all have at least one point of S_n on their boundaries and that two spheres are (combinatorially) equivalent of they have exactly the same points of S_n on their boundaries (even if they contain different points in their interiors).

The reason for dispensing with the assumption of 'Voronoiness' is that this makes the analysis much easier. This was the motivation for introducing the concepts of x-good and x-bad spheres.

5.1 Sphere center p *outside or on* $\mathcal{P}$

If p is *outside or on* $\mathcal{P}$, let $p' = NN(p, \mathcal{P})$. The convexity of $\mathcal{P}$ guarantees that p' is unique. Let $S = S(p, r)$ be a $\frac{\log n}{\sqrt{n}}$-good Type-III sphere that is defined by 2/3/4 points of S_n on its boundary. For $p' \notin Skel(\mathcal{P})$ let F be the unique face of $\mathcal{P}$ such that $p' \in F$; let Π be F's supporting plane. Note that in this case $B = S \cap \Pi$ is a closed disk on Π with center $p' \in F$; let r' be the radius of this circle. Then $B = \overline{D_\Pi}(p', r')$.

Now set $c_1 = \frac{2}{c'\sqrt{\pi}}$ where c' is the constant defined in Lemma 6. If $p' \notin Bd\left(\frac{c_1 \log n}{\sqrt{n}}\right)$, then $d(p', Skel(\mathcal{P})) > \frac{c_1 \log n}{\sqrt{n}}$. Thus Lemma 6 states that if S intersects any other face of $\mathcal{P}$ then $D_\Pi\left(p', \frac{1}{\sqrt{\pi}}\frac{\log n}{\sqrt{n}}\right) \subseteq S \cap \mathcal{P}$. Since Area $\left(D_\Pi\left(p', \frac{1}{\sqrt{\pi}}\frac{\log n}{\sqrt{n}}\right)\right) = \frac{\log^2 n}{n}$ this implies that if $p' \notin Bd\left(\frac{c_1 \log n}{\sqrt{n}}\right)$ and S intersects some other face of $\mathcal{P}$ besides F then S is a $\frac{\log n}{\sqrt{n}}$-bad sphere.

Our approach is to divide the problem into three cases.

Case 1 $p' \notin Bd\left(\frac{c_1 \log n}{\sqrt{n}}\right)$ (Figure 11).

Case 2 $p' \in Bd(\frac{c_1 \log n}{\sqrt{n}})$ but $p' \notin Skel(\mathcal{P})$ (Figure 12).

Case 3 $p' \in Skel(\mathcal{P})$ (Figure 13).

We now work through the cases.

Case 1. $p' \notin Bd\left(\frac{c_1 \log n}{\sqrt{n}}\right)$.

Let $p' \notin Bd\left(\frac{c_1 \log n}{\sqrt{n}}\right)$. Since we are only counting good spheres the the discussion above shows that definition of c_1 immediately implies that $S \cap \mathcal{P} \subseteq F$, i.e., S does not intersect any other face of $\mathcal{P}$. But then S is a Type-I sphere and need not be examined here.

Case 2. $p' \in Bd(\frac{c_1 \log n}{\sqrt{n}})$ but $p' \notin Skel(\mathcal{P})$

This case is illustrated by Figure 12. The analysis is based on the following lemma:

Lemma 10 *If $p' \in Bd\left(\frac{c_1 \log n}{\sqrt{n}}\right)$ but $p' \notin Skel(\mathcal{P})$, then $\exists q$ on some edge of $\mathcal{P}$, q dependent upon p, such that if $u \in S \cap \mathcal{P}$ then $d(u,q) \le c_2 \frac{\log n}{\sqrt{n}}$ where $c_2 = c_0 + c_1$ (c_0 is the constant introduced in Definition 7).*

Proof. Let r and r' be, respectively, the radii of S and B. The $\frac{\log n}{\sqrt{n}}$-goodness of S implies $r' < c_0 \frac{\log n}{\sqrt{n}}$ since otherwise, from Definition 7,

$$\text{Area}(S \cap \mathcal{P}) \ge \text{Area}(D_\Pi(p', r') \cap F) \ge \left(\frac{\log n}{\sqrt{n}}\right)^2,$$

contradicting the definition of a good sphere. Let $q \in Skel(\mathcal{P})$ be a nearest neighbor to p' on $Skel(\mathcal{P})$. Since $p' \in Bd\left(\frac{c_1 \log n}{\sqrt{n}}\right)$, $d(p', q) \le c_1 \frac{\log n}{\sqrt{n}}$.

We will now show that for $u \in S \cap \mathcal{P}$, $d(u, p') \le r'$. The lemma will follow since

$$d(u,q) \le d(u,p') + d(p',q) \le (c_0 + c_1)\frac{\log n}{\sqrt{n}} = c_2 \frac{\log n}{\sqrt{n}}$$

where $c_2 = c_0 + c_1$.

First note that since p is outside $\mathcal{P}$, the convexity of $\mathcal{P}$ and $p' = NN(p, \mathcal{P})$ together imply $\angle pp'u \ge 90°$. The law of cosines states

$$d(p',u)^2 + d(p,p')^2 - d(p,u)^2 = 2d(p',u)d(p,p')\cos\angle pp'u.$$

Thus, using the fact that $d(p,p')^2 + (r')^2 = r^2$,

$$\begin{aligned} d(p',u)^2 &= d(p,u)^2 - d(p,p')^2 + 2d(p',u)d(p,p')\cos\angle pp'u \\ &\le r^2 - d(p,p')^2 + 2d(p',u)d(p,p')\cos\angle pp'u \\ &= (r')^2 + 2d(p',u)d(p,p')\cos\angle pp'u \le (r')^2 \end{aligned}$$

so $d(u,p') \le r'$ and we are done. □

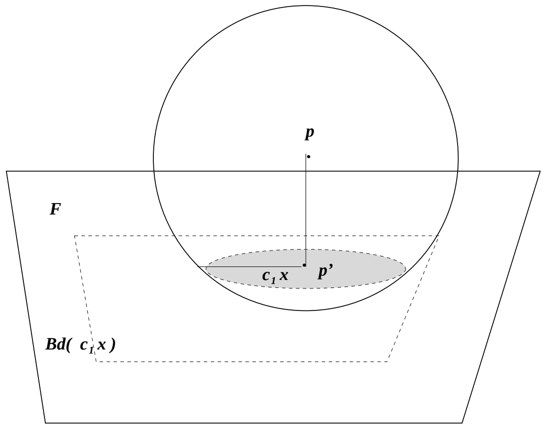


Figure 11: Case 1 for x-good Type-III Spheres with p outside or on $\mathcal{P}$. Notice that the intersection of S with $\mathcal{P}$ is a circle completely contained in F. If the intersection contains a part of some other face then the intersection circle on F would have to grow to be so big that S would no longer be x-good. This is a consequence of p' being at least distance c_1x from the border.

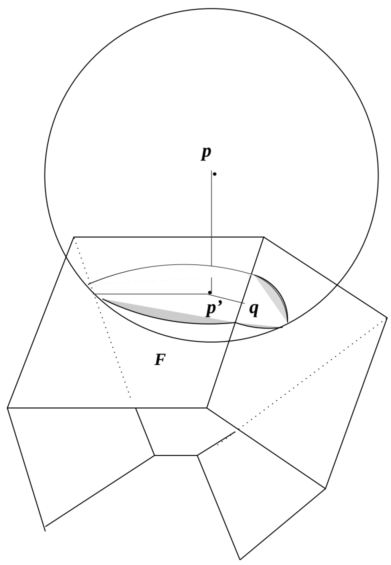
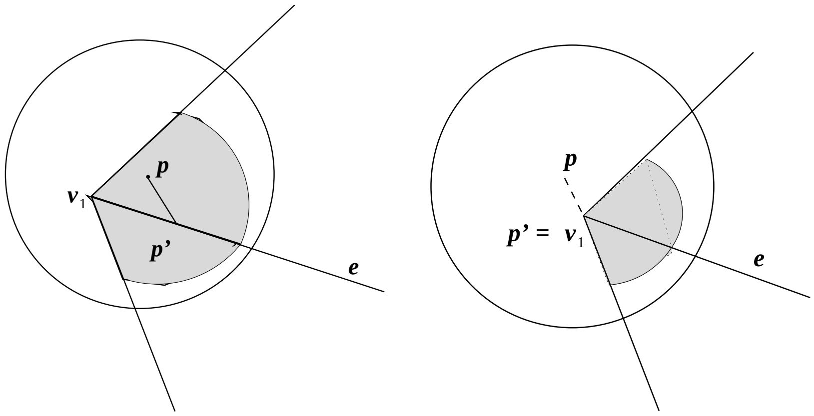


Figure 12: Case 2 for x-good Type-III Spheres with p outside or on $\mathcal{P}$. $p' \in Bd(\frac{c_1 \log n}{\sqrt{n}})$ but $p' \notin Skel(\mathcal{P})$. Let $q \in Skel(\mathcal{P})$ be a nearest neighbor to p' on $Skel(\mathcal{P})$. Notice that, in this example, the center of the circle formed by the intersection of S with the supporting plane of the second face, (i.e., the face not containing p') is not on that face.



(a) $p' = NN(p, \mathcal{P})$ is on an edge e

(b) $p' = NN(p, \mathcal{P})$ is a vertex v

Figure 13: Case 3 for x-good Type-III Spheres with p outside or on $\mathcal{P}$.

We can use this lemma to show that the expected total number of $\frac{\log n}{\sqrt{n}}$-good Type-III spheres $S(p, r)$ with $p' \in Bd\left(\frac{c_1 \log n}{\sqrt{n}}\right)$ but $p' \notin Skel(\mathcal{P})$ is $O(n)$:

Let L be the total length of all of the edges of $\mathcal{P}$. We partition up the edges of P into $O(L/\frac{\log n}{\sqrt{n}})$ line segments each of length $\leq \frac{\log n}{\sqrt{n}}$, e.g., all segments will have length $\frac{\log n}{\sqrt{n}}$ except for, possibly, one segment per edge (which contains one of that edge's endpoints).

Now let s be any edge segment and $N(s) = \{u \in \mathcal{P} : d(u, s) \leq c_2 \frac{\log n}{\sqrt{n}}\}$ be the set of all points on $\mathcal{P}$ within distance $c_2 \frac{\log n}{\sqrt{n}}$ of s. For any sphere center p let q be the corresponding edge point defined by Lemma 10 and s the segment that q belongs to. From Lemma 10 and the definition of $N(s)$, the 2/3/4 points of S_n that define a $\frac{\log n}{\sqrt{n}}$-good sphere $S(p, r)$ must be in $N(s)$ so the *total* number of $\frac{\log n}{\sqrt{n}}$-good spheres $S(p, r)$ such that p correspond to some point $q \in s$ is upperbounded by $|N(s) \cap S_n|^4$.

Now $Area(N(s)) \leq c'_2 (\frac{\log n}{\sqrt{n}})^2 = c'_2 \frac{\log^2 n}{n}$ for some constant c'_2 dependent only upon $\mathcal{P}$. Plugging into the Poisson distribution (with rate n) we find from Lemma 3 that $E\left(|N(s) \cap S_n|^4\right) = O(\log^8 n)$. Thus, for a fixed segment s on $Skel(\mathcal{P})$, the expected total number of $\frac{\log n}{\sqrt{n}}$-good Type-III spheres $S(p, r)$ such that $NN(p, \mathcal{P}) = p' \notin Skel(\mathcal{P})$ and $NN(p', Skel(\mathcal{P})) \in s$ is $O(\log^8 n)$.

But *every* $\frac{\log n}{\sqrt{n}}$-good Type-III sphere $S(p, r)$ with $NN(p, \mathcal{P}) = p' \notin Skel(\mathcal{P})$ must have $NN(p', Skel(\mathcal{P})) \in s'$ for *some* interval s' so the total expected number of spheres of this type is upperbounded by the number of such segments times $O(\log^8 n)$. That is, $O(L/\frac{\log n}{\sqrt{n}}) \times O(\log^8 n) = O(\sqrt{n} \log^7 n) = O(n)$. This shows that the expected total number of $\frac{\log n}{\sqrt{n}}$-good Type-III spheres $S(p, r)$ with

$NN(p,\mathcal{P}) = p' \in Bd\left(\frac{c_1 \log n}{\sqrt{n}}\right)$ but $p' \notin Skel(\mathcal{P})$ is $O(n)$ and we have completed Case 2.

Case 3. $p' \in Skel(\mathcal{P})$ (Figure 13).

Note that this case differs in a major way from that of Case 2. This is because, unlike in Case 2, the line pp' here is not necessarily perpendicular to the plane Π. This means that $NN(p,\Pi) \neq p'$ so p' is not the center of the circle $B = \Pi \cap S$. Consequentially, even if S is $\frac{\log n}{\sqrt{n}}$-good it is quite possible that the points in $S \cap F$ are very far away from p'' the center of B. This can occur because, even if B is quite large, the intersection $B \cap F$ can be quite small. This, for example, means that we can not use the triangle inequality in the same way as we did in Case 2, since we will not be able to bound $d(p', u)$ for $u \in B \cap F$. See Figure 14.

To sidestep the difficulties caused by these differences we introduce the following definition:

Definition 11 *For $p' \in Skel(\mathcal{P})$ define*

$$\begin{aligned} M(p') = \{q \in \mathcal{P} : \ & q \in S(p,r) \text{ for some } \tfrac{\log n}{\sqrt{n}}\text{-good Type-III sphere } S(p,r) \\ & \text{such that } p' = NN(p,\mathcal{P}) \end{aligned}$$

For s a segment of an edge in $Skel(\mathcal{P})$

$$M(s) = \cup_{p' \in s} M(p').$$

These definitions will be useful since they will permit us to restrict the number of Voronoi spheres associated with a segment; by partitioning $Skel(\mathcal{P})$ into a small number of appropriately sized segments we will be able to bound the expected number of spheres in Case 3.

Our goal will be to prove that $\text{Area}(M(s))$ is small. If we can do this then we will be able to use the technique at the end of Part 2 to show that the expected number of $\frac{\log n}{\sqrt{n}}$-good Type-III spheres with $p' \in Skel(\mathcal{P})$ is small. The important thing to keep in mind when reading the lemmas and proofs is that points in $M(s)$ might actually be quite far from s. We will therefore need something stronger than the triangle inequality to reach our goal.

Lemma 11 *Let s be a segment of an edge e in $Skel(\mathcal{P})$ with $length(s) \leq \frac{\log n}{\sqrt{n}}$. Then*

$$Area(M(s)) \leq c_3 \frac{\log^3 n}{n}$$

for some c_3 dependent only upon $\mathcal{P}$.

Before proving this Lemma we explain why it implies that expected number of $\frac{\log n}{\sqrt{n}}$-good Type-III spheres $S(p,r)$ with $p' = NN(p,\mathcal{P}) \in Skel(\mathcal{P})$ is $O(n)$. The analysis is very similar to that just performed at the end of Case 2.

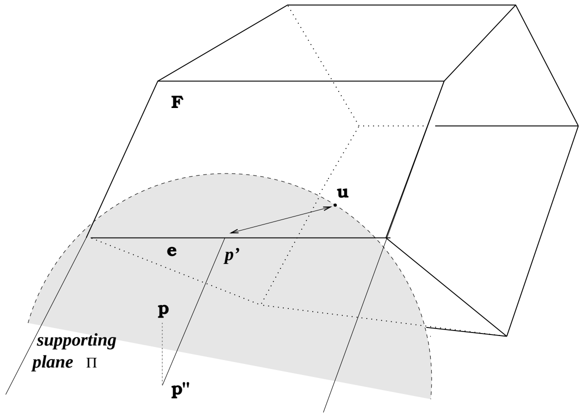


Figure 14: Good Type-III Sphere S with p outside $\mathcal{P}$. In this case no matter how large the radius of S is, the area of the intersection of S with $\mathcal{P}$ can be arbitrarily small. This means that even if S is a good sphere, it is quite possible that the points in $S \cap F$ are very far away from p'', the center of the circle $B = \Pi \cap S$. For example, in this figure, we are not able to bound $d(p', u)$ for $u \in B \cap F$.

Let L be the total length of all of the edges of $\mathcal{P}$. We partition up the edges of P into $O(L/\frac{\log n}{\sqrt{n}})$ line segments each of length $\leq \frac{\log n}{\sqrt{n}}$, e.g., all segments will have length $\frac{\log n}{\sqrt{n}}$ except for, possibly, one segment per edge (which contains one of that edge's endpoints).

For any sphere center p, let $p' = NN(p, \mathcal{P}) \in Skel(\mathcal{P})$ and let s be the segment such that $p' \in s$. By the definition of $M(s)$ the 2/3/4 points of S_n that define a $\frac{\log n}{\sqrt{n}}$-good sphere $S(p,r)$ must be in $M(s)$ so the *total* number of $\frac{\log n}{\sqrt{n}}$-good spheres $S(p,r)$ such that $NN(p, \mathcal{P}) = p'$ is upperbounded by $|M(s) \cap S_n|^4$.

Lemma 11 tells us that $Area(M(s)) \leq c_3 \frac{\log^3 n}{n}$ for some constant c_3 dependent only upon $\mathcal{P}$. Plugging into the Poisson distribution (with rate n) we find from Lemma 3 that $E\left(|M(s) \cap S_n|^4\right) = O(\log^{12} n)$. Thus, for a fixed segment s on $Skel(\mathcal{P})$, the expected total number of $\frac{\log n}{\sqrt{n}}$-good Type-III spheres $S(p,r)$ such that $NN(p, \mathcal{P}) = p' \in s$ is $O(\log^{12} n)$.

But *every* $\frac{\log n}{\sqrt{n}}$-good Type-III sphere $S(p,r)$ with $NN(p, \mathcal{P}) \in Skel(\mathcal{P})$ must have $NN(p, \mathcal{P}) \in s'$ for *some* interval s' so the total expected number of spheres of this type is upperbounded by the number of such segments times $O(\log^{12} n)$. That is, $O(L/\frac{\log n}{\sqrt{n}}) \times O(\log^{12} n) = O(\sqrt{n} \log^{11} n) = O(n)$. This shows that the expected total number of $\frac{\log n}{\sqrt{n}}$-good Type-III spheres $S(p,r)$ with $p' = NN(p, \mathcal{P}) \in Skel(\mathcal{P})$ is $O(n)$.

We now prove the lemma.

Proof. [Proof of Lemma 11]

The proof is split into three parts. In parts 1 and 2 we examine the possible locations of $q \in M(p')$ for $\forall p' \in Skel(\mathcal{P})$; part 1 assumes that p' a point that is

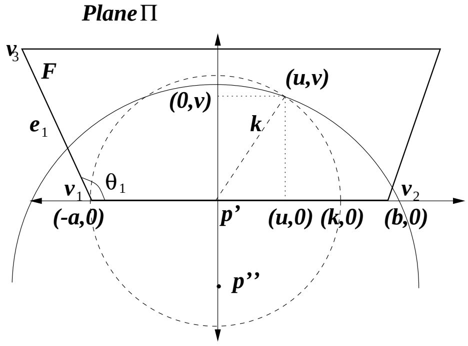


Figure 15: None of the points $(0,0)$, $(0,\pm v)$, $(\pm u,0)$, $(\pm\sqrt{u^2+v^2},0)$ are further away from p'' than (u,v) is.

not vertex of $\mathcal{P}$, part 2 assumes that p' is a vertex of $\mathcal{P}$. In part 3 we actually calculate Area$(M(s))$ for a *segment* s.

1. Let p' be on an edge e but not a vertex of e. Let v_1 and v_2 be the two endpoints of e. We split the case into two parts: in **(i)** we examine $M(p')\cap F$ for F one of the two faces *incident* upon e, i.e., one of the two faces having e on their boundary; in **(ii)** we examine $M(p')\cap F$ when F is a face of $\mathcal{P}$ *not incident* upon e.

(i) Let F be either of the two faces incident upon e and Π its supporting plane. Let $p''=NN(p,\Pi)$. Note that $p''p'\perp e$ and $p'\neq p''$ (except in the extreme case in which line $pp'\perp\Pi$). We do have, though, that $p'=NN(p'',F)$. This follows from the fact that $pp''\perp\Pi$ so $\forall u\in\Pi$, $d(p,u)=\sqrt{d(p,p'')^2+d(p'',u)^2}$ leading to the general observation that

$$\forall u,v\in\Pi \quad d(p,u)\le d(p,v) \quad \text{if and only if} \quad d(p'',u)\le d(p'',v). \tag{51}$$

In particular, since $p'=NN(p,F)$ this implies that $p'=NN(p'',F)$.

Consider the orthogonal coordinate system (u,v) of Π whose origin is p' and whose u-axis lies on the line defined by extending e. W.L.O.G we will assume that F lies in the half plane $v\ge 0$ since the other case is symmetric. Note that by definition, the point $(u'',v'')=p''$ has $u''=0$, $v''\le 0$. The first is because $p'=NN(p'',F)$ and p' is not a vertex of F (since p' is not a vertex of e) so $p''p'\perp e$. The second follows from the first plus the fact that if $v''>0$ there would be some point *inside* F closer to p'' than p' is, forcing a contradiction.

Let $q=(u,v)$ be any point of F in some $\frac{\log n}{\sqrt{n}}$-good Type-III sphere $S=S(p,r)$ with $p'=NN(p,\mathcal{P})$. Straightforward geometric arguments show that *none* of the points $(0,0)$, $(0,\pm v)$, $(\pm u,0)$, $(\pm\sqrt{u^2+v^2},0)$ are further away from p'' than (u,v) is. See Figure 15. Thus, (51) immediately implies that all of these points are in S. Also, since $p'=NN(p,\mathcal{P})$ we have by definition that $p'\in S\cap F$.

S and F are both convex and thus $S\cap F$ is convex as well. In particular, if points $q_1,q_2,q_3\in S\cap F$ then, by convexity, the triangle T with vertices q_1,q_2,q_3 is totally contained within $S \cap F$. Although simple, this will be an extremely powerful tool. As a first application, we prove that $v \leq 2\frac{\log n}{\sqrt{n}}$. If $\sqrt{u^2+v^2} \leq 2\frac{\log n}{\sqrt{n}}$, then v must be less than or equal to $\sqrt{2}\frac{\log n}{\sqrt{n}}$, so we are done. If $\sqrt{u^2+v^2} > 2\frac{\log n}{\sqrt{n}}$, then $\left(\pm\frac{\log n}{\sqrt{n}}, 0\right) \in S$ immediately follows from $(\pm\sqrt{u^2+v^2}, 0) \in S$. For large enough n, at least one of $\left(\frac{\log n}{\sqrt{n}}, 0\right)$ or $\left(-\frac{\log n}{\sqrt{n}}, 0\right)$ must be in F, since otherwise the edge e has length less than $2\frac{\log n}{\sqrt{n}}$, becoming infinitesimally small. Thus at least one of the triangles T_1 with vertices $(0,0),(u,v)$, and $\left(\frac{\log n}{\sqrt{n}}, 0\right)$, or T_2 with vertices $(0,0),(u,v)$, and $\left(-\frac{\log n}{\sqrt{n}}, 0\right)$, is totally contained in $S \cap F$.

If triangle $T \subseteq S \cap F$ where S is $\frac{\log n}{\sqrt{n}}$-good then $\mathrm{Area}(T) \leq \left(\frac{\log n}{\sqrt{n}}\right)^2$ since if $\mathrm{Area}(T) \geq \left(\frac{\log n}{\sqrt{n}}\right)^2$ then $\mathrm{Area}(S \cap \mathcal{P}) \geq \left(\frac{\log n}{\sqrt{n}}\right)^2$ contradicting the definition of $\frac{\log n}{\sqrt{n}}$-goodness. Since $Area(T_1) = Area(T_2) = \frac{1}{2}v\frac{\log n}{\sqrt{n}}$ and one of T_1, T_2 is contained in $S \cap F$ we have proven that

$$\text{If } (u,v) \in S \cap F \text{ then } v \leq 2\frac{\log n}{\sqrt{n}}. \tag{52}$$

As another application we can show that

$$\text{If } (u,v) \in S \cap F, \text{ then } v \leq \tfrac{2C}{|u|}\left(\frac{\log n}{\sqrt{n}}\right)^2 \tag{53}$$

for some constant $C > 0$ dependent only upon $\mathcal{P}$.

To prove this, label the "left" endpoint of e as v_1; the "right" endpoint as v_2. See Figure 16. Let e_1, e_2 be the two edges of F incident upon e, respectively, at v_1 and v_2. Let θ_1, θ_2 be the angles between e and v_1, v_2. Finally, let $a, b \geq 0$ be such that, in our coordinate system, $v_1 = (-a, 0)$ and $v_2 = (0, b)$. Recall that $p' = (0,0)$. Recall too, that we saw before that $(u,v) \in S$ implies $(\pm u, 0), (0, v) \in S$.

If $-a \leq u \leq b$ then $(u, 0) \in F$. Therefore the triangle T with endpoints (u,v), $(0,0)$, $(u,0)$ has all endpoints in $S \cap F$ so $T \subseteq S \cap F$. By the $\frac{\log n}{\sqrt{n}}$-goodness of S, $\mathrm{Area}(T) = \frac{1}{2}|u|v \leq \left(\frac{\log n}{\sqrt{n}}\right)^2$ proving (53) with $C = 1$. See Figure 16 (a).

Suppose now that $u \notin [-a, b]$. Without loss of generality we will assume that $u > b$ ($u < -a$ is symmetric). Now additionally assume that $(0, v) \in F$. This implies that the triangle T with endpoints (u,v), $(0,0)$, $(0,v)$ has all endpoints in $S \cap F$ so $T \subseteq S \cap F$. By the $\frac{\log n}{\sqrt{n}}$-goodness of S, $\mathrm{Area}(T) = \frac{1}{2}|u|v \leq \left(\frac{\log n}{\sqrt{n}}\right)^2$, again proving (53) with $C = 1$. See Figure 16 (b).

The only case left to examine is when $u > b$ and $(0, v) \notin F$; Figure 16 (c). First note that this implies that $\theta_2 > 90°$ since otherwise it would be impossible for $(u,v) \in F$ with $u > b$. Next recall that $(u,v) \in S \cap F$ so (52) implies $v \leq 2\frac{\log n}{\sqrt{n}}$. Since $\theta_2 > 90°$, in order for $(v, 0) \notin F$ we must have that edge e_1

intersects the line segment connecting p' and $(0,v)$ at some point $(0,v')$ with $v' \le 2\frac{\log n}{\sqrt{n}}$ so $\theta_1 \le 90°$ as well. This means that $a = \frac{1}{\tan\theta_1}v' \le \frac{2}{\tan\theta_1}\frac{\log n}{\sqrt{n}}$. Thus, for large enough n, $a \le \frac{b}{2}$ so $b \ge \frac{length(e)}{2}$. Finally, since $(u,0) \in S$, we have $(b,0) \in S$ so $(b,0) \in S \cap F$. This implies that the triangle T with endpoints (u,v), $(0,0)$, $(b,0)$ has all endpoints in $S \cap F$ so $T \subseteq S \cap F$. By the $\frac{\log n}{\sqrt{n}}$-goodness of S, $\text{Area}(T) = \frac{1}{2}bv \le \left(\frac{\log n}{\sqrt{n}}\right)^2$ so

$$v \le \frac{2}{b}\left(\frac{\log n}{\sqrt{n}}\right)^2 \le \frac{4}{length(e)}\left(\frac{\log n}{\sqrt{n}}\right)^2 \le \frac{4}{u}\frac{diameter(F)}{length(e)}\left(\frac{\log n}{\sqrt{n}}\right)^2$$

proving (53) with $C = \frac{2\,diameter(F)}{length(e)}$.

Combining (52), and (53) then proves that, for F incident upon e and the constant $C = \max\left\{1, \frac{2\,diameter(F)}{length(e)}\right\}$,

$$M(p') \cap F \subseteq \left\{(u,v) \in F \,:\, v \le \min\left(2\frac{\log n}{\sqrt{n}}, \frac{2C}{|u|}\left(\frac{\log n}{\sqrt{n}}\right)^2\right)\right\}. \tag{54}$$

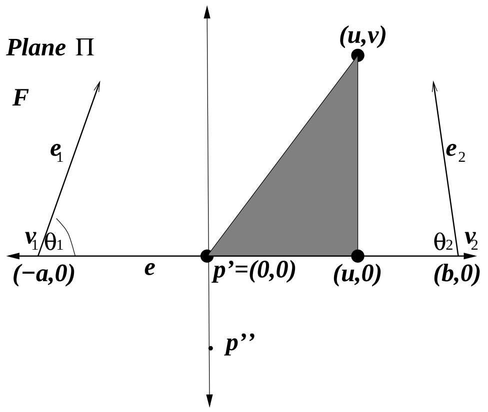
(ii) Now let F be a face of $\mathcal{P}$ which is not incident upon e. To bound $\text{Area}(M(p') \cap F)$ we find a region containing $M(p') \cap F$.

Let L_e be the cylinder of radius $c_0\frac{\log n}{\sqrt{n}}$ around the line defined by extending e. Figure 17(a). Our claim is that

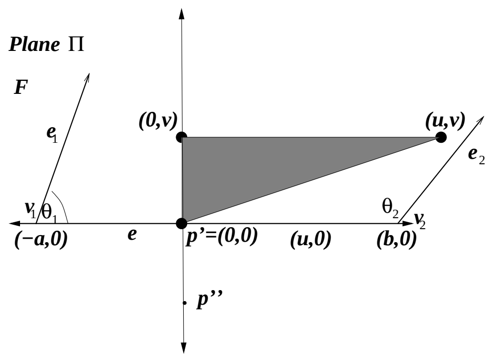
$$\text{If } q \in M(p') \cap F, \text{ then } q \text{ is inside or on } L_e. \tag{55}$$

The proof is by contradiction. Assume then, that $\exists q \in M(p') \cap F$ such that q is outside L_e. Let Π_q be the plane containing q and e and set $l \subset \Pi_q$ to be the line perpendicular to e passing through p'. Consider the coordinate system (u,v) of Π_q such that the origin is p', the u- and v-axis are respectively e and l, and every point of $\mathcal{P}$ has nonnegative v-coordinates (this coordinate system is defined in the same fashion as in **(i)**). See Figure 17(b). Let $q = (u,v)$ and set $q' = (u',v')$ to be the projection of q onto the line l. Note that $u' = 0$ because all points on l have $u' = 0$ and $v = v' > c_0\frac{\log n}{\sqrt{n}}$ because $q \notin L_e$. Let $S = S(p,r)$ be a $\frac{\log n}{\sqrt{n}}$-good Type-III sphere such that $NN(p,\mathcal{P}) = p'$ and $q \in S$, and let $p'' = NN(p,\Pi_q)$. From $pp'' \perp \Pi_q$ and $pp' \perp e$ we have $p''p' \perp e$, so $p'' = (u'',v'') \in l$, $u'' = 0$ and $v'' \le 0$. Note that $length(p''q') < length(p''q)$ which implies that $length(pq') < length(pq)$ and thus $q' \in S$.

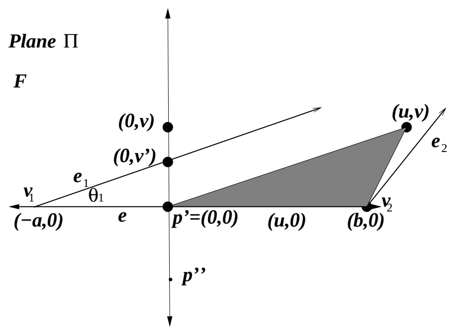
The two faces of $\mathcal{P}$ incident upon e are separated by the plane Π_q. Let F'' be the face that lies on the same side of Π_q as p and Π'' its supporting plane. Let $q'' \in \Pi''$ be a point such that $p'q'' \perp e$, $length(p'q'') = length(p'q')$, and q'' lies on the opposite side of e to $NN(p,\Pi'')$. Note that in the coordinate system (u,v) given in **(i)**, $NN(p,\Pi'')$ has coordinate $(0,v)$ for some $v < 0$ and q'' has



(a) $-a \leq u \leq b$.

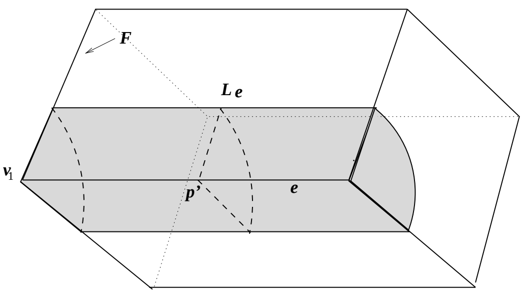


(b) $u > b$ and $(0, v) \in S$

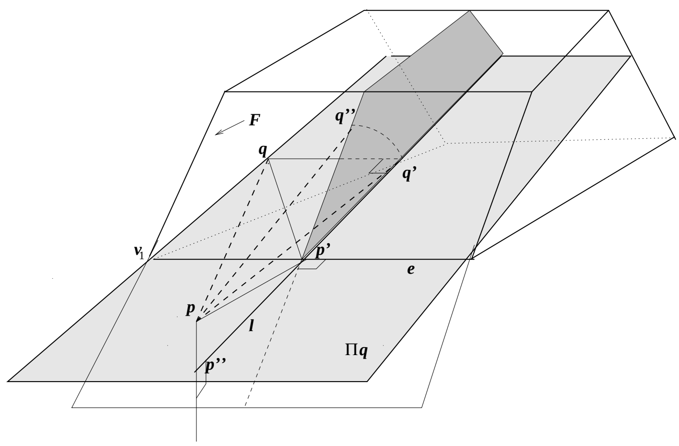


(c) $u > b$ and $(0, v) \notin S$

Figure 16: From the proof of Lemma 11 **(i)**. Edge e is $v_1 v_2$. It is known that $(u, v) \in S \cap F$ and $(u, 0), (0, v) \in S$. The shaded region is triangle T.



(a)



(b)

Figure 17: Let F be a face of $\mathcal{P}$ incident upon v_1, $p' \in e$ and L_e the cylinder of radius $c_0 \frac{\log n}{\sqrt{n}}$ around the line defined by extending e. If $q \in M(p') \cap F$, then q must be inside or on L_e.

coordinate $(0, v)$ for $v = length(p'q') = v' > c_0 \frac{\log n}{\sqrt{n}}$. This means that if $q'' \in S$, then $D_{\Pi''}(p', length(p'q'')) \subset S$.

But, since $90^\circ \leq \angle pp'q'' < \angle pp'q' < 180^\circ$ and $length(p'q'') = length(p'q')$, $q' \in S$ implies that $q'' \in S$. Thus $D_{\Pi''}(p', length(p'q'')) \subset S$ contradicting the goodness of S by the fact that $length(p'q'') > c_0 \frac{\log n}{\sqrt{n}}$ and Definition 7. We have therefore proven (55).

Note that (55) implies that if $M(p') \cap F \neq \emptyset$ then $d(e, F) \leq c_0 \frac{\log n}{\sqrt{n}}$. Thus, for large enough n we have that if $M(p') \cap F \neq \emptyset$ then $F \cap e \neq emptyset$, i.e., F must contain one of the two endpoints of e as a vertex.

2. In part 1 we assumed that $p' = NN(p, \mathcal{P}) \in Skel(\mathcal{P})$ but that p' *was not* a vertex of $\mathcal{P}$. In this section we examine the case in which p' *is* a vertex of $\mathcal{P}$.

Let F be one of the faces incident upon p', Π its supporting plane, e and e_1

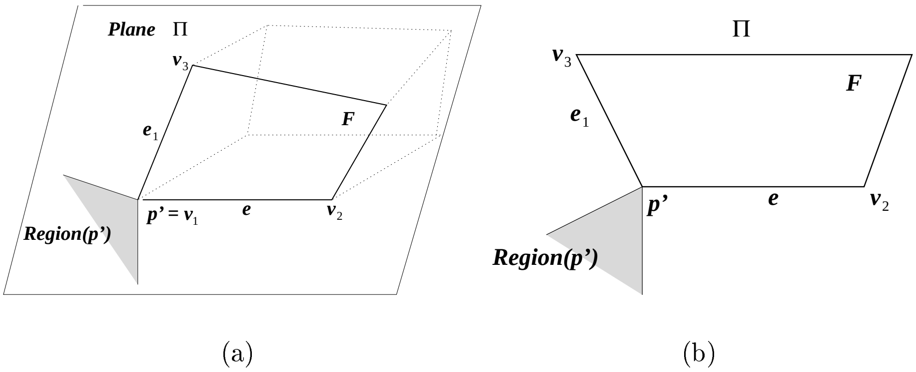


Figure 18: $Region(p')$.

the edges on F incident upon $p' = v_1$, v_2 and v_3 the other endpoints of e and e_1 respectively.

Working backwards, let $Region(p')$ be the set of possible locations of p'' such that $p' = NN(p'', F)$. See Figure 18. Given $p'' \in Region(p')$, let $A(p'')$ and $B(p'')$ be the closed subsets of F lying on the right and the left of the half line $\overrightarrow{p''p'}$. One of $A(p'')$ or $B(p'')$ might be empty but $A(p'') \cup B(p'') = F$. That is, for any point $q \in F$, q is either in $A(p'')$ or in $B(p'')$. Moreover, we have the property that

$$\begin{cases} \text{if } A(p'') \neq \emptyset, \text{ then } e \subset A(p'') \\ \text{if } B(p'') \neq \emptyset, \text{ then } e_1 \subset B(p''). \end{cases} \tag{56}$$

See Figure 19.

Let q be any point of F in some $\frac{\log n}{\sqrt{n}}$-good Type-III sphere $S = S(p, r)$ with $p' = NN(p, \mathcal{P})$. We divide our analysis into two cases: (i) $q \in S \cap A(p'')$ and (ii) $q \in S \cap B(p'')$.

(i) $q \in S \cap A(p'')$: Consider the orthogonal coordinate system (u, v) of Π whose origin is p' and whose u-axis lies on the line defined by extending e. W.L.O.G we may assume that F lies in the half plane $v \geq 0$ and $v_2 = (b, 0)$ is the other endpoint of e for some $b > 0$. Note that $q = (u, v)$ has $u \geq 0$ and $v \geq 0$. See Figure 19 for the possible cases.

Firstly we show that $q' = (\sqrt{u^2 + v^2}, 0) \in S$: It is no longer necessarily true that $p''p' \perp e$, so the proof of this fact is no longer as straightforward as it was in part 1.

Since $p'' \in Region(p')$ and $q \in A(p'')$, the angles $\angle p''p'q$ and $\angle p''p'q'$ satisfy that

$$90° \leq \angle p''p'q' < \angle p''p'q < 180°. \tag{57}$$

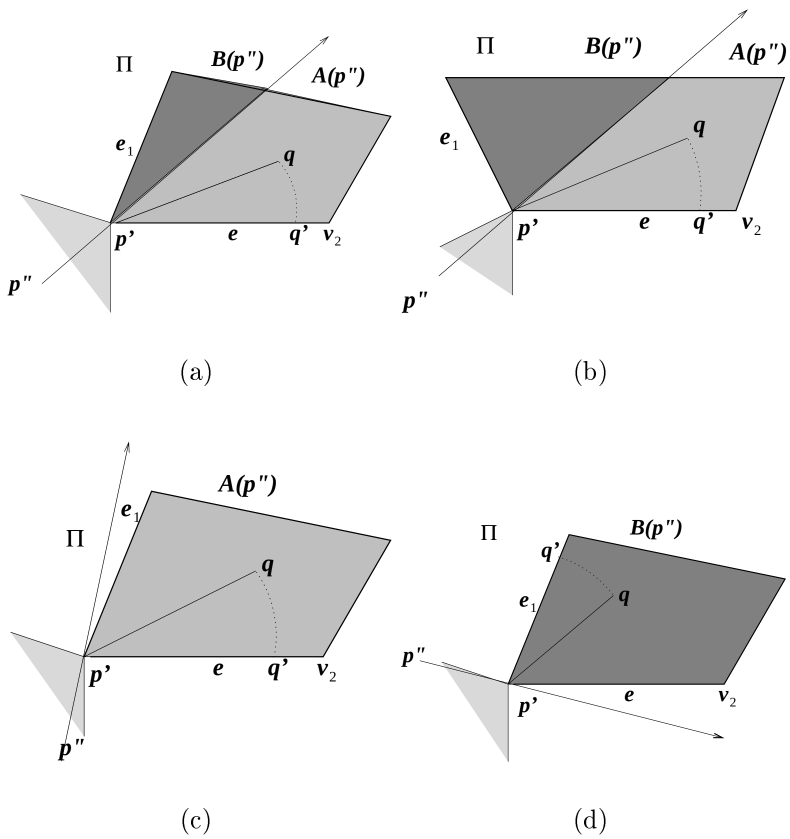


Figure 19: $A(p'')$ and $B(p'')$

The law of cosines gives us that

$$\left\{\begin{array}{ll} d(p'',q)^2 & = d(p'',p')^2 + d(p',q)^2 - 2\cos(\angle p''p'q) d(p'',p') d(p',q), \\ d(p'',q')^2 & = d(p'',p')^2 + d(p',q')^2 - 2\cos(\angle p''p'q') d(p'',p') d(p',q'). \end{array}\right.$$

Thus $d(p'',q) > d(p'',q')$ immediately follows from $d(p',q) = d(p',q') = \sqrt{u^2+v^2}$ and $\cos(\angle p''p'q) < \cos(\angle p''p'q') \leq 0$ (by (57)). Hence $q' \in S$ follows from (51).

Using the fact that $q' = (\sqrt{u^2+v^2}, 0) \in S$, we prove, in a fashion similar to (52), that

$$\text{If } q = (u,v) \in S \cap A(p''), \text{ then } v \leq 2\frac{\log n}{\sqrt{n}}. \tag{58}$$

If $\sqrt{u^2+v^2} \leq 2\frac{\log n}{\sqrt{n}}$, then v must be $\leq 2\frac{\log n}{\sqrt{n}}$, so we are done. If $\sqrt{u^2+v^2} > 2\frac{\log n}{\sqrt{n}}$, then we have $\left(\frac{\log n}{\sqrt{n}}, 0\right) \in S \cap F$: $\left(\frac{\log n}{\sqrt{n}}, 0\right) \in S$ follows from $(\sqrt{u^2+v^2}, 0) \in S$, and we may assume that $\left(\frac{\log n}{\sqrt{n}}, 0\right) \in F$ since otherwise for large enough n, the length of e is less than $\frac{\log n}{\sqrt{n}}$ which becomes infinitesimally small. Thus the triangle T with vertices $(0,0), (u,v)$, and $\left(\frac{\log n}{\sqrt{n}}, 0\right)$ is totally contained in $S \cap F$. By the $\frac{\log n}{\sqrt{n}}$-goodness of S, $Area(T) = \frac{1}{2} v \frac{\log n}{\sqrt{n}} \leq \left(\frac{\log n}{\sqrt{n}}\right)^2$ which implies that $v \leq 2\frac{\log n}{\sqrt{n}}$ and we are done.

We now prove that for the constant C of (54),

$$\left\{\begin{array}{l} \text{If } 0 \leq u \leq b, \text{ then } v \leq \frac{2C}{|u|}\left(\frac{\log n}{\sqrt{n}}\right)^2. \\ \text{If } u > b, \text{ then } v \leq \frac{2C}{b}\left(\frac{\log n}{\sqrt{n}}\right)^2. \end{array}\right. \tag{59}$$

Since $q' = (\sqrt{u^2+v^2}, 0) \in S$, $(u,0) \in S$. If $0 \leq u \leq b$, then $(u,0) \in F$, so $(u,0) \in S \cap F$. This implies that the triangle T_1 with vertices $(0,0), (u,v)$, and $(u,0)$ is contained in $S \cap F$ so by the $\frac{\log n}{\sqrt{n}}$-goodness of S, $Area(T_1) = \frac{1}{2}|u|v \leq \left(\frac{\log n}{\sqrt{n}}\right)^2$. Hence we have proven that for $0 \leq u \leq b$, $v \leq \frac{2C}{|u|}\left(\frac{\log n}{\sqrt{n}}\right)^2$ since by definition $C \geq 1$. If $u > b$, then $(b,0) \in S$ since $(u,0) \in S$ and $(b,0) = v_2 \in F$. Hence the triangle T_2 with vertices $(0,0), (u,v)$, and $(b,0)$ is contained in $S \cap F$, so again by the goodness of S $Area(T_2) = \frac{1}{2}bv \leq \left(\frac{\log n}{\sqrt{n}}\right)^2$. Thus for $u > b$, $v \leq \frac{2C}{b}\left(\frac{\log n}{\sqrt{n}}\right)^2$.

Combining (58) and (59), we have, for $q = (u,v) \in S \cap A(p'')$,

$$\left\{\begin{array}{l} \text{If } 0 \leq u \leq b, \text{ then } v \leq \min\left(2\frac{\log n}{\sqrt{n}}, \frac{2C}{|u|}\left(\frac{\log n}{\sqrt{n}}\right)^2\right). \\ \text{If } u > b, \text{ then } v \leq \min\left(2\frac{\log n}{\sqrt{n}}, \frac{2C}{b}\left(\frac{\log n}{\sqrt{n}}\right)^2\right). \end{array}\right. \tag{60}$$

Therefore

$$\begin{aligned} M(p') \cap A(p'') &\subseteq \{(u,v) \in A(p'') : (u,v) \text{ satisfies } (60)\} \\ &\subseteq \{(u,v) \in F : (u,v) \text{ satisfies } (60)\}. \end{aligned} \tag{61}$$

Note that the region on the right-hand side of (61) is not dependent on p'' but only upon e.

(ii) $q \in S \cap B(p'')$: Consider the new orthogonal coordinate system (u, v) of Π whose origin is p' and whose u-axis lies on the line defined by extending e_1. Let $v_3 = (-b', 0)$ for some $b' > 0$. Note that $q = (u, v) \in S \cap B(p'')$ satisfies $u \le 0$ and $v \ge 0$. See Figure 19.

By analogs of the proofs given in (i), we have, for $q = (u, v) \in S \cap B(p'')$

$$\begin{cases} \text{If } -b' \le u \le 0, \text{ then } v \le \min\left(2\frac{\log n}{\sqrt{n}}, \frac{2C}{|u|}\left(\frac{\log n}{\sqrt{n}}\right)^2\right). \\ \text{If } u < -b', \text{ then } v \le \min\left(2\frac{\log n}{\sqrt{n}}, \frac{2C}{b'}\left(\frac{\log n}{\sqrt{n}}\right)^2\right). \end{cases} \tag{62}$$

Therefore

$$\begin{aligned} M(p') \cap B(p'') &\subseteq \{(u,v) \in B(p'') : (u,v) \text{ satisfies } (62)\} \\ &\subseteq \{(u,v) \in F : (u,v) \text{ satisfies } (62)\}. \end{aligned} \tag{63}$$

Note that the region on the right-hand side of (63) is not dependent on p'' but only upon e_1.

Combining **(i)** and **(ii)**, we have

$$M(p') \cap F \subseteq \{q \in F : q \text{ satisfies either } (60) \text{ or } (62)\}. \tag{64}$$

Note that coordinate systems of (60) and (62) are different from each other, though, for convenience, we denote both by (u, v).

3. Up to now we have derived the structure of $M(p')$ for point $p' \in Skel(\mathcal{P})$. Now we will examine $M(s)$ and $Area(M(s))$ where $s \subset e$ is a line segment with $length(s) \le \frac{\log n}{\sqrt{n}}$. For any segment $s \subset e$ of length $\le \frac{\log n}{\sqrt{n}}$, we calculate $Area(M(s))$ by $Area(M(s)) = Area\left(\cup_{p' \in s} M(p')\right)$ and show that $Area(M(s)) \le c_3 \frac{\log^3 n}{n}$. To do this, we divide our analysis into three cases: **(i)** $v_1 \notin s$, $v_2 \notin s$; **(ii)** v_1 is the leftmost point of s; **(iii)** v_2 is the rightmost point of s. Note that at most one of $v_1 \in s$ or $v_2 \in s$ is possible because, if $v_1 \in s$ and $v_2 \in s$ then $length(e) \le length(s) \le \frac{\log n}{\sqrt{n}}$, which implies that for large enough n, $length(e)$ becomes infinitesimally small. Since **(iii)** is symmetric to **(ii)**, we investigate only

(i) and **(ii)**.

(i) $v_1 \notin s,\ v_2 \notin s$: Let F be either of the two faces incident upon e with its supporting plane Π. Consider the orthogonal coordinate system (u,v) of Π such that the origin is v_1, the u-axis lies on the line defined by extending e and v_2 corresponds to the point with coordinate $(b,0)$ where b is the length of e. Let $(u_1,0)$ be the leftmost point of s and $(u_2,0)$ the rightmost point of s. Since we will consider only cases (i) and (ii), we may assume that $u_2 < b$.

Note that for all $p' \in s$, $M(p') \cap F$ satisfies (54). Let t be such that $p' = (t,0)$. Then $0 < u_1 \le t \le u_2 < b$. Define the functions $f(u)$ and $f_t(u)$ by

$$\begin{aligned} f(u) &= \begin{cases} 2\frac{\log n}{\sqrt{n}}, & |u| \le C\frac{\log n}{\sqrt{n}} \\ \frac{2C}{|u|}(\frac{\log n}{\sqrt{n}})^2, & |u| \ge C\frac{\log n}{\sqrt{n}}, \end{cases} && (65) \\ f_t(u) &= f(u-t) \text{ for } t > 0. && (66) \end{aligned}$$

See Figure 20. Note that (a) $f(u) = f(-u)$, (b) f is nonincreasing, and (c) f is continuous. (c) follows from the fact that at $u = C\frac{\log n}{\sqrt{n}}$, $2\frac{\log n}{\sqrt{n}} = \frac{2C}{|u|}(\frac{\log n}{\sqrt{n}})^2$. The function f_t is the function f translated by t. Using these functions, (54) can be rewritten as

$$M(p') \cap F \subseteq \{(u,v) \in F \,:\, 0 \le v \le f_t(u)\}.$$

Since $(t,0) = p' \in s$ implies $u_1 \le t \le u_2$, we have

$$M(s) \cap F \subseteq \left\{(u,v) \in F \,:\, 0 \le v \le \sup_{u_1 \le t \le u_2} f_t(u)\right\}. \tag{67}$$

Let $H_s(u) = \sup_{u_1 \le t \le u_2} f_t(u)$. By the definition of $H_s(u)$ and $f_t(u)$,

$$H_s(u) = \sup_{u_1 \le t \le u_2} f_t(u) = \begin{cases} f_{u_1}(u) & u \le u_1 \\ 2\frac{\log n}{\sqrt{n}} & u_1 \le u \le u_2 \\ f_{u_2}(u) & u \ge u_2 \end{cases} \tag{68}$$

See Figure 21(a). Let L be the diameter of F. Then

$$\begin{aligned} Area(M(s) \cap F) &\le \int_{u_1 - L}^{u_1} f_{u_1}(u)du + \int_{u_1}^{u_2} 2\frac{\log n}{\sqrt{n}}du + \int_{u_2}^{u_2+L} f_{u_2}(u)du \\ &= 2\int_0^L f(u)du + \int_{u_1}^{u_2} 2\frac{\log n}{\sqrt{n}}du \quad \text{by (66)} && (69) \\ &= O\left(\frac{\log^3 n}{n}\right) \end{aligned}$$

where the constant implicit in $O()$ is dependent only upon $\mathcal{P}$. Hence we just upperbounded $Area(M(s) \cap F)$ for either of the two faces incident upon e.

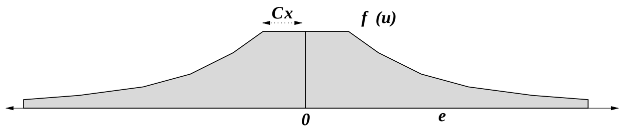
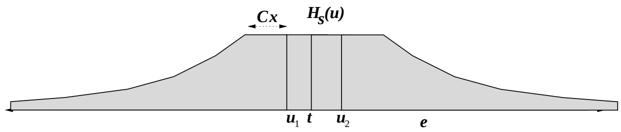
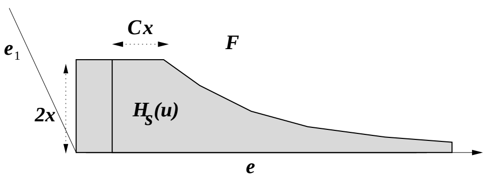


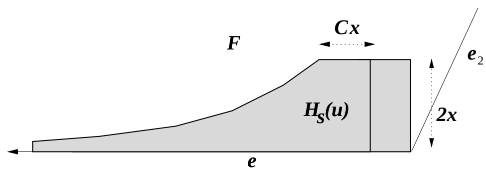
Figure 20: Function $f(u)$



(a) $H_s(u) = \sup_{u_1 \le t \le u_2} f_t(u)$ when $v_1 \notin s$, $v_2 \notin s$



(b) Right half of $H_s(u)$ when v_1 is the leftmost point of s



(c) Left half of $H_s(u)$ when v_2 is the rightmost point of s

Figure 21: $H_s(u) = \sup_{u_1 \le t \le u_2} f_t(u)$ for a segment s of edge e with $length(s) \le \frac{\log n}{\sqrt{n}}$.

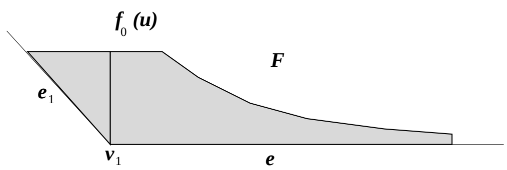


Figure 22: Function $f_0(u)$ describing $M(p')$ with $p' = v_1 = (0, 0)$.

Now let F be a face of $\mathcal{P}$ which is not incident upon e. Then $M(p') \cap F$ must satisfy (55) and $M(p') \cap F = \emptyset$ if F is incident upon neither v_1 nor v_2. Let F be incident upon v_1, θ' the angle between e and F. Note that $\forall q \in F$ lying inside or on L_e, $q \in S\left(v_1, \frac{1}{\sin\theta'} c_0 \frac{\log n}{\sqrt{n}}\right)$. From this fact and (55), we have

$$M(p') \cap F \subseteq S\left(v_1, \frac{1}{\sin\theta'} c_0 \frac{\log n}{\sqrt{n}}\right) \cap F.$$

Since the right-hand side of the above inequality is independent of p',

$$M(s) \cap F \subseteq S\left(v_1, \frac{1}{\sin\theta'} c_0 \frac{\log n}{\sqrt{n}}\right) \cap F,$$

so

$$Area\,(M(s) \cap F) = O\left(\frac{\log^2 n}{n}\right) \tag{70}$$

where the constant in the notation $O()$ is dependent only upon c_0 and θ'.

(ii) v_1 is the leftmost point of s; Let F be either of the two faces incident upon e with its supporting plane Π. Recall that $Area(M(s) \cap F) = Area\left(\cup_{p' \in s} M(p') \cap F\right)$. Note that for all $p' \in s$ except for the leftmost point, $M(p') \cap F$ satisfies (54). For simplicity, we bound $Area(M(s) \cap F)$ as follows:

$$Area(M(s) \cap F) \leq Area(M(v_1) \cap F) + Area\left(\cup_{p' \in s, p' \neq v_1} M(p') \cap F\right).$$

The analysis in **(i)** shows that,

$$Area\left(\cup_{p' \in s, p' \neq v_1} M(p') \cap F\right) = O\left(\frac{\log^3 n}{n}\right).$$

We therefore only have to bound $Area(M(v_1) \cap F)$, for which we will use (64).

Recall that $b = length(e)$ and $b' = length(e_1)$. We may assume that $b \geq C\frac{\log n}{\sqrt{n}}$ and $b' \geq C\frac{\log n}{\sqrt{n}}$ since otherwise, e or e_1 have length less than $C\frac{\log n}{\sqrt{n}}$. Thus in (60) and (62), the inequalities

$$v \leq \min\left(2\frac{\log n}{\sqrt{n}}, \frac{2C}{b}\left(\frac{\log n}{\sqrt{n}}\right)^2\right) \text{ and } v \leq \min\left(2\frac{\log n}{\sqrt{n}}, \frac{2C}{b'}\left(\frac{\log n}{\sqrt{n}}\right)^2\right)$$

become, respectively,

$$v \leq \frac{2C}{b}\left(\frac{\log n}{\sqrt{n}}\right)^2 \text{ and } v \leq \frac{2C}{b'}\left(\frac{\log n}{\sqrt{n}}\right)^2.$$

Setting $b'' = \min(b, b')$, we can replace (61) by

$$M(v_1) \cap A(p'') \subseteq \left\{ (u,v) \in F : (u,v) \text{satisfies} \begin{pmatrix} \text{If } 0 \le u \le b'', \text{ then } v \le \frac{2C}{|u|}\left(\frac{\log n}{\sqrt{n}}\right)^2. \\ \text{If } u > b'', \text{ then } v \le \frac{2C}{b''}\left(\frac{\log n}{\sqrt{n}}\right)^2. \end{pmatrix} \right\} \tag{71}$$

and (63) by

$$M(v_1) \cap B(p'') \subseteq \left\{ (u,v) \in F : (u,v) \text{satisfies} \begin{pmatrix} \text{If } -b'' \le u \le 0, \text{ then } v \le \frac{2C}{|u|}\left(\frac{\log n}{\sqrt{n}}\right)^2. \\ \text{If } u < -b'', \text{ then } v \le \frac{2C}{b''}\left(\frac{\log n}{\sqrt{n}}\right)^2. \end{pmatrix} \right\} \tag{72}$$

To analyze $Area(M(v_1) \cap F)$, we define function $f_0(u)$ by

$$f_0(u) = \begin{cases} 2\frac{\log n}{\sqrt{n}} & |u| \le C\frac{\log n}{\sqrt{n}}, \\ \frac{2C}{|u|}\left(\frac{\log n}{\sqrt{n}}\right)^2, & C\frac{\log n}{\sqrt{n}} \le |u| \le b'' \\ \frac{2C}{b''}\left(\frac{\log n}{\sqrt{n}}\right)^2, & |u| \ge b''. \end{cases} \tag{73}$$

See Figure 22. Using this definition, (71), and (72), we can rewrite (64) as

$$M(v_1) \cap F \subseteq \{(u,v) \in F : 0 \le v \le f_0(u)\}.$$

Let L be the diameter of F. Then

$$\begin{aligned} Area(M(v_1) \cap F) &\le \int_{-L}^{L} f_0(u)du = 2\int_0^L f_0(u)du \\ &= 2\left(\int_0^{C\frac{\log n}{\sqrt{n}}} 2\frac{\log n}{\sqrt{n}}du + \int_{C\frac{\log n}{\sqrt{n}}}^{b''} \frac{2C}{|u|}\left(\frac{\log n}{\sqrt{n}}\right)^2 du\right) \\ &\quad +2\int_{b''}^{L} \frac{2C}{b''}\left(\frac{\log n}{\sqrt{n}}\right)^2 du \quad \text{by (73)} \\ &= O\left(\frac{\log^3 n}{n}\right) \end{aligned} \tag{74}$$

where the constant implicit in $O()$ is dependent only upon L and F.

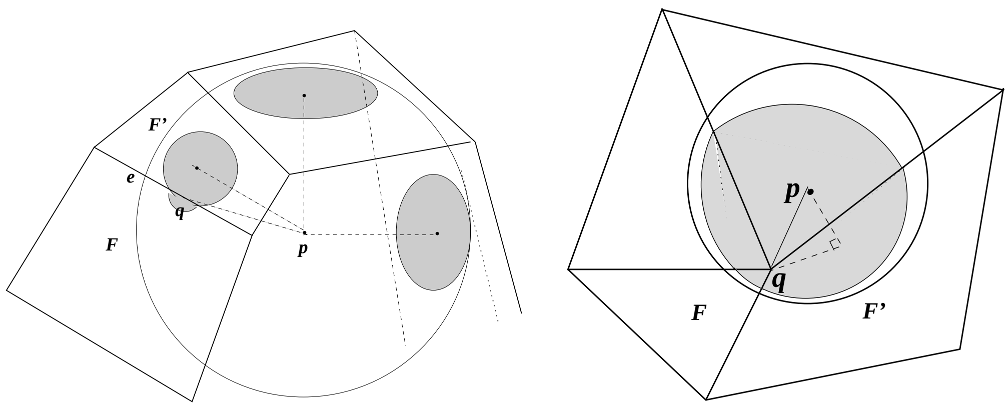
Now let F be one of the faces incident upon v_1. Then in

$$Area(M(s) \cap F) \le Area(M(v_1) \cap F) + Area\left(\cup_{p' \in s, p' \ne v_1} M(p') \cap F\right),$$

$Area(M(v_1) \cap F) = O\left(\frac{\log^3 n}{n}\right)$ by the same analysis as before and

$$Area\left(\cup_{p' \in s, p' \ne v_1} M(p') \cap F\right) = O\left(\frac{\log^2 n}{n}\right)$$

by (70), so we've done. $\square$



(a) $q = NN(p, F)$ is on the edge e (b) $q = NN(p, F)$ is a vertex

Figure 23: Type-III spheres $S = S(p, r)$ with p *inside* $\mathcal{P}$. Note that the nearest neighbor of p in F is q which is on the boundary of F. The line segment pq is not perpendicular to F which implies that the center of the intersection circle of S with the supporting plane of F is not in F.

This concludes the analyses of Case 3.

Combining the proven $O(n)$ bounds for Case 1, Case 2, and Case 3, we see that the expected total number of $\frac{\log n}{\sqrt{n}}$-good Type-III spheres $S(p, r)$ with p *outside or on* $\mathcal{P}$ is $O(n)$ and we are done with this part.

5.2 Sphere center p *inside* $\mathcal{P}$

We have just analyzed the number of $\frac{\log n}{\sqrt{n}}$-good Type-III spheres $S = S(p, r)$ when p is *outside* $\mathcal{P}$. In this subsection we analyze the case when p is *inside* $\mathcal{P}$. In this case, from the definition of Type-III spheres, we know that $\exists$ a face F of $\mathcal{P}$, such that $F \cap S \neq \emptyset$ but $p'' = NN(p, \Pi)$ is not in F, where Π is the supporting plane of F.

The main lemma we prove here is

Lemma 12 *Let $S = S(p, r)$ be a $\frac{\log n}{\sqrt{n}}$-good Type-III sphere with p inside $\mathcal{P}$. Furthermore, suppose $\exists$ a face F of $\mathcal{P}$, such that $F \cap S \neq \emptyset$ but $p'' = NN(p, \Pi)$ is not in F, where Π is the supporting plane of F. Then $\exists q$ on the boundary of F such that, $\forall u \in S(p, r)$, $d(q, u) \leq c_4 \frac{\log n}{\sqrt{n}}$, where c_4 is dependent only upon $\mathcal{P}$.*

The proof of this lemma involves showing that if $S(p, r)$ satisfies the given conditions then r is actually some small multiple of $\frac{\log n}{\sqrt{n}}$, so the distance between q, the nearest point to p on F, and any point in $S(p, r)$ is $\leq c_4 \frac{\log n}{\sqrt{n}}$.

Given Lemma 12 we can use *exactly* the same type of segment partitioning analysis as was employed at the end of Case 2 of subsection 5.1 (note the similarity of Lemma 12 to Lemma 10) to show that the expected number of $\frac{\log n}{\sqrt{n}}$-good Type-III spheres $S(p,r)$ with p *inside* $\mathcal{P}$ is $O(n)$. It therefore only remains to prove Lemma 12.

Proof. [Proof of Lemma 12]

Let F be the face such that $F \cap S \neq \emptyset$ but $p'' = NN(p,\Pi) \notin F$, and let $q = NN(p,F)$. Note that $pq \not\perp F$ and q must be on the boundary of F. Since $F \cap S \neq \emptyset$, $q \in S$. See Figure 23

1. We first assume that q is on some edge $e \in F$ but is not a vertex of F. Let F' be the other face incident to e and Π' its corresponding supporting plane. Let T be the plane passing through q and perpendicular to e. Since $pq \perp e$, $p \in T$. Orient T so that segment $F' \cap T$ is followed by segment $F \cap T$ in the counterclockwise order of segments of $\mathcal{P} \cap T$. See Figure 24(a).

Let θ be the angle between F and F' measured inside $\mathcal{P}$. Note that $pq \perp e$. Since $T \perp e = F \cap F'$, angle θ is exactly the angle between $F \cap T$ and $F' \cap T$ measured inside $\mathcal{P} \cap T$ on T. See Figure 24(b). We first show that $\theta > 90°$; suppose by contradiction that $\theta \leq 90°$. Let $\angle FqF'$ be the angle between the two half lines $\overrightarrow{qF}$ and $\overrightarrow{qF'}$, defined by extending $F \cap T$ and $F' \cap T$, respectively. Let p''' be the perpendicular dropped from p onto $\overrightarrow{qF}$. Since $\overrightarrow{qF}$ is an extension of $F \cap T$, there must be points $q' \in \overrightarrow{qF}$ such that $q' \neq q$, $qq' \subseteq qp'''$, and $qq' \subseteq F \cap T$. The first and second condition guarantee that $length(pq') < length(pq)$, and the third one implies that $q' \in F$, so the existence of such q' contradicts $q = NN(p,F)$. Thus $\theta > 90°$. See Figure 24(c).

Now we consider the plane T' such that $e \subset T'$ and $T' \perp F$; $(T \cap T') \perp (F \cap T)$ and p is to the left of $T \cap T'$ on T. See Figure 24(b). In the following we will work only in the plane T. Let h be the perpendicular dropped from p onto $\Pi' \cap T$. Let $\theta' = \angle pqh$. Note that $0 \leq \theta' < \theta - 90°$ where $\theta > 90°$. There are two cases: either $h \in F' \cap T$ or $h \notin F' \cap T$. In both cases, if we show that $length(qh) \leq C\frac{\log n}{\sqrt{n}}$ for some constant C dependent only upon θ and c_0, the constant from Definition 7, then using the facts that $length(pq) = \frac{1}{\cos\theta'} length(qh)$ and $\theta' < \theta - 90°$, we will have

$$\begin{aligned} length(pq) &= \tfrac{1}{\cos\theta'} length(qh) \\ &< \tfrac{1}{\cos(\theta-90°)} length(qh) = \tfrac{1}{\sin\theta} length(qh) < \tfrac{C}{\sin\theta}\tfrac{\log n}{\sqrt{n}}. \end{aligned} \tag{75}$$

This will imply that

$$\forall u \in S(p,r),\ d(q,u) \leq 2\left(\frac{C}{\sin\theta} + c_0\right)\frac{\log n}{\sqrt{n}}, \tag{76}$$

which proves the lemma by setting $c_4 = 2\left(\frac{C}{\sin\theta} + c_0\right)$. To see why (75) implies (76) consider the sphere centered at q with radius $c_0\frac{\log n}{\sqrt{n}}$. As usual we denote this sphere by $S(q, c_0\frac{\log n}{\sqrt{n}})$. Definition 7 tells us that $Area\left(S(q, c_0\frac{\log n}{\sqrt{n}}) \cap F\right) \geq$

$\left(\frac{\log n}{\sqrt{n}}\right)^2$. Since $S = S(p,r)$ is $\frac{\log n}{\sqrt{n}}$-good, $S(q, c_0\frac{\log n}{\sqrt{n}}) \not\subseteq S$. By the triangle inequality, if $r \geq d(p,q) + c_0\frac{\log n}{\sqrt{n}}$, then $S(q, c_0\frac{\log n}{\sqrt{n}}) \subseteq S$. Thus

$$r < d(p,q) + c_0\frac{\log n}{\sqrt{n}} < \left(\frac{C}{\sin\theta} + c_0\right)\frac{\log n}{\sqrt{n}}.$$

Since

$$\forall u \in S(p,r), \quad d(q,u) \leq d(q,p) + d(p,u) \leq 2r \leq 2\left(\frac{C}{\sin\theta} + c_0\right)\frac{\log n}{\sqrt{n}},$$

this proves (76). Hence to complete the proof of part 1 of the lemma it only remains to prove that $length(qh) \leq C\frac{\log n}{\sqrt{n}}$ for some constant C dependent only upon c_0 and θ. See Figure 24(b).

If $h \in F' \cap T$, note that by the definition of h and T, we have $ph \perp \Pi'$ where Π' is the supporting plane of F'. This means that $h = NN(p, \Pi')$ so $S \cap \Pi' = D_{\Pi'}(h, r')$ for some $r' > 0$. Since $q \in F'$ this implies $r' > length(qh)$. But, from Definition 7 we know that $r' < c_0\frac{\log n}{\sqrt{n}}$ since otherwise

$$\text{Area}(S \cap \mathcal{P}) \geq \text{Area}\left(D_{\Pi'}(h, r') \cap F'\right) \geq \left(\frac{\log n}{\sqrt{n}}\right)^2, \tag{77}$$

contradicting the $\frac{\log n}{\sqrt{n}}$-goodness of S. Thus $h \in F' \cap T$ implies $length(qh) \leq c_0\frac{\log n}{\sqrt{n}}$.

If $h \notin F' \cap T$, then we claim that $length(qh) \leq \frac{c_0}{1+\cos\theta}\frac{\log n}{\sqrt{n}}$; Suppose that $length(qh) > \frac{c_0}{1+\cos\theta}\frac{\log n}{\sqrt{n}}$. Then we have

$$\begin{array}{rcll} length(pq) - length(ph) & = & \frac{1}{\cos\theta'}length(qh) - (\tan\theta')length(qh) & \\ & = & \frac{1-\sin\theta'}{\cos\theta'}length(qh) & \\ & > & \frac{1-\sin\theta'}{\cos\theta'}\frac{c_0}{1+\cos\theta}\frac{\log n}{\sqrt{n}} & \\ & > & \frac{1+\cos\theta}{\cos\theta'}\frac{c_0}{1+\cos\theta}\frac{\log n}{\sqrt{n}} & \text{since } \sin\theta' < -\cos\theta \\ & > & c_0\frac{\log n}{\sqrt{n}}. & \end{array} \tag{78}$$

Since $h \notin F' \cap T$, h must be outside $\mathcal{P} \cap T$. Since p is inside $\mathcal{P} \cap T$ and h is outside $\mathcal{P} \cap T$, there must be a point h' such that $h' \in ph \cap \mathcal{P} \cap T$. For such h', (78) gives that

$$length(pq) - length(ph') > length(pq) - length(ph) > c_0\frac{\log n}{\sqrt{n}}.$$

Since S has radius $\geq length(pq)$, this means that the disc centered at h' with radius $c_0\frac{\log n}{\sqrt{n}}$ is in S. By Definition 7 , this contradicts the goodness of S. Hence we have proven that if $h \in F' \cap T$, then $length(qh) \leq c_0\frac{\log n}{\sqrt{n}}$ and if $h \notin F' \cap T$, then

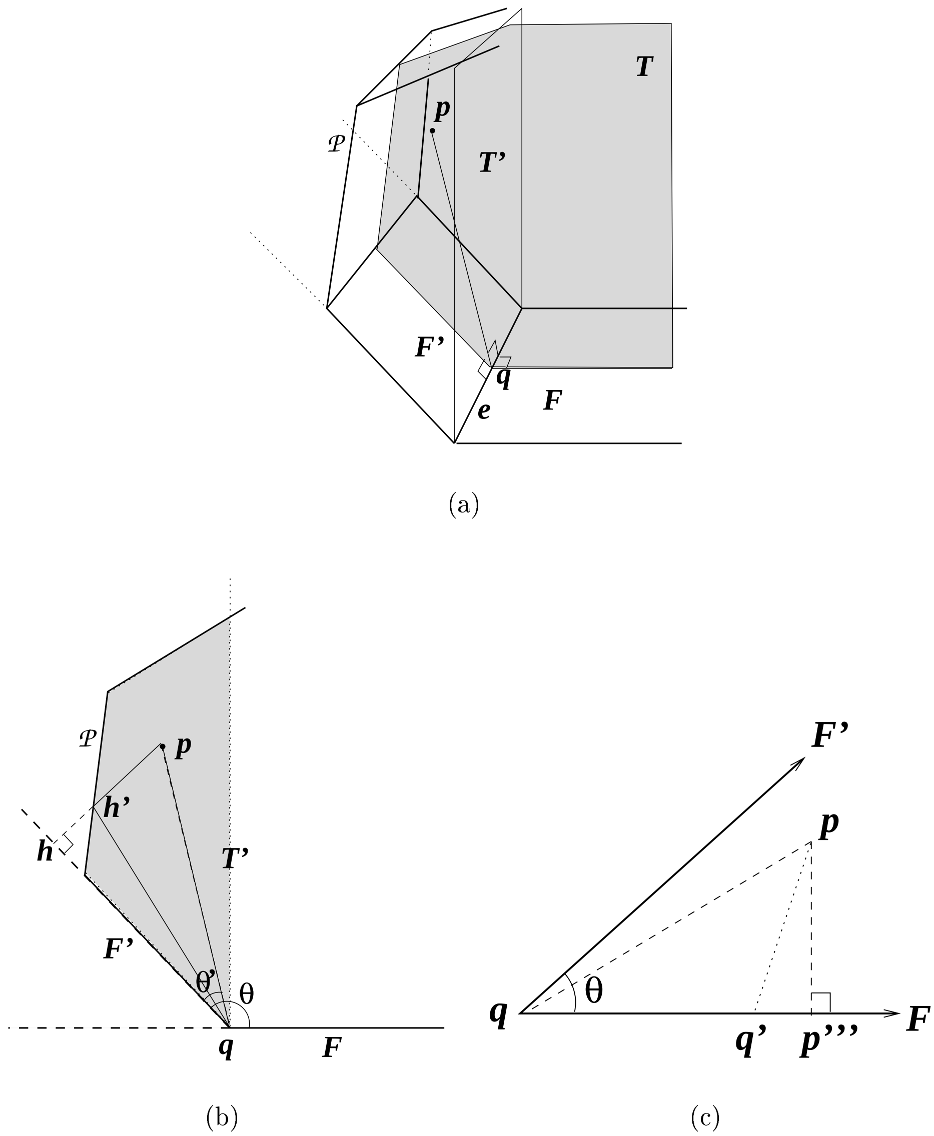
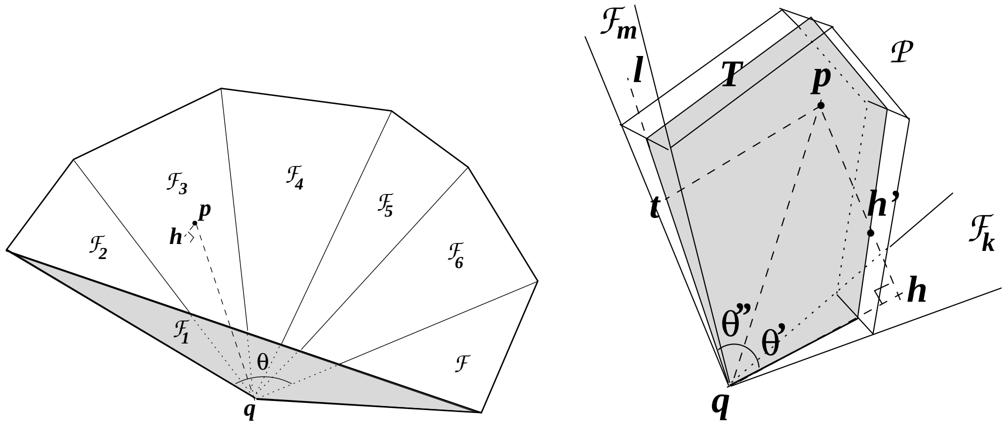


Figure 24: Proof of Lemma 12 when $q = NN(p, F)$ is on the edge e.



(a) $\mathcal{F}_k = \mathcal{F}_3$ since $h \in \mathcal{F}_3$.

(b) $\theta' \leq \frac{\theta}{2}$ where $\theta = \sup_{x,y \in \mathcal{C}} \angle xqy$ and $h = NN(p, \mathcal{C})$.

Figure 25: Proof of Lemma 12 when $q = NN(p, F)$ is a vertex.

$length(qh) \leq \frac{c_0}{1+\cos\theta} \frac{\log n}{\sqrt{n}}$. By letting $C = \max\{c_0, \frac{c_0}{1+\cos\theta}\}$, $length(qh) \leq C \frac{\log n}{\sqrt{n}}$, so we are done.

2. Assume that q is a vertex of F and thus of $\mathcal{P}$ as well. Let $\mathcal{C}$ be the cone defined by extending the faces of $\mathcal{P}$ incident upon q. Let $F_1, \cdots, F_n$ be the other faces of $\mathcal{P}$ incident upon q and $\mathcal{F}_1, \cdots, \mathcal{F}_n$ their corresponding faces of $\mathcal{C}$, respectively. Let $\mathcal{F}$ be the face of $\mathcal{C}$ corresponding to F. Note that $F \subset \mathcal{F} \subset \Pi$. Let $\theta = \sup_{x,y \in \mathcal{C}} \angle xqy$. Then $0 < \theta < 180°$.

Let $h = NN(p, \mathcal{C})$. Since $\mathcal{C}$ is a convex body and p is inside $\mathcal{C}$, h must be in some face of $\mathcal{C}$ but be neither on an edge nor on the vertex q; suppose, to the contrary, that h is on an edge e or the vertex q of $\mathcal{C}$. If h is on an edge e, then $ph \perp e$. Let H be the plane perpendicular to e and passing through h. Note that H contains p and $H \cap \mathcal{C}$ is a convex polygon such that p lies in its interior and h is one of the vertices. Since $h = NN(p, \mathcal{C})$, $h = NN(p, H \cap \mathcal{C})$, which contradicts Lemma 5. If $h = q$, then let H be any plane passing through h and p. Then $H \cap \mathcal{C}$ is a dihedral angle composed of two half lines incident upon h and with q lying inside. Since $h = NN(p, \mathcal{C})$, $h = NN(p, H \cap \mathcal{C})$, which again contradicts Lemma 5.

We also have that $h \notin \mathcal{F}$, since if $h \in \mathcal{F}$, there must be points $q' \in qh$ such that $q' \neq q$, $qq' \subseteq qh$, and $qq' \subseteq F$, and this yields a contradiction (the first and the second conditions guarantee that $length(pq') < length(pq)$, and the third one implies that $q' \in F$, so the existence of such q' contradicts $q = NN(p, F)$). Let $\mathcal{F}_k \neq \mathcal{F}$ be the face of $\mathcal{C}$ such that $h \in \mathcal{F}_k$. See Figure 25(a).

Let T be the plane passing through points q, h, and p. $T \cap \mathcal{C}$ is composed of two half lines rooted at vertex q. One is $\overrightarrow{qh}$ and let l be the other half line. Let $\mathcal{F}_m$ be the face of $\mathcal{C}$ such that $l \subset \mathcal{F}_m$. See Figure 25(b). Let $\theta' = \angle pqh$ and θ'' the angle between $\overrightarrow{qh}$ and l measured inside $\mathcal{P}$. Note that $\theta' + \theta'' \leq \theta < 180°$. We claim that $\theta' \leq \frac{\theta}{2}$. Suppose by contradiction that $\theta' > \frac{\theta}{2}$. Then $\theta'' \leq \theta - \theta' < \frac{\theta}{2} < 90°$. This means that the perpendicular t, dropped from p onto the line defined by extending l, is on l and thus on $\mathcal{C}$. Since $\theta'' < \frac{\theta}{2} < \theta'$, we have $d(p, \mathcal{C}) \leq d(p,t) < d(p,h)$, which contradicts the definition of h.

We will now prove, in a fashion similar to part 1, that $length(qh) \leq C\frac{\log n}{\sqrt{n}}$ for some constant C dependent only upon θ and c_0. Using the facts that $length(pq) = \frac{1}{\cos\theta'}length(qh)$ and $\theta' \leq \frac{\theta}{2}$, we will then have

$$length(pq) = \frac{1}{\cos\theta'}length(qh) \leq \frac{1}{\cos\frac{\theta}{2}}length(qh) \leq \frac{C}{\cos\frac{\theta}{2}}\frac{\log n}{\sqrt{n}}.$$

As in the proof of part 1 we know that if $r \geq d(p,q) + c_0\frac{\log n}{\sqrt{n}}$ then $S(q, c_0\frac{\log n}{\sqrt{n}}) \subseteq S$. But, from Definition 7 the $\frac{\log n}{\sqrt{n}}$-goodness of S forces $S(q, c_0\frac{\log n}{\sqrt{n}}) \not\subseteq S$. Thus $r < d(p,q) + c_0\frac{\log n}{\sqrt{n}}$. This in turn implies, again as in the proof of part 1, that

$$\forall u \in S(p,r), \quad d(q,u) \leq d(q,p) + d(p,u) \leq 2r \leq 2\left(\frac{C}{\cos\frac{\theta}{2}} + c_0\right)\frac{\log n}{\sqrt{n}}. \tag{79}$$

Setting $c_4 = 2\left(\frac{C}{\cos\frac{\theta}{2}} + c_0\right)$, we will be done.

We now prove that $length(qh) \leq C\frac{\log n}{\sqrt{n}}$. See Figure 25(b). Recall that $h \in \mathcal{F}_k$. There are two cases; either $h \in F_k$, or $h \notin F_k$.

If $h \in F_k$, then note that $q = NN(p,F) \in S \cap F_k$. By definition $ph \perp \Pi_k$, the supporting plane of F_k. Thus $h = NN(p, \Pi_k)$ so $S \cap \Pi_k = D_{\Pi_k}(h, r')$ for some $r' > 0$. Since $q \in \Pi_k \cap S$ we have $r' > length(qh)$. But, from Definition 7, the $\frac{\log n}{\sqrt{n}}$-goodness of S and the fact that $h \in F_k$, we find that $r' \leq c_0\frac{\log n}{\sqrt{n}}$ proving that $length(qh) < c_0\frac{\log n}{\sqrt{n}}$.

If $h \notin F_k$, then we claim that $length(qh) \leq \frac{c_0}{1-\sin\frac{\theta}{2}}\frac{\log n}{\sqrt{n}}$; Suppose that $length(qh) > \frac{c_0}{1-\sin\frac{\theta}{2}}\frac{\log n}{\sqrt{n}}$. Then we have

$$\begin{array}{rcll} length(pq) - length(ph) & = & \frac{1}{\cos\theta'}length(qh) - (\tan\theta')length(qh) & \\ & = & \frac{1-\sin\theta'}{\cos\theta'}length(qh) & \\ & > & \frac{1-\sin\theta'}{\cos\theta'}\frac{c_0}{1-\sin\frac{\theta}{2}}\frac{\log n}{\sqrt{n}} & \\ & \geq & \frac{1-\sin\frac{\theta}{2}}{\cos\theta'}\frac{c_0}{1-\sin\frac{\theta}{2}}\frac{\log n}{\sqrt{n}} & \text{since } \sin\theta' \leq \sin\frac{\theta}{2} \\ & \geq & c_0\frac{\log n}{\sqrt{n}}. & \end{array} \tag{80}$$

Since $h \in \mathcal{F}_k \setminus F_k$, h must be outside $\mathcal{P}$. Since p is inside $\mathcal{P}$ and h is outside $\mathcal{P}$, there must be a point h' such that $h' \in ph \cap \mathcal{P}$. For such h', (80) gives that

$$length(pq) - length(ph') > length(pq) - length(ph) > c_0\frac{\log n}{\sqrt{n}},$$

which means that the disc centered at h' with radius $c_0 \frac{\log n}{\sqrt{n}}$ is in S. By Definition 7, this contradicts the goodness of S. Hence we have proven that if $h \in F_k$, then $length(qh) \leq c_0 \frac{\log n}{\sqrt{n}}$ and if $h \notin F_k$, then $length(qh) \leq \frac{c_0}{1-\sin\frac{\theta}{2}} \frac{\log n}{\sqrt{n}}$. By letting $C = \max\{c_0, \frac{c_0}{1-\sin\frac{\theta}{2}}\}$, $length(qh) \leq C\frac{\log n}{\sqrt{n}}$, so we are done. $\square$

6 Review and Open Problems

In this paper we proved that if points S_n were chosen from the surface of a convex polytope $\mathcal{P}$ with a Poisson process of rate n then the expected complexity of the Voronoi diagram of S_n, i.e., the expected number of Voronoi vertices, faces and edges, is $O(n)$. Equivalently, the expected complexity of the dual of the Voronoi diagram, the Delaunay triangulation, is $O(n)$. This means that the expected number of Delaunay tetrahedra, faces and edges is $O(n)$.

As stated in the first section, for reasons of mathematical simplicity we proved our result for the Poisson distribution but the result will still hold if the n points are chosen IID from the uniform distribution over the surface of $\mathcal{P}$. That is, for n points chosen IID from the uniform distribution over $\mathcal{P}$ the expected complexity of their Voronoi diagram is also $O(n)$.

To the best of our knowledge this is the first analysis of the expected complexity of the the 3-dimensional Voronoi triangulation of points chosen from any 2-dimensional surface. In fact, the problem of analyzing the complexity, be it average or worst case, for such points, seems quite a new problem with the only known results being the very recent ones of Attali and Boissonnat [1] and Erikson [7] who discuss the worst case complexity when n points are 'well-sampled' from certain types of surfaces. Attali and Boissonnat show that in their case the complexity is $O(n^{7/4})$. Erikson proves that there is a set of n points on the cylinder with Voronoi complexity $\Omega(n^{3/2})$.

This paucity of research is probably due to the fact that it is only relatively recently that the practical problem of constructing Voronoi triangulations for such point sets has become important in the general geometric community, e.g., for surface reconstruction and modelling. The lack of work in this area means that most problems remain open. One very general problem would be, *given $\mathcal{M}$, a 2-dimensional manifold in 3-dimensional space, choose a set of points from the Poisson distribution with rate n over $\mathcal{M}$ or a set of n points IID from the uniform distribution over $\mathcal{M}$ Give an expression for how the complexity of the Voronoi diagram grows as a function of n.* Of course, this growth would depend upon the particular manifold $\mathcal{M}$; an interesting problem would be to attack this problem for different classes of $\mathcal{M}$. In this paper we solved this problem for the class $\mathcal{M}$ of boundaries of convex polytope. Another extension would be to fix $\mathcal{M}$, and do the analysis for different distributions over $\mathcal{M}$ that are not uniform but depend somehow on features of $\mathcal{M}$, e.g., its curvature.

We end with a few more comments on our results. Our analysis was only of the *expected* complexity of the Voronoi diagram. It said nothing about how *concentrated* the complexity is around its $O(n)$ mean. We state without proof the fact that it is possible to straightforwardly modify our results to show that if points are chosen from the Poisson distribution with rate n over convex polytope $\mathcal{P}$ then *the probability that the complexity of the Voronoi diagram is greater than* $n \log^2 n$ *is* $n^{-\Omega(\log n)}$. (The $n \log^2 n$ is quite loose and it is very possible that it can be improved. We do not prove this concentration theorem here because this paper is already quite long and introducing this new analysis would not introduce any new techniques or ideas, just many new long equations.)

Finally, we discuss a possible different approach towards solving our problem. An upper level view of our analysis is that we bounded the complexity of the Voronoi diagram of point set S_n by partitioning the Voronoi vertices and their corresponding spheres into two parts and bounding each part separately. The first part consisted of spheres whose centers are outside the polytope or inside the polytope but very close to the edges. We showed that the expected number of such spheres is $o(n)$, i.e., very small. The second part consisted of those spheres whose centers were inside the polytope and not that close to the edges. The number of such spheres was the dominant term. We analyzed this number by working backwards, and identifying where such spheres could intersect the polytope on a particular face. Given the intersection region we then identified which points, both on that face and other faces, could be on the border of a sphere without causing the sphere's intersection with $\mathcal{P}$ to be so large that it must contain some points and therefore not be a Voronoi sphere.

Although we did not mention it at the time there is another way of attempting to attack the enumeration of the second type of Voronoi spheres. Let p be a point inside of $\mathcal{P}$ and $r_p = d(p, \mathcal{P})$. The *Medial Axis* of polytope $\mathcal{P}$ is the set of points p inside $\mathcal{P}$ such that the sphere $S(p, r_p)$ touches at least two points of $\mathcal{P}$. Intuitively, if a point p is very far from the medial axis then any sphere $S(p, r)$ that intersects two or more faces of $\mathcal{P}$ would have to intersect at least one of them in a large area and therefore is likely to contain points from S_n so the sphere is not a Voronoi sphere. Thus, at least intuitively, the Voronoi vertices should be concentrated along the medial axis. This phenomenon is actually observable in Figure 1.

Another way of attempting to enumerate the second type of Voronoi vertices might to use the above observation as follows: (i) prove that the probability of a Voronoi sphere center being far from the medial axis is negligible. Then (ii) identify the medial axis of $\mathcal{P}$, partition it into small regions, and for each region enumerate the expected number of Voronoi spheres whose centers are near that region and (iii) add up all of these values. While we did not employ this approach in this paper it might be useful in extensions in which the manifold $\mathcal{M}$ is smooth and our techniques can not be used. We note, that Attali and Boissonnat have already used a Medial axis based approach in [1] to achieve their $O(n^{7/4})$ bound.

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Appendix: Proof of Lemma 1

In this appendix we sketch the proof of Lemma 1 which states that if S_n is a set of points chosen from the standard 2-dimensional Poisson distribution on $\mathcal{P}$ with rate n then

$$\Pr\left(\text{there exists a } \tfrac{\log n}{\sqrt{n}}\text{-}bad \text{ sphere } S \text{ such that } S_n \cap S = \emptyset\right) = n^{-\Omega(\log n)}. \quad (81)$$

We will assume that $\text{Area}(\mathcal{P}) = 1$; if not, then scaling $\mathcal{P}$ so that its area is 1 will enable us to prove the lemma.

Until stated otherwise we will change our distribution and assume that $S_n = \{p_1, p_2, \ldots, p_n\}$ are n points chosen IID from the uniform distribution over $\mathcal{P}$. For $q_1, q_2, q_3, q_4 \in \mathbb{R}^3$ let $S(q_1, q_2, q_3, q_4)$ be the sphere that has the q_i on its boundary. Then set

$$A(q_1, q_2, q_3, q_4) = \text{Area}\left(S(q_1, q_2, q_3, q_4) \cap \mathcal{P}\right)$$

and

$$\begin{aligned} f(q_1, q_2, q_3, q_4) &= \Pr\left(S(q_1, q_2, q_3, q_4) \cap \{p_5, p_6, \ldots, p_n\} = \emptyset\right) \\ &= \left(1 - A(q_1, q_2, q_3, q_4)\right)^{n-4} \end{aligned}$$

Note that the sphere $S(p_1, p_2, p_3, p_4)$ is a Voronoi sphere if and only if $S(p_1, p_2, p_3, p_4) \cap \{p_5, p_6, \ldots, p_n\} = \emptyset$. Now define $\mathbf{A}$ to be the set

$$\mathbf{A} = \left\{ (p_1, p_2, p_3, p_4) \;:\; A(p_1, p_2, p_3, p_4) \geq \frac{\log^2 n}{n} \right\}$$

Note that $\mathbf{A}$ is measurable. Then

$$\begin{aligned} & \Pr\left(S(p_1, p_2, p_3, p_4) \text{ is a Voronoi sphere and } A(p_1, p_2, p_3, p_4) \geq \tfrac{\log^2 n}{n}\right) \\ = \;& \int_{\mathbf{A}} f(p_1, p_2, p_3, p_4)\, dp_1\, dp_2\, dp_3\, dp_4 \\ \leq \;& \int_{\mathbf{A}} \left(1 - \frac{\log^2 n}{n}\right)^{n-4} dp_1\, dp_2\, dp_3\, dp_4 \\ \leq \;& \left(1 - \frac{\log^2 n}{n}\right)^{n-4} \int_{\mathbf{A}} dp_1\, dp_2\, dp_3\, dp_4 \\ \leq \;& \left(1 - \frac{\log^2 n}{n}\right)^{n-4} = n^{-\Omega(\log n)}. \end{aligned}$$

By a symmetry argument we have that

$$\Pr\left(S(p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}) \text{ is a Voronoi sphere and } A(p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}) \geq \tfrac{\log^2 n}{n}\right) = n^{-\Omega(\log n)}$$

for *any* four points $p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}$ where $i_1, i_2, i_3, i_4 \in \{1, 2, \ldots, n\}$. Summing over *all* $\binom{n}{4}$ possible choices of $p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}$ we have proven that the probability that $\exists i_1, i_2, i_3, i_4 \in \{1, 2, \ldots, n\}$ such that $S(p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4})$ is a Voronoi sphere and $A(p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}) \geq \frac{\log^2 n}{n}$ is $\leq \binom{n}{4} n^{-\Omega(\log n)} = n^{-\Omega(\log n)}$. This last statement is exactly a reformulation of (81) so we have just proven (81) for n points chosen IID from the uniform distribution over $\mathcal{P}$.

To prove (81) for S_n chosen from the Poisson process with rate n over $\mathcal{P}$ we note that, conditioned on the event $S_n = m$, S_n has the same distribution as m

points chosen IID from the uniform distribution over $\mathcal{P}$. This means that, with S_n chosen from the Poisson process we have

$$\begin{aligned}
& \Pr\left(\text{there exists a } \tfrac{\log n}{\sqrt{n}}\text{-}\textit{bad} \text{ sphere } S \text{ such that } S_n \cap S = \emptyset\right) \\
= \; & \sum_{m \geq 0} \Pr(S_n = m) m^{-\Omega(\log m)} \\
= \; & n^{-\Omega(\log n)}
\end{aligned}$$

where the last equality comes from the fact that

$$\Pr\left(|S_n - n| \geq \frac{n}{2}\right) = n^{-\Omega(\log n)}.$$

The proof is completed.