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On the Proofs of Two Lemmas Describing the Intersections of Spheres with the Boundary of a Convex Polytope*

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Abstract

Let $\mathcal{P}$ be the boundary of a convex polytope and S_n be a set of points drawn from the 2-dimensional Poisson distribution with rate n over $\mathcal{P}$. In a companion paper [1] the authors show that the expected complexity of the 3-dimensional Voronoi Diagram of S_n is $O(n)$. In the derivation of that fact [1] used two lemmas describing the geometric structure of the intersection of various types of spheres with $\mathcal{P}$. In this note we provide the proofs of those two lemmas.

1 Introduction

Let $\mathcal{P}$ be the boundary of any fixed 3-dimensional convex polytope. In [1] it was proven that, if S_n is a set of points drawn from the 2-dimensional Poisson distribution over $\mathcal{P}$, then the expected complexity of the 3-dimensional Voronoi diagram of S_n grows as $O(n)$. The proof technique used was to count the expected

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number of *Voronoi* or *defining Spheres* of S_n. These are spheres that have points of S_n on their boundaries but no points of S_n in their interiors.

In [1] it was noted that if the intersection of sphere S with $\mathcal{P}$ is "large" then there is a high probability that the interior of S contains a point of S_n and thus S is not a Voronoi Sphere. The analysis there therefore required a good understanding of the structure of the intersection of spheres with the boundaries of convex polytopes. In particular, that paper used two lemmas bounding the areas of the intersections. The proofs of those two lemmas, while quite long, are actually straightforward geometric case-by-case analyses that do not contain much new in the way of techniques. They were therefore not included in that paper. For completeness' sake this note presents the full proofs.

Section 2 introduces some definitions and utility lemmas. Section 3 proves the first geometric lemma from [1]. Section 4 proves the second geometric lemma from [1].

2 Definitions and Utility Lemmas

This section contains pertinent definitions from [1]. It also provides two utility lemmas that will be needed later.

For a point $p \in \mathbb{R}^3$ and any closed or finite set set $X \subseteq \mathbb{R}^3$ extend the Euclidean distance function so that $d(p, X) = \min_{q \in X} d(p, q)$. Now define

Definition 1 *Let* $p \in \mathbb{R}^3$, $X \subseteq \mathbb{R}^3$ *and* $r \geq 0$:

- $S(p, r) = \{q \in \mathbb{R}^3 \,:\, d(q, p) \leq r\}$ *is the closed sphere of radius* r *around* p.
- *For point* $p \in \mathbb{R}^3$, $NN(p, X)$, *will denote a nearest neighbor* q *to* p *in* X, *i.e., a* $q \in X$ *such that*

$$\forall q' \in X,\ d(p, q) \leq d(p, q').$$

In this paper all of the sets X *used will either be finite or closed. Thus such a* q *will always exist although it might not always be unique.*

Definition 2 *Let* $\mathcal{P}$ *be the boundary of a convex polytope. A sphere* S *in* $\mathbb{R}^3$ *is* x-bad *(with respect to* $\mathcal{P}$*) if*

$$Area(S \cap \mathcal{P}) \geq x^2.$$

A sphere S *in* $\mathbb{R}^3$ *is* x-good *(with respect to* $\mathcal{P}$*) if it is not* x-bad.

Definition 3 *Let* $F \subseteq \mathbb{R}^3$ *be a 2-dimensional planar object in* $\mathbb{R}^3$. *Its* supporting plane *is the unique plane* $\Pi \subset \mathbb{R}^3$ *such that* $F \subseteq \Pi$.

Definition 4 *Let* Π *be a 2-dimensional plane in* $\mathbb{R}^3$; C_Π, D_Π *and* $\overline{D_\Pi}$ *will be the circle, open and closed disks*

$$\begin{aligned} C_\Pi(p,r) &= \{q \in \Pi : d(q,p) = r\}. \\ D_\Pi(p,r) &= \{q \in \Pi : d(q,p) < r\}. \\ \overline{D_\Pi}(p,r) &= \{q \in \Pi : d(q,p) \le r\} = C_\Pi(p,r) \cup D_\Pi(p,r). \end{aligned}$$

Definition 5 *We also define the* Skeleton *and* r-boundary *of* $\mathcal{P}$*:*

$$\begin{aligned} Skel(\mathcal{P}) &= \{u \in \mathcal{P} : \ u \text{ is on some edge of } \mathcal{P}\} \\ Bd(r) &= \{u \in \mathcal{P} : \exists \text{point } v \in Skel(\mathcal{P}) \text{ such that } d(u,v) < r\}. \end{aligned}$$

Thus $Bd(r)$ is the set of points on $\mathcal{P}$ within distance r of an edge or vertex of $\mathcal{P}$.

Finally, we will later need the following two basic geometric lemmas and definition so we state them here:

Lemma 1 *Let* F *be a convex polygon and* Π *its supporting plane. Then there exist* $\sigma \ge 0$ *(*σ *is a function of the angles of* F*) and* $K \ge 0$ *such that*

- $\forall r \le K, \forall p \in F, \quad Area(F \cap D_\Pi(p,r)) \ge \sigma r^2.$
- $\forall r \ge K, \forall p \in F, \quad Area(F \cap D_\Pi(p,r)) \ge \sigma.$

The proof of this lemma is straightforward so we omit it. The lemma permits us to introduce the following definition:

Definition 6 *Let* $\mathcal{P}$ *be a convex polytope,* F_i, $i = 1,\ldots,k$, *its faces and* Π_i, $i = 1,\ldots,k,$, *their respective supporting planes. Let* σ_i *and* K_i *be the* σ *and* K *associated with* F_i *in Lemma 1. Set*

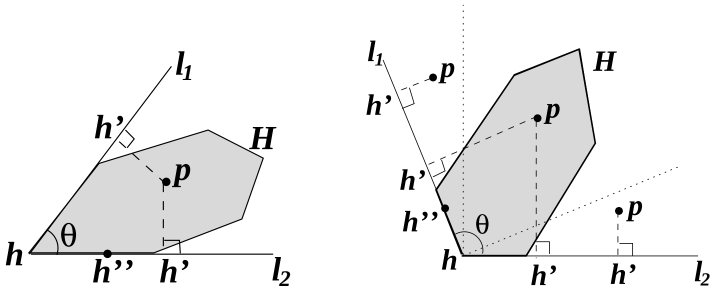
$$c_0 = \frac{1}{\sqrt{\max_i \sigma_i}}$$

and $K_0 = \min_i K_i$. *We note that this directly implies that if* $p \in F_i$ *for some* F_i *then*

$$\forall r \le K_0, \quad Area(F_i \cap D_{\Pi_i}(p, c_0 r)) \ge r^2.$$

Lemma 2 *Let* H *be the boundary of a (possibly unbounded) 2-dimensional convex polygon. Let* p *be inside* H *and* $h = NN(p,H)$. *Then* h *is not a vertex of* H.

Proof. Suppose that $h = NN(p,H)$ is a vertex of H. Let e_1 and e_2 be the edges of H incident upon h and θ the angle between them. Let l_1 and l_2 be the half lines defined by extending e_1 and e_2, respectively. There exists two cases: $\theta \le 90°$ and $\theta > 90°$. In each case at least one, and possibly both, of the perpendiculars dropped from p onto the line supporting e_1, e_2 must be on l_1, l_2. Let h' be a point such that the perpendicular dropped from p onto e_1 or e_2 is on l_1 or l_2. See Figure 1. For every point $h'' \in hh'$ we have $length(ph'') < length(ph)$. Since there must exist some such point $h'' \in H$, this contradicts $h = NN(p,H)$ and we are done. $\square$



(a) $\theta \leq 90^\circ$

(b) $\theta > 90^\circ$

Figure 1: A vertex h of H cannot be $NN(p, H)$.

3 The First Lemma

We start by examining the structure of the intersection of $\mathcal{P}$ with spheres whose centers are *outside* of P and whose nearest neighbor $NN(p, \mathcal{P}) \in Skel(\mathcal{P})$, i.e., is on one of the edges of P.

Definition 7 *For $p' \in Skel(\mathcal{P})$ define*

$$M(p') = \left\{ \begin{array}{rl} q \in \mathcal{P}: & q \in S(p,r) \text{ for some } \frac{\log n}{\sqrt{n}}\text{-good sphere } S(p,r) \\ & \text{such that } p \text{ is outside of } \mathcal{P} \text{ and } p' = NN(p, \mathcal{P}) \end{array} \right\}$$

Now, for s a segment of an edge in $Skel(\mathcal{P})$,

$$M(s) = \cup_{p' \in s} M(p').$$

Our goal, in the next lemma, will be to prove that Area$(M(s)$ is small. The important fact to keep in mind when reading the lemma and proof is that points in $M(s)$ might actually be quite far from s. We will therefore need something stronger than the triangle inequality to reach our goal.

Lemma 3 *Let s be a segment of an edge e in $Skel(\mathcal{P})$ with length$(s) \leq \frac{\log n}{\sqrt{n}}$. Then*

$$Area(M(s)) \leq c_3 \frac{\log^3 n}{n}$$

for some c_3 dependent only upon $\mathcal{P}$.

Proof. The proof is split into three parts. In parts 1 and 2 we examine the structure of $M(p')$ for p' a point on $Skel(\mathcal{P})$. Part 1 assumes that p' *is not* a vertex of $\mathcal{P}$, part 2 assumes that p' *is* a vertex of $\mathcal{P}$. In part 3 we actually calculate Area$(M(s))$ for a *segment* s.

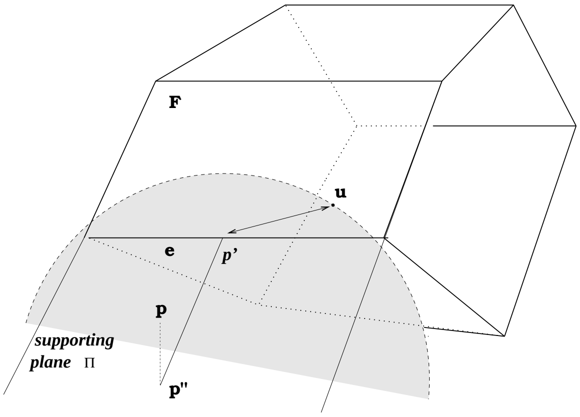


Figure 2: $\frac{\log n}{\sqrt{n}}$-good sphere S with p outside $\mathcal{P}$. No matter how large the radius of S is, the area of the intersection of S with $\mathcal{P}$ can be arbitrarily small. This means that even if S is a good sphere, it is quite possible that the points in $S \cap F$ are very far away from p'', the center of the circle $B = \Pi \cap S$. For example, in this figure, we are not able to bound $d(p', u)$ for $u \in B \cap F$.

1. Suppose that $p' \in Skel(\mathcal{P})$ but p' is not a *vertex* of $\mathcal{P}$. Let e be the edge of $\mathcal{P}$ upon which p' lies and let v_1 and v_2 be the two endpoints of e. We split the case into two parts: in **(i)** we examine $M(p') \cap F$ when F is one of the two faces *incident* upon e, i.e., one of the two faces having e as a boundary; in **(ii)** we examine $M(p') \cap F$ when F is a face of $\mathcal{P}$ *not incident* upon e.

(i) Let F be either of the two faces incident upon e and Π its supporting plane. Let $p'' = NN(p, \Pi)$. Note that $p''p' \perp e$ and $p' \neq p''$ (except in the extreme case in which line $pp' \perp \Pi$). We do have, though, that $p' = NN(p'', F)$. This follows from the fact that $pp'' \perp \Pi$ so $\forall u \in \Pi$, $d(p, u) = \sqrt{d(p, p'')^2 + d(p'', u)^2}$ leading to the general observation that

$$\forall u, v \in \Pi \quad d(p, u) \leq d(p, v) \quad \text{ if and only if } \quad d(p'', u) \leq d(p'', v). \tag{1}$$

In particular, since $p' = NN(p, F)$ this implies that $p' = NN(p'', F)$.

Consider the orthogonal coordinate system (u, v) of Π whose origin is p' and whose u-axis lies on the line defined by extending e. Without loss of generality (WLOG) we will assume that F lies in the half plane $v \geq 0$ since the other case is symmetric. Note that by definition, the point $(u'', v'') = p''$ has $u'' = 0$, $v'' \leq 0$. The first is because $p' = NN(p'', F)$ and p' is not a vertex of F (since p' is not a vertex of e) so $p''p' \perp e$. The second follows from the first plus the fact that if $v'' > 0$ there would be some point *inside* F closer to p'' than p' is, forcing a contradiction.

Let $q = (u, v)$ be any point of F in some $\frac{\log n}{\sqrt{n}}$-good sphere $S = S(p, r)$ with $p' = NN(p, \mathcal{P})$. Straightforward geometric arguments show that *none* of the

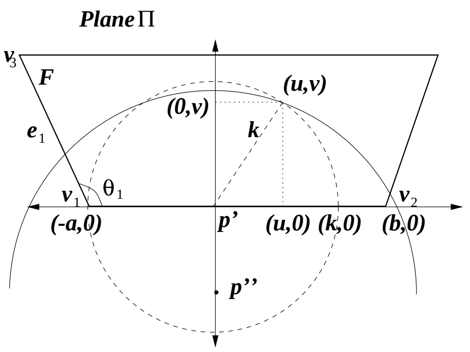


Figure 3: None of the points $(0,0)$, $(0,\pm v)$, $(\pm u, 0)$, $(\pm\sqrt{u^2+v^2}, 0)$ are further away from p'' than (u,v) is.

points $(0,0)$, $(0,\pm v)$, $(\pm u,0)$, $(\pm\sqrt{u^2+v^2},0)$ are further away from p'' than (u,v) is. See Figure 3. Thus, (1) immediately implies that all of these points are in S. Also, since $p' = NN(p, \mathcal{P})$ we have by definition that $p' \in S \cap F$.

S and F are both convex and thus $S \cap F$ is convex as well. In particular, if points $q_1, q_2, q_3 \in S \cap F$ then, by convexity, the triangle T with vertices q_1, q_2, q_3 is totally contained within $S \cap F$. Although simple, this will be an extremely powerful tool. As a first application, we prove that $v \le 2\frac{\log n}{\sqrt{n}}$. If $\sqrt{u^2+v^2} \le 2\frac{\log n}{\sqrt{n}}$, then v must be less than or equal to $\sqrt{2}\frac{\log n}{\sqrt{n}}$, so we are done. If $\sqrt{u^2+v^2} > 2\frac{\log n}{\sqrt{n}}$, then $\left(\pm\frac{\log n}{\sqrt{n}}, 0\right) \in S$ immediately follows from $\left(\pm\sqrt{u^2+v^2}, 0\right) \in S$. For large enough n, at least one of $\left(\frac{\log n}{\sqrt{n}}, 0\right)$ or $\left(-\frac{\log n}{\sqrt{n}}, 0\right)$ must be in F, since otherwise the edge e has length less than $2\frac{\log n}{\sqrt{n}}$, becoming infinitesimally small. Thus at least one of the triangles T_1 with vertices $(0,0), (u,v)$, and $\left(\frac{\log n}{\sqrt{n}}, 0\right)$, or T_2 with vertices $(0,0), (u,v)$, and $\left(-\frac{\log n}{\sqrt{n}}, 0\right)$, is totally contained in $S \cap F$.

If triangle $T \subseteq S \cap F$ where S is $\frac{\log n}{\sqrt{n}}$-good then $\text{Area}(T) \le \left(\frac{\log n}{\sqrt{n}}\right)^2$ since if $\text{Area}(T) \ge \left(\frac{\log n}{\sqrt{n}}\right)^2$ then $\text{Area}(S \cap \mathcal{P}) \ge \left(\frac{\log n}{\sqrt{n}}\right)^2$ contradicting the definition of $\frac{\log n}{\sqrt{n}}$-goodness. Since $Area(T_1) = Area(T_2) = \frac{1}{2}v\frac{\log n}{\sqrt{n}}$ and one of T_1, T_2 is contained in $S \cap F$ we have proven that

$$\text{If } (u,v) \in S \cap F \text{ then } v \le 2\frac{\log n}{\sqrt{n}}. \tag{2}$$

As another application we can show that

$$\text{If } (u,v) \in S \cap F\text{, then } v \le \frac{2C}{|u|}\left(\frac{\log n}{\sqrt{n}}\right)^2 \tag{3}$$

for some constant $C > 0$ dependent only upon $\mathcal{P}$.

To prove this, label the "left" endpoint of e as v_1; the "right" endpoint as v_2. See Figure 4. Let e_1, e_2 be the two edges of F incident upon e, respectively, at

v_1 and v_2. Let θ_1, θ_2 be the angles between e and v_1, v_2. Finally, let $a, b \geq 0$ be such that, in our coordinate system, $v_1 = (-a, 0)$ and $v_2 = (0, b)$. Recall that $p' = (0,0)$. Recall too, that we saw before that $(u, v) \in S$ implies $(\pm u, 0), (0, v) \in S$.

If $-a \leq u \leq b$ then $(u, 0) \in F$. Therefore the triangle T with endpoints (u, v), $(0,0)$, $(u, 0)$ has all endpoints in $S \cap F$ so $T \subseteq S \cap F$. By the $\frac{\log n}{\sqrt{n}}$-goodness of S, $\text{Area}(T) = \frac{1}{2}|u|v \leq \left(\frac{\log n}{\sqrt{n}}\right)^2$ proving (3) with $C = 1$. See Figure 4 (a).

Suppose now that $u \notin [-a, b]$. Without loss of generality we will assume that $u > b$ ($u < -a$ is symmetric). Now additionally assume that $(0, v) \in F$. This implies that the triangle T with endpoints (u, v), $(0,0)$, $(0, v)$ has all endpoints in $S \cap F$ so $T \subseteq S \cap F$. By the $\frac{\log n}{\sqrt{n}}$-goodness of S, $\text{Area}(T) = \frac{1}{2}|u|v \leq \left(\frac{\log n}{\sqrt{n}}\right)^2$, again proving (3) with $C = 1$. See Figure 4 (b).

The only case left to examine is when $u > b$ and $(0, v) \notin F$; Figure 4 (c). First note that this implies that $\theta_2 > 90°$ since otherwise it would be impossible for $(u, v) \in F$ with $u > b$. Next recall that $(u, v) \in S \cap F$ so (2) implies $v \leq 2\frac{\log n}{\sqrt{n}}$. Since $\theta_2 > 90°$, in order for $(v, 0) \notin F$ we must have that edge e_1 intersects the line segment connecting p' and $(0, v)$ at some point $(0, v')$ with $v' \leq 2\frac{\log n}{\sqrt{n}}$ so $\theta_1 \leq 90°$ as well. This means that $a = \frac{1}{\tan\theta_1}v' \leq \frac{2}{\tan\theta_1}\frac{\log n}{\sqrt{n}}$. Thus, for large enough n, $a \leq \frac{b}{2}$ so $b \geq \frac{length(e)}{2}$. Finally, since $(u, 0) \in S$, we have $(b, 0) \in S$ so $(b, 0) \in S \cap F$. This implies that the triangle T with endpoints (u, v), $(0,0)$, $(b, 0)$ has all endpoints in $S \cap F$ so $T \subseteq S \cap F$. By the $\frac{\log n}{\sqrt{n}}$-goodness of S, $\text{Area}(T) = \frac{1}{2}bv \leq \left(\frac{\log n}{\sqrt{n}}\right)^2$ so

$$v \leq \frac{2}{b}\left(\frac{\log n}{\sqrt{n}}\right)^2 \leq \frac{4}{length(e)}\left(\frac{\log n}{\sqrt{n}}\right)^2 \leq \frac{4}{u}\frac{diameter(F)}{length(e)}\left(\frac{\log n}{\sqrt{n}}\right)^2$$

proving (3) with $C = \frac{2\,diameter(F)}{length(e)}$.

Combining (2), and (3) then proves that, for F incident upon e and the constant $C = \max\left\{1, \frac{2\,diameter(F)}{length(e)}\right\}$,

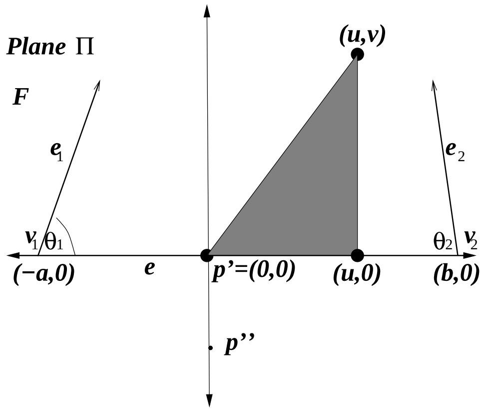
$$M(p') \cap F \subseteq \left\{(u, v) \in F \,:\, v \leq \min\left(2\frac{\log n}{\sqrt{n}}, \frac{2C}{|u|}\left(\frac{\log n}{\sqrt{n}}\right)^2\right)\right\}. \tag{4}$$

(ii) Now let F be a face of $\mathcal{P}$ which is not incident upon e. To bound $\text{Area}(M(p') \cap F)$ we find a region containing $M(p') \cap F$.

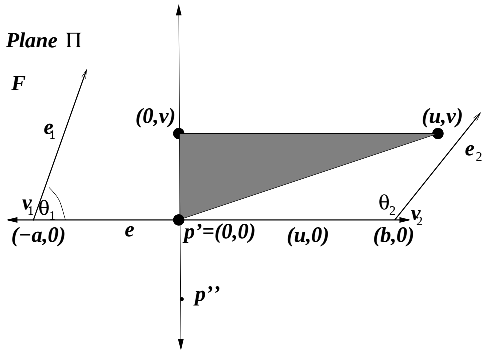
Let L_e be the cylinder of radius $c_0\frac{\log n}{\sqrt{n}}$ around the line defined by extending e. Figure 5(a). Our claim is that

$$\text{If } q \in M(p') \cap F, \text{ then } q \text{ is inside or on } L_e. \tag{5}$$

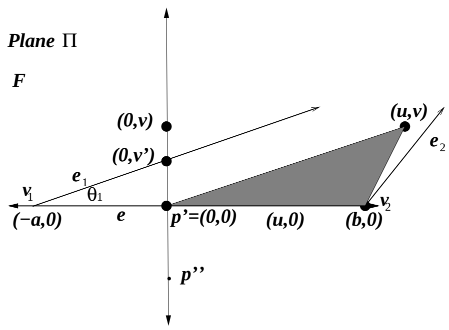
The proof is by contradiction. Assume then, that $\exists q \in M(p') \cap F$ such that q is outside L_e. Let Π_q be the plane containing q and e and set $l \subset \Pi_q$ to be the line



(a) $-a \le u \le b$.



(b) $u > b$ and $(0, v) \in S$



(c) $u > b$ and $(0, v) \notin S$

Figure 4: From the proof of Lemma 3 **(i)**. Edge e is v_1v_2. It is known that $(u, v) \in S \cap F$ and $(u, 0), (0, v) \in S$. The shaded region is triangle T.

perpendicular to e passing through p'. Consider the coordinate system (u, v) of Π_q such that the origin is p', the u- and v-axis are respectively e and l, and every point of $\mathcal{P}$ has nonnegative v-coordinates (this coordinate system is defined in the same fashion as in **(i)**). See Figure 5(b). Let $q = (u, v)$ and set $q' = (u', v')$ to be the projection of q onto the line l. Note that $u' = 0$ because all points on l have $u' = 0$ and $v = v' > c_0 \frac{\log n}{\sqrt{n}}$ because $q \notin L_e$. Let $S = S(p, r)$ be a $\frac{\log n}{\sqrt{n}}$-good sphere such that $NN(p, \mathcal{P}) = p'$ and $q \in S$, and let $p'' = NN(p, \Pi_q)$. From $pp'' \perp \Pi_q$ and $pp' \perp e$ we have $p''p' \perp e$, so $p'' = (u'', v'') \in l$, $u'' = 0$ and $v'' \leq 0$. Note that $length(p''q') < length(p''q)$ which implies that $length(pq') < length(pq)$ and thus $q' \in S$.

The two faces of $\mathcal{P}$ incident upon e are separated by the plane Π_q. Let F'' be the face that lies on the same side of Π_q as p and Π'' its supporting plane. Let $q'' \in \Pi''$ be a point such that $p'q'' \perp e$, $length(p'q'') = length(p'q')$, and q'' lies on the opposite side of e to $NN(p, \Pi'')$. Note that in the coordinate system (u, v) given in **(i)**, $NN(p, \Pi'')$ has coordinate $(0, v)$ for some $v < 0$ and q'' has coordinate $(0, v)$ for $v = length(p'q') = v' > c_0 \frac{\log n}{\sqrt{n}}$. This means that if $q'' \in S$, then $D_{\Pi''}(p', length(p'q'')) \subset S$.

But, since $90^\circ \leq \angle pp'q'' < \angle pp'q' < 180^\circ$ and $length(p'q'') = length(p'q')$, $q' \in S$ implies that $q'' \in S$. Thus $D_{\Pi''}(p', length(p'q'')) \subset S$ contradicting the goodness of S by the fact that $length(p'q'') > c_0 \frac{\log n}{\sqrt{n}}$ and Definition 6. We have therefore proven (5).

Note that (5) implies that if $M(p') \cap F \neq \emptyset$ then $d(e, F) \leq c_0 \frac{\log n}{\sqrt{n}}$. Thus, for large enough n we have that if $M(p') \cap F \neq \emptyset$ then $F \cap e \neq \emptyset$, i.e., F must contain one of the two endpoints of e as a vertex.

2. In part 1 we assumed that $p' = NN(p, \mathcal{P}) \in Skel(\mathcal{P})$ but that p' *was not* a vertex of $\mathcal{P}$. In this section we examine the case in which p' *is* a vertex of $\mathcal{P}$. See Figure 6.

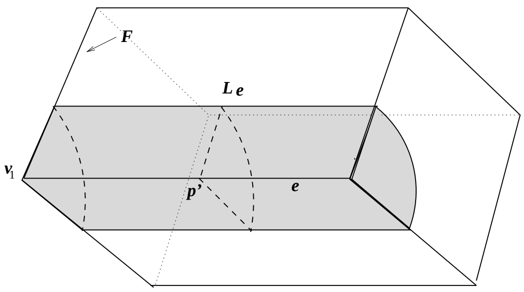
Let F be one of the faces incident upon p', Π be its supporting plane and e and e_1 be the edges on F incident upon $p' = v_1$. Set v_2 and v_3 to be, respectively, the other endpoints of e and e_1.

Working backwards, let $Region(p')$ be the set of possible locations of p'' such that $p' = NN(p'', F)$. Given $p'' \in Region(p')$, let $A(p'')$ and $B(p'')$ be the closed subsets of F lying on the right and the left of the half line $\overrightarrow{p''p'}$. One of $A(p'')$ or $B(p'')$ might be empty but $A(p'') \cup B(p'') = F$. That is, for any point $q \in F$, q is either in $A(p'')$ or in $B(p'')$. Moreover, we have the property that

$$\begin{cases} \text{if } A(p'') \neq \emptyset, \text{ then } e \subset A(p'') \\ \text{if } B(p'') \neq \emptyset, \text{ then } e_1 \subset B(p''). \end{cases} \tag{6}$$

See Figure 7.

Let q be any point of F in some $\frac{\log n}{\sqrt{n}}$-good sphere $S = S(p, r)$ with $p' = NN(p, \mathcal{P})$. We divide our analysis into two cases: **(i)** $q \in S \cap A(p'')$ and **(ii)**



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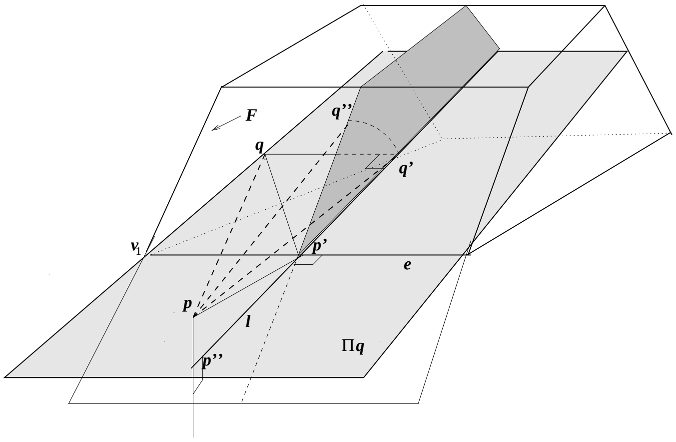
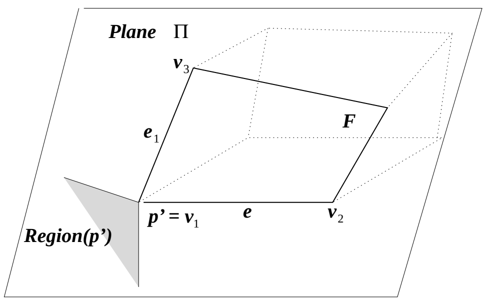


(b)

Figure 5: Let F be a face of $\mathcal{P}$ incident upon v_1, $p' \in e$ and L_e the cylinder of radius $c_0 \frac{\log n}{\sqrt{n}}$ around the line defined by extending e. If $q \in M(p') \cap F$, then q must be inside or on L_e.



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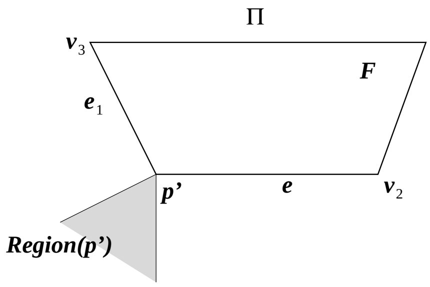


(b)

Figure 6: $Region(p')$.

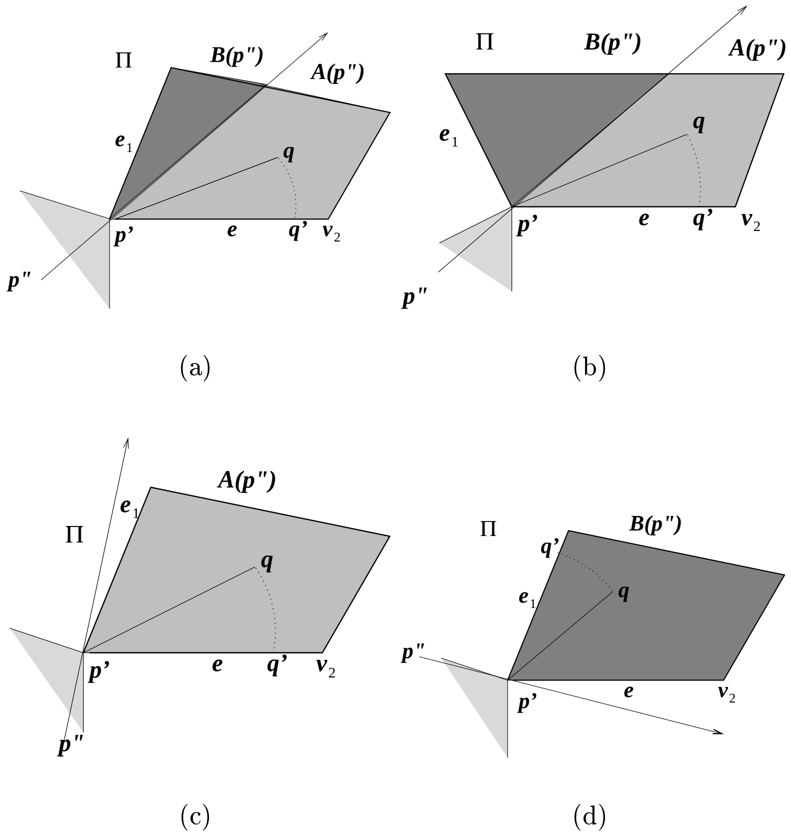


Figure 7: $A(p'')$ and $B(p'')$

$q \in S \cap B(p'')$.

(i) $q \in S \cap A(p'')$: Consider the orthogonal coordinate system (u, v) of Π whose origin is p' and whose u-axis lies on the line defined by extending e. W.L.O.G we may assume that F lies in the half plane $v \geq 0$ and $v_2 = (b, 0)$ is the other endpoint of e for some $b > 0$. Note that $q = (u, v)$ has $u \geq 0$ and $v \geq 0$. See Figure 7 for the possible cases.

Firstly we show that $q' = (\sqrt{u^2 + v^2}, 0) \in S$: It is no longer necessarily true that $p''p' \perp e$, so the proof of this fact is no longer as straightforward as it was in part 1.

Since $p'' \in Region(p')$ and $q \in A(p'')$, the angles $\angle p''p'q$ and $\angle p''p'q'$ satisfy that

$$90^\circ \leq \angle p''p'q' < \angle p''p'q < 180^\circ. \tag{7}$$

The law of cosines gives us that

$$\begin{cases} d(p'', q)^2 & = d(p'', p')^2 + d(p', q)^2 - 2\cos(\angle p''p'q)d(p'', p')d(p', q), \\ d(p'', q')^2 & = d(p'', p')^2 + d(p', q')^2 - 2\cos(\angle p''p'q')d(p'', p')d(p', q'). \end{cases}$$

Thus $d(p'', q) > d(p'', q')$ immediately follows from $d(p', q) = d(p', q') = \sqrt{u^2 + v^2}$ and $\cos(\angle p''p'q) < \cos(\angle p''p'q') \leq 0$ (by (7)). Hence $q' \in S$ follows from (1).

Using the fact that $q' = (\sqrt{u^2 + v^2}, 0) \in S$, we prove, in a fashion similar to (2), that

$$\text{If } q = (u, v) \in S \cap A(p''), \text{ then } v \leq 2\tfrac{\log n}{\sqrt{n}}. \tag{8}$$

If $\sqrt{u^2 + v^2} \leq 2\frac{\log n}{\sqrt{n}}$, then v must be $\leq 2\frac{\log n}{\sqrt{n}}$, so we are done. If $\sqrt{u^2 + v^2} > 2\frac{\log n}{\sqrt{n}}$, then we have $\left(\frac{\log n}{\sqrt{n}}, 0\right) \in S \cap F$: $\left(\frac{\log n}{\sqrt{n}}, 0\right) \in S$ follows from $(\sqrt{u^2 + v^2}, 0) \in S$, and we may assume that $\left(\frac{\log n}{\sqrt{n}}, 0\right) \in F$ since otherwise for large enough n, the length of e is less than $\frac{\log n}{\sqrt{n}}$ which becomes infinitesimally small. Thus the triangle T with vertices $(0, 0), (u, v)$, and $\left(\frac{\log n}{\sqrt{n}}, 0\right)$ is totally contained in $S \cap F$. By the $\frac{\log n}{\sqrt{n}}$-goodness of S, $Area(T) = \frac{1}{2}v\frac{\log n}{\sqrt{n}} \leq \left(\frac{\log n}{\sqrt{n}}\right)^2$ which implies that $v \leq 2\frac{\log n}{\sqrt{n}}$ and we are done.

We now prove that for the constant C of (4),

$$\begin{cases} \text{If } 0 \leq u \leq b, \text{ then } v \leq \frac{2C}{|u|}\left(\frac{\log n}{\sqrt{n}}\right)^2. \\ \text{If } u > b, \text{ then } v \leq \frac{2C}{b}\left(\frac{\log n}{\sqrt{n}}\right)^2. \end{cases} \tag{9}$$

Since $q' = (\sqrt{u^2 + v^2}, 0) \in S$, $(u, 0) \in S$. If $0 \leq u \leq b$, then $(u, 0) \in F$, so $(u, 0) \in S \cap F$. This implies that the triangle T_1 with vertices $(0, 0), (u, v)$, and $(u, 0)$ is contained in $S \cap F$ so by the $\frac{\log n}{\sqrt{n}}$-goodness of S, $Area(T_1) = \frac{1}{2}|u|v \leq \left(\frac{\log n}{\sqrt{n}}\right)^2$.

Hence we have proven that for $0 \le u \le b$, $v \le \frac{2C}{|u|}\left(\frac{\log n}{\sqrt{n}}\right)^2$ since by definition $C \ge 1$. If $u > b$, then $(b,0) \in S$ since $(u,0) \in S$ and $(b,0) = v_2 \in F$. Hence the triangle T_2 with vertices $(0,0)$, (u,v), and $(b,0)$ is contained in $S \cap F$, so again by the goodness of S $Area(T_2) = \frac{1}{2}bv \le \left(\frac{\log n}{\sqrt{n}}\right)^2$. Thus for $u > b$, $v \le \frac{2C}{b}\left(\frac{\log n}{\sqrt{n}}\right)^2$.

Combining (8) and (9), we have, for $q = (u,v) \in S \cap A(p'')$,

$$\begin{cases} \text{If } 0 \le u \le b, \text{ then } v \le \min\left(2\frac{\log n}{\sqrt{n}}, \frac{2C}{|u|}\left(\frac{\log n}{\sqrt{n}}\right)^2\right). \\ \text{If } u > b, \text{ then } v \le \min\left(2\frac{\log n}{\sqrt{n}}, \frac{2C}{b}\left(\frac{\log n}{\sqrt{n}}\right)^2\right). \end{cases} \tag{10}$$

Therefore

$$\begin{aligned} M(p') \cap A(p'') &\subseteq \{(u,v) \in A(p'') : (u,v) \text{ satisfies } (10)\} \\ &\subseteq \{(u,v) \in F : (u,v) \text{ satisfies } (10)\}. \end{aligned} \tag{11}$$

Note that the region on the right-hand side of (11) is not dependent on p'' but only upon e.

(ii) $q \in S \cap B(p'')$: Consider the new orthogonal coordinate system (u,v) of Π whose origin is p' and whose u-axis lies on the line defined by extending e_1. Let $v_3 = (-b', 0)$ for some $b' > 0$. Note that $q = (u,v) \in S \cap B(p'')$ satisfies $u \le 0$ and $v \ge 0$. See Figure 7.

By analogs of the proofs given in (i), we have, for $q = (u,v) \in S \cap B(p'')$

$$\begin{cases} \text{If } -b' \le u \le 0, \text{ then } v \le \min\left(2\frac{\log n}{\sqrt{n}}, \frac{2C}{|u|}\left(\frac{\log n}{\sqrt{n}}\right)^2\right). \\ \text{If } u < -b', \text{ then } v \le \min\left(2\frac{\log n}{\sqrt{n}}, \frac{2C}{b'}\left(\frac{\log n}{\sqrt{n}}\right)^2\right). \end{cases} \tag{12}$$

Therefore

$$\begin{aligned} M(p') \cap B(p'') &\subseteq \{(u,v) \in B(p'') : (u,v) \text{ satisfies } (12)\} \\ &\subseteq \{(u,v) \in F : (u,v) \text{ satisfies } (12)\}. \end{aligned} \tag{13}$$

Note that the region on the right-hand side of (13) is not dependent on p'' but only upon e_1.

Combining **(i)** and **(ii)**, we have

$$M(p') \cap F \subseteq \{q \in F : q \text{ satisfies either } (10) \text{ or } (12)\}. \tag{14}$$

Note that coordinate systems of (10) and (12) are different from each other, though, for convenience, we denote both by (u,v).

3. Up to now we have derived the structure of $M(p')$ for point $p' \in Skel(\mathcal{P})$. Now we will examine $M(s)$ and $Area(M(s))$ where $s \subset e$ is a line segment with $length(s) \le \frac{\log n}{\sqrt{n}}$. For any segment $s \subset e$ of length $\le \frac{\log n}{\sqrt{n}}$, we calculate $Area(M(s))$ by $Area(M(s)) = Area\left(\cup_{p' \in s} M(p')\right)$ and show that $Area(M(s)) \le c_3 \frac{\log^3 n}{n}$. To do this, we divide our analysis into three cases: **(i)** $v_1 \notin s$, $v_2 \notin s$; **(ii)** v_1 is the leftmost point of s; **(iii)** v_2 is the rightmost point of s. Note that at most one of $v_1 \in s$ or $v_2 \in s$ is possible because, if $v_1 \in s$ and $v_2 \in s$ then $length(e) \le length(s) \le \frac{\log n}{\sqrt{n}}$, which implies that for large enough n, $length(e)$ becomes infinitesimally small. Since **(iii)** is symmetric to **(ii)**, we investigate only **(i)** and **(ii)**.

(i) $v_1 \notin s$, $v_2 \notin s$: Let F be either of the two faces incident upon e with its supporting plane Π. Consider the orthogonal coordinate system (u, v) of Π such that the origin is v_1, the u-axis lies on the line defined by extending e and v_2 corresponds to the point with coordinate $(b, 0)$ where b is the length of e. Let $(u_1, 0)$ be the leftmost point of s and $(u_2, 0)$ the rightmost point of s. Since we will consider only cases (i) and (ii), we may assume that $u_2 < b$.

Note that for all $p' \in s$, $M(p') \cap F$ satisfies (4). Let t be such that $p' = (t, 0)$. Then $0 < u_1 \le t \le u_2 < b$. Define the functions $f(u)$ and $f_t(u)$ by

$$\begin{aligned} f(u) &= \begin{cases} 2\frac{\log n}{\sqrt{n}}, & |u| \le C\frac{\log n}{\sqrt{n}} \\ \frac{2C}{|u|}\left(\frac{\log n}{\sqrt{n}}\right)^2, & |u| \ge C\frac{\log n}{\sqrt{n}}, \end{cases} && (15) \\ f_t(u) &= f(u-t) \text{ for } t > 0. && (16) \end{aligned}$$

See Figure 8. Note that (a) $f(u) = f(-u)$, (b) f is nonincreasing, and (c) f is continuous. (c) follows from the fact that at $u = C\frac{\log n}{\sqrt{n}}$, $2\frac{\log n}{\sqrt{n}} = \frac{2C}{|u|}\left(\frac{\log n}{\sqrt{n}}\right)^2$. The function f_t is the function f translated by t. Using these functions, (4) can be rewritten as

$$M(p') \cap F \subseteq \{(u, v) \in F \,:\, 0 \le v \le f_t(u)\}.$$

Since $(t, 0) = p' \in s$ implies $u_1 \le t \le u_2$, we have

$$M(s) \cap F \subseteq \left\{(u, v) \in F \,:\, 0 \le v \le \sup_{u_1 \le t \le u_2} f_t(u)\right\}. \tag{17}$$

Let $H_s(u) = \sup_{u_1 \le t \le u_2} f_t(u)$. By the definition of $H_s(u)$ and $f_t(u)$,

$$H_s(u) = \sup_{u_1 \le t \le u_2} f_t(u) = \begin{cases} f_{u_1}(u) & u \le u_1 \\ 2\frac{\log n}{\sqrt{n}} & u_1 \le u \le u_2 \\ f_{u_2}(u) & u \ge u_2 \end{cases} \tag{18}$$

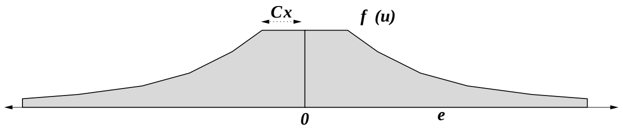
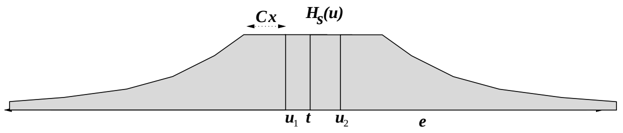
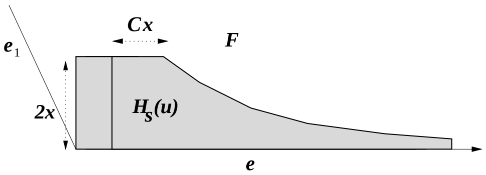


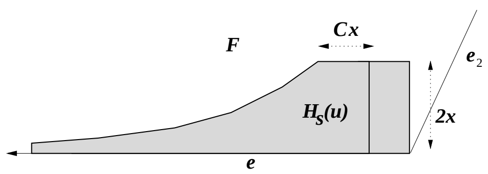
Figure 8: Function $f(u)$



(a) $H_s(u) = \sup_{u_1 \leq t \leq u_2} f_t(u)$ when $v_1 \notin s$, $v_2 \notin s$



(b) Right half of $H_s(u)$ when v_1 is the leftmost point of s



(c) Left half of $H_s(u)$ when v_2 is the rightmost point of s

Figure 9: $H_s(u) = \sup_{u_1 \leq t \leq u_2} f_t(u)$ for a segment s of edge e with $length(s) \leq \frac{\log n}{\sqrt{n}}$.

See Figure 9(a). Let L be the diameter of F. Then

$$\begin{aligned} Area(M(s) \cap F) &\leq \int_{u_1 - L}^{u_1} f_{u_1}(u)du + \int_{u_1}^{u_2} 2\frac{\log n}{\sqrt{n}}du + \int_{u_2}^{u_2+L} f_{u_2}(u)du \\ &= 2\int_0^L f(u)du + \int_{u_1}^{u_2} 2\frac{\log n}{\sqrt{n}}du \quad \text{by (16)} \\ &= O\left(\frac{\log^3 n}{n}\right) \end{aligned} \tag{19}$$

where the constant implicit in $O()$ is dependent only upon $\mathcal{P}$. Hence we just upperbounded $Area(M(s) \cap F)$ for either of the two faces incident upon e.

Now let F be a face of $\mathcal{P}$ which is not incident upon e. Then $M(p') \cap F$ must satisfy (5) and $M(p') \cap F = \emptyset$ if F is incident upon neither v_1 nor v_2. Let F be incident upon v_1 and θ' be the angle between e and F. Note that $\forall q \in F$ lying

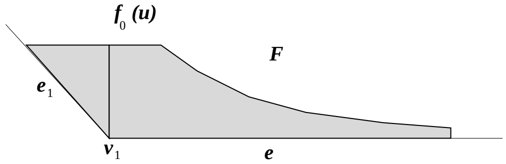


Figure 10: Function $f_0(u)$ describing $M(p')$ with $p' = v_1 = (0,0)$.

inside or on L_e, $q \in S\left(v_1, \frac{1}{\sin\theta'} c_0 \frac{\log n}{\sqrt{n}}\right)$. From this fact and (5), we have

$$M(p') \cap F \subseteq S\left(v_1, \frac{1}{\sin\theta'} c_0 \frac{\log n}{\sqrt{n}}\right) \cap F.$$

Since the right-hand side of the above inequality is independent of p',

$$M(s) \cap F \subseteq S\left(v_1, \frac{1}{\sin\theta'} c_0 \frac{\log n}{\sqrt{n}}\right) \cap F,$$

so

$$Area\,(M(s) \cap F) = O\left(\frac{\log^2 n}{n}\right) \tag{20}$$

where the constant in the notation $O()$ is dependent only upon c_0 and θ'.

(ii) v_1 is the leftmost point of s; Let F be either of the two faces incident upon e with its supporting plane Π. Recall that $Area(M(s) \cap F) = Area\left(\cup_{p' \in s} M(p') \cap F\right)$. Note that for all $p' \in s$ except for the leftmost point, $M(p') \cap F$ satisfies (4). For simplicity, we bound $Area(M(s) \cap F)$ as follows:

$$Area(M(s) \cap F) \leq Area(M(v_1) \cap F) + Area\left(\cup_{p' \in s, p' \neq v_1} M(p') \cap F\right).$$

The analysis in **(i)** shows that,

$$Area\left(\cup_{p' \in s, p' \neq v_1} M(p') \cap F\right) = O\left(\frac{\log^3 n}{n}\right).$$

We therefore only have to bound $Area(M(v_1) \cap F)$, for which we will use (14).

Recall that $b = length(e)$ and $b' = length(e_1)$. We may assume that $b \geq C\frac{\log n}{\sqrt{n}}$ and $b' \geq C\frac{\log n}{\sqrt{n}}$ since otherwise, e or e_1 have length less than $C\frac{\log n}{\sqrt{n}}$ which, for large enough n, is impossible. Thus in (10) and (12), the inequalities

$$v \leq \min\left(2\frac{\log n}{\sqrt{n}}, \frac{2C}{b}\left(\frac{\log n}{\sqrt{n}}\right)^2\right) \text{ and } v \leq \min\left(2\frac{\log n}{\sqrt{n}}, \frac{2C}{b'}\left(\frac{\log n}{\sqrt{n}}\right)^2\right)$$

become, respectively,

$$v \leq \frac{2C}{b}\left(\frac{\log n}{\sqrt{n}}\right)^2 \text{ and } v \leq \frac{2C}{b'}\left(\frac{\log n}{\sqrt{n}}\right)^2.$$

Setting $b'' = \min(b, b')$, we can replace (11) by

$$M(v_1) \cap A(p'') \subseteq \left\{ (u,v) \in F : (u,v) \text{satisfies} \begin{pmatrix} \text{If } 0 \le u \le b'', \text{ then } v \le \frac{2C}{|u|}\left(\frac{\log n}{\sqrt{n}}\right)^2. \\ \text{If } u > b'', \text{ then } v \le \frac{2C}{b''}\left(\frac{\log n}{\sqrt{n}}\right)^2. \end{pmatrix} \right\} \tag{21}$$

and (13) by

$$M(v_1) \cap B(p'') \subseteq \left\{ (u,v) \in F : (u,v) \text{satisfies} \begin{pmatrix} \text{If } -b'' \le u \le 0, \text{ then } v \le \frac{2C}{|u|}\left(\frac{\log n}{\sqrt{n}}\right)^2. \\ \text{If } u < -b'', \text{ then } v \le \frac{2C}{b''}\left(\frac{\log n}{\sqrt{n}}\right)^2. \end{pmatrix} \right\} \tag{22}$$

To analyze $Area(M(v_1) \cap F)$, we define function $f_0(u)$ by

$$f_0(u) = \begin{cases} 2\frac{\log n}{\sqrt{n}} & |u| \le C\frac{\log n}{\sqrt{n}}, \\ \frac{2C}{|u|}\left(\frac{\log n}{\sqrt{n}}\right)^2, & C\frac{\log n}{\sqrt{n}} \le |u| \le b'' \\ \frac{2C}{b''}\left(\frac{\log n}{\sqrt{n}}\right)^2, & |u| \ge b''. \end{cases} \tag{23}$$

See Figure 10. Using this definition, (21), and (22), we can rewrite (14) as

$$M(v_1) \cap F \subseteq \{(u,v) \in F : 0 \le v \le f_0(u)\}.$$

Let L be the diameter of F. Then

$$\begin{aligned} Area(M(v_1) \cap F) &\le \int_{-L}^{L} f_0(u)du = 2\int_0^L f_0(u)du \\ &= 2\left(\int_0^{C\frac{\log n}{\sqrt{n}}} 2\frac{\log n}{\sqrt{n}}du + \int_{C\frac{\log n}{\sqrt{n}}}^{b''} \frac{2C}{|u|}\left(\frac{\log n}{\sqrt{n}}\right)^2 du\right) \\ &\quad +2\int_{b''}^{L} \frac{2C}{b''}\left(\frac{\log n}{\sqrt{n}}\right)^2 du \quad \text{by (23)} \\ &= O\left(\frac{\log^3 n}{n}\right) \end{aligned} \tag{24}$$

where the constant implicit in $O()$ is dependent only upon L and F.

Now let F be one of the faces incident upon v_1. Then in

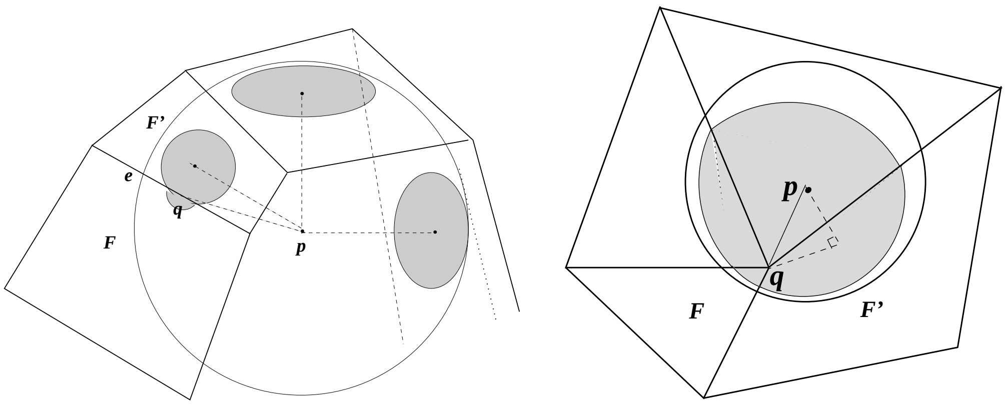
$$Area(M(s) \cap F) \le Area(M(v_1) \cap F) + Area\left(\cup_{p' \in s, p' \ne v_1} M(p') \cap F\right),$$

$Area(M(v_1) \cap F) = O\left(\frac{\log^3 n}{n}\right)$ by the same analysis as before and

$$Area\left(\cup_{p' \in s, p' \ne v_1} M(p') \cap F\right) = O\left(\frac{\log^2 n}{n}\right)$$

by (20).

Combining the results of cases **(i)** and **(ii)** and using the fact that case **(iii)** is symmetric to case **(ii)** completes the proof. □



(a) $q = NN(p, F)$ is on the edge e (b) $q = NN(p, F)$ is a vertex

Figure 11: Spheres $S = S(p, r)$ with p *inside* $\mathcal{P}$. Note that the nearest neighbor of p in F is q which is on the boundary of F. The line segment pq is not perpendicular to F which implies that the center of the intersection circle of S with the supporting plane of F is not in F.

4 The Second Lemma

We have just analyzed the structure of the intersection of $\mathcal{P}$ with special $\frac{\log n}{\sqrt{n}}$-good spheres $S = S(p, r)$ whose center p is *outside* $\mathcal{P}$. In this section we analyze the structure of the intersection of $\mathcal{P}$ with particular $\frac{\log n}{\sqrt{n}}$-good spheres $S = S(p, r)$ whose center p is *inside* $\mathcal{P}$. In particular we will require that $\exists$ a face F of $\mathcal{P}$, such that $F \cap S \neq \emptyset$ but $p'' = NN(p, \Pi)$ is not in F, where Π is the supporting plane of F. The reason for this requirement is that in [1] it is shown that the other case is easy to analyze (the other case is when p is inside $\mathcal{P}$ and for all faces F of P, $F \cap S \neq \emptyset$ implies $p'' = NN(p, \Pi) \in F$).

The main lemma we prove here is

Lemma 4 *Let $S = S(p, r)$ be a $\frac{\log n}{\sqrt{n}}$-good sphere with p inside $\mathcal{P}$. Furthermore, suppose $\exists$ a face F of $\mathcal{P}$, such that $F \cap S \neq \emptyset$ but $p'' = NN(p, \Pi)$ is not in F, where Π is the supporting plane of F. Then $\exists q$ on the boundary of F such that, $\forall u \in S(p, r)$, $d(q, u) \leq c_4 \frac{\log n}{\sqrt{n}}$, where c_4 is dependent only upon $\mathcal{P}$.*

The proof of this lemma involves showing that if $S(p, r)$ satisfies the given conditions then r is actually some small multiple of $\frac{\log n}{\sqrt{n}}$, so the distance between q, the nearest point to p on F, and any point in $S(p, r)$ is $\leq c_4 \frac{\log n}{\sqrt{n}}$.

Proof. Let F be the face such that $F \cap S \neq \emptyset$ but $p'' = NN(p, \Pi) \notin F$, and let $q = NN(p, F)$. Note that $pq \not\perp F$ and q must be on the boundary of F. Since $F \cap S \neq \emptyset$, $q \in S$. See Figure 11

1. We first assume that q is on some edge $e \in F$ but is not a vertex of F. Let F' be the other face incident to e and Π' its corresponding supporting plane. Let T be the plane passing through q and perpendicular to e. Since $pq \perp e$, $p \in T$. Orient T so that segment $F' \cap T$ is followed by segment $F \cap T$ in the counterclockwise order of segments of $\mathcal{P} \cap T$. See Figure 12(a).

Let θ be the angle between F and F' measured inside $\mathcal{P}$. Note that $pq \perp e$. Since $T \perp e = F \cap F'$, angle θ is exactly the angle between $F \cap T$ and $F' \cap T$ measured inside $\mathcal{P} \cap T$ on T. See Figure 12(b). We first show that $\theta > 90°$; suppose by contradiction that $\theta \leq 90°$. Let $\angle FqF'$ be the angle between the two half lines $\overrightarrow{qF}$ and $\overrightarrow{qF'}$, defined by extending $F \cap T$ and $F' \cap T$, respectively. Let p''' be the perpendicular dropped from p onto $\overrightarrow{qF}$. Since $\overrightarrow{qF}$ is an extension of $F \cap T$, there must be points $q' \in \overrightarrow{qF}$ such that $q' \neq q$, $qq' \subseteq qp'''$, and $qq' \subseteq F \cap T$. The first and second condition guarantee that $length(pq') < length(pq)$, and the third one implies that $q' \in F$, so the existence of such q' contradicts $q = NN(p, F)$. Thus $\theta > 90°$. See Figure 12(c).

Now we consider the plane T' such that $e \subset T'$ and $T' \perp F$; $(T \cap T') \perp (F \cap T)$ and p is to the left of $T \cap T'$ on T. See Figure 12(b). In the following we will work only in the plane T. Let h be the perpendicular dropped from p onto $\Pi' \cap T$. Let $\theta' = \angle pqh$. Note that $0 \leq \theta' < \theta - 90°$ where $\theta > 90°$. There are two cases: either $h \in F' \cap T$ or $h \notin F' \cap T$. In both cases, if we show that $length(qh) \leq C\frac{\log n}{\sqrt{n}}$ for some constant C dependent only upon θ and c_0, the constant from Definition 6, then using the facts that $length(pq) = \frac{1}{\cos\theta'} length(qh)$ and $\theta' < \theta - 90°$, we will have

$$\begin{aligned} length(pq) &= \tfrac{1}{\cos\theta'} length(qh) \\ &< \tfrac{1}{\cos(\theta - 90°)} length(qh) = \tfrac{1}{\sin\theta} length(qh) < \tfrac{C}{\sin\theta}\tfrac{\log n}{\sqrt{n}}. \end{aligned} \tag{25}$$

This will imply that

$$\forall u \in S(p, r),\ d(q, u) \leq 2\left(\frac{C}{\sin\theta} + c_0\right)\frac{\log n}{\sqrt{n}}, \tag{26}$$

which proves the lemma by setting $c_4 = 2\left(\frac{C}{\sin\theta} + c_0\right)$. To see why (25) implies (26) consider the sphere centered at q with radius $c_0\frac{\log n}{\sqrt{n}}$. As usual we denote this sphere by $S(q, c_0\frac{\log n}{\sqrt{n}})$. Definition 6 tells us that $Area\left(S(q, c_0\frac{\log n}{\sqrt{n}}) \cap F\right) \geq \left(\frac{\log n}{\sqrt{n}}\right)^2$. Since $S = S(p, r)$ is $\frac{\log n}{\sqrt{n}}$-good, $S(q, c_0\frac{\log n}{\sqrt{n}}) \not\subseteq S$. By the triangle inequality, if $r \geq d(p, q) + c_0\frac{\log n}{\sqrt{n}}$, then $S(q, c_0\frac{\log n}{\sqrt{n}}) \subseteq S$. Thus

$$r < d(p, q) + c_0\frac{\log n}{\sqrt{n}} < \left(\frac{C}{\sin\theta} + c_0\right)\frac{\log n}{\sqrt{n}}.$$

Since

$$\forall u \in S(p, r), \quad d(q, u) \leq d(q, p) + d(p, u) \leq 2r \leq 2\left(\frac{C}{\sin\theta} + c_0\right)\frac{\log n}{\sqrt{n}},$$

this proves (26). Hence to complete the proof of part 1 of the lemma it only remains to prove that $length(qh) \le C\frac{\log n}{\sqrt{n}}$ for some constant C dependent only upon c_0 and θ. See Figure 12(b).

If $h \in F' \cap T$, note that by the definition of h and T, we have $ph \perp \Pi'$ where Π' is the supporting plane of F'. This means that $h = NN(p, \Pi')$ so $S \cap \Pi' = D_{\Pi'}(h, r')$ for some $r' > 0$. Since $q \in F'$ this implies $r' > length(qh)$. But, from Definition 6 we know that $r' < c_0\frac{\log n}{\sqrt{n}}$ since otherwise

$$\text{Area}(S \cap \mathcal{P}) \ge \text{Area}\left(D_{\Pi'}(h, r') \cap F'\right) \ge \left(\frac{\log n}{\sqrt{n}}\right)^2, \tag{27}$$

contradicting the $\frac{\log n}{\sqrt{n}}$-goodness of S. Thus $h \in F' \cap T$ implies $length(qh) \le c_0\frac{\log n}{\sqrt{n}}$.

If $h \notin F' \cap T$, then we claim that $length(qh) \le \frac{c_0}{1+\cos\theta}\frac{\log n}{\sqrt{n}}$; Suppose that $length(qh) > \frac{c_0}{1+\cos\theta}\frac{\log n}{\sqrt{n}}$. Then we have

$$\begin{array}{rcll} length(pq) - length(ph) & = & \frac{1}{\cos\theta'}length(qh) - (\tan\theta')length(qh) & \\ & = & \frac{1-\sin\theta'}{\cos\theta'}length(qh) & \\ & > & \frac{1-\sin\theta'}{\cos\theta'}\frac{c_0}{1+\cos\theta}\frac{\log n}{\sqrt{n}} & \\ & > & \frac{1+\cos\theta}{\cos\theta'}\frac{c_0}{1+\cos\theta}\frac{\log n}{\sqrt{n}} & \text{since } \sin\theta' < -\cos\theta \\ & > & c_0\frac{\log n}{\sqrt{n}}. & \end{array} \tag{28}$$

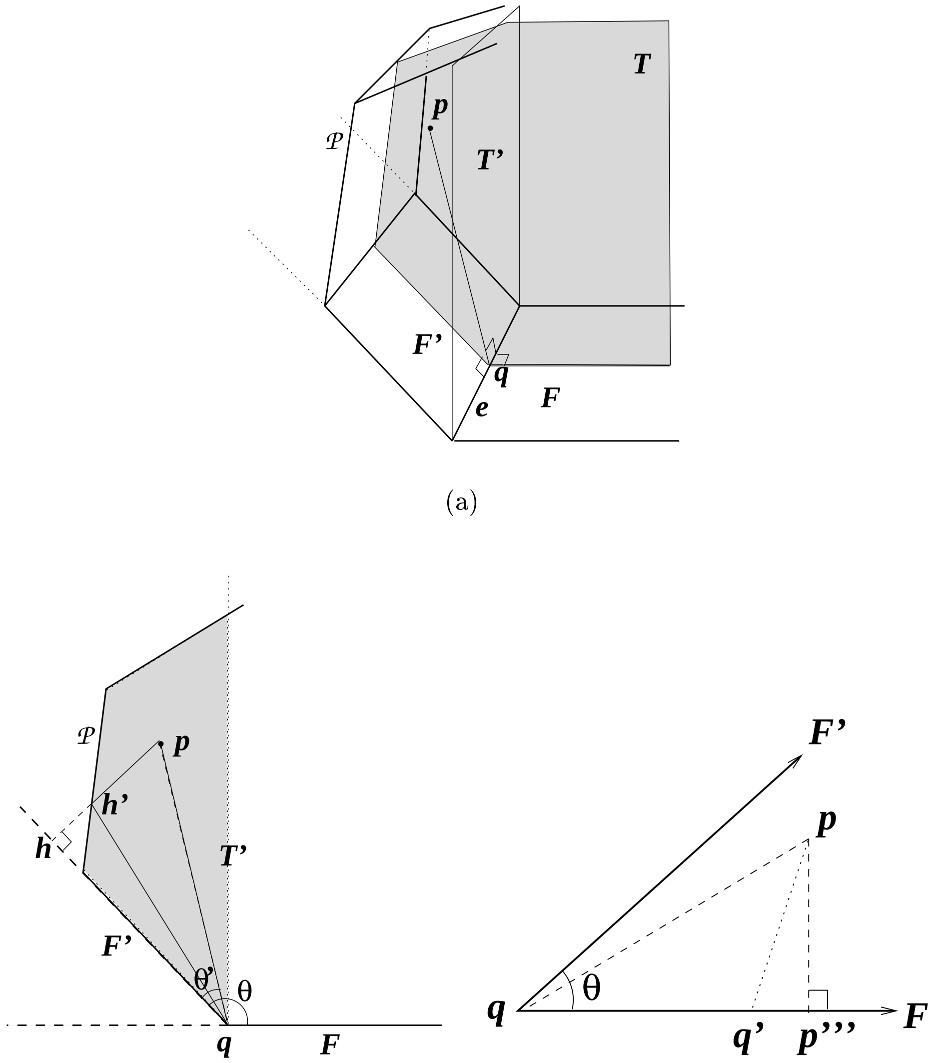
Since $h \notin F' \cap T$, h must be outside $\mathcal{P} \cap T$. Since p is inside $\mathcal{P} \cap T$ and h is outside $\mathcal{P} \cap T$, there must be a point h' such that $h' \in ph \cap \mathcal{P} \cap T$. For such h', (28) gives that

$$length(pq) - length(ph') > length(pq) - length(ph) > c_0\frac{\log n}{\sqrt{n}}.$$

Since S has radius $\ge length(pq)$, this means that the disc centered at h' with radius $c_0\frac{\log n}{\sqrt{n}}$ is in S. By Definition 6 , this contradicts the goodness of S. Hence we have proven that if $h \in F' \cap T$, then $length(qh) \le c_0\frac{\log n}{\sqrt{n}}$ and if $h \notin F' \cap T$, then $length(qh) \le \frac{c_0}{1+\cos\theta}\frac{\log n}{\sqrt{n}}$. By letting $C = \max\{c_0, \frac{c_0}{1+\cos\theta}\}$, $length(qh) \le C\frac{\log n}{\sqrt{n}}$, so we are done.

2. Assume that q is a vertex of F and thus of $\mathcal{P}$ as well. Let $\mathcal{C}$ be the cone defined by extending the faces of $\mathcal{P}$ incident upon q. Let $F_1, \cdots, F_n$ be the other faces of $\mathcal{P}$ incident upon q and $\mathcal{F}_1, \cdots, \mathcal{F}_n$ their corresponding faces of $\mathcal{C}$, respectively. Let $\mathcal{F}$ be the face of $\mathcal{C}$ corresponding to F. Note that $F \subset \mathcal{F} \subset \Pi$. Let $\theta = \sup_{x,y\in\mathcal{C}} \angle xqy$. Then $0 < \theta < 180°$.

Let $h = NN(p, \mathcal{C})$. Since $\mathcal{C}$ is a convex body and p is inside $\mathcal{C}$, h must be in some face of $\mathcal{C}$ but be neither on an edge nor on the vertex q; suppose, to the contrary, that h is on an edge e or the vertex q of $\mathcal{C}$. If h is on an edge e, then

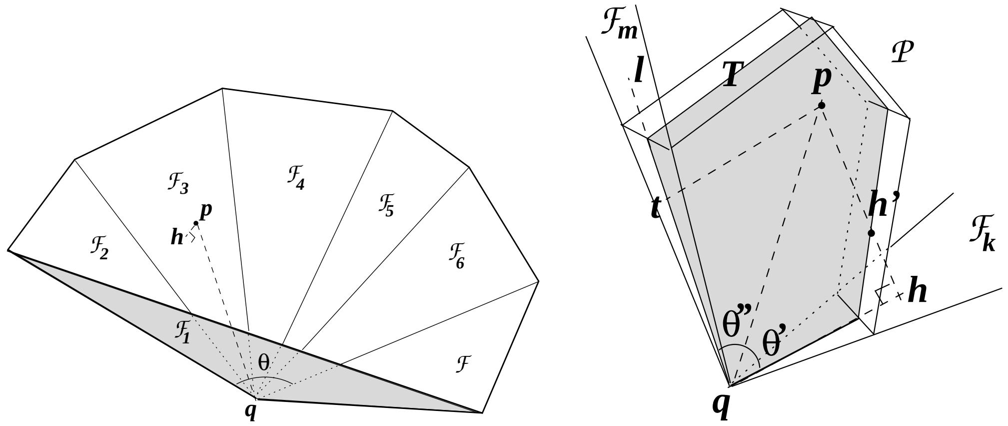


(a)

(b)

(c)

Figure 12: Proof of Lemma 4 when $q = NN(p, F)$ is on the edge e.



(a) $\mathcal{F}_k = \mathcal{F}_3$ since $h \in \mathcal{F}_3$.

(b) $\theta' \leq \frac{\theta}{2}$ where $\theta = \sup_{x,y\in\mathcal{C}} \angle xqy$ and $h = NN(p, \mathcal{C})$.

Figure 13: Proof of Lemma 4 when $q = NN(p, F)$ is a vertex.

$ph \perp e$. Let H be the plane perpendicular to e and passing through h. Note that H contains p and $H \cap \mathcal{C}$ is a convex polygon such that p lies in its interior and h is one of the vertices. Since $h = NN(p, \mathcal{C})$, $h = NN(p, H \cap \mathcal{C})$, which contradicts Lemma 2. If $h = q$, then let H be any plane passing through h and p. Then $H \cap \mathcal{C}$ is a dihedral angle composed of two half lines incident upon h and with q lying inside. Since $h = NN(p, \mathcal{C})$, $h = NN(p, H \cap \mathcal{C})$, which again contradicts Lemma 2.

We also have that $h \notin \mathcal{F}$, since if $h \in \mathcal{F}$, there must be points $q' \in qh$ such that $q' \neq q$, $qq' \subseteq qh$, and $qq' \subseteq F$, and this yields a contradiction (the first and the second conditions guarantee that $length(pq') < length(pq)$, and the third one implies that $q' \in F$, so the existence of such q' contradicts $q = NN(p, F)$). Let $\mathcal{F}_k \neq \mathcal{F}$ be the face of $\mathcal{C}$ such that $h \in \mathcal{F}_k$. See Figure 13(a).

Let T be the plane passing through points q, h, and p. $T \cap \mathcal{C}$ is composed of two half lines rooted at vertex q. One is $\overrightarrow{qh}$ and let l be the other half line. Let $\mathcal{F}_m$ be the face of $\mathcal{C}$ such that $l \subset \mathcal{F}_m$. See Figure 13(b). Let $\theta' = \angle pqh$ and θ'' the angle between $\overrightarrow{qh}$ and l measured inside $\mathcal{P}$. Note that $\theta' + \theta'' \leq \theta < 180°$. We claim that $\theta' \leq \frac{\theta}{2}$. Suppose by contradiction that $\theta' > \frac{\theta}{2}$. Then $\theta'' \leq \theta - \theta' < \frac{\theta}{2} < 90°$. This means that the perpendicular t, dropped from p onto the line defined by extending l, is on l and thus on $\mathcal{C}$. Since $\theta'' < \frac{\theta}{2} < \theta'$, we have $d(p, \mathcal{C}) \leq d(p, t) < d(p, h)$, which contradicts the definition of h.

We will now prove, in a fashion similar to part 1, that $length(qh) \leq C\frac{\log n}{\sqrt{n}}$ for some constant C dependent only upon θ and c_0. Using the facts that $length(pq) =$

$\frac{1}{\cos\theta'}length(qh)$ and $\theta' \le \frac{\theta}{2}$, we will then have

$$length(pq) = \frac{1}{\cos\theta'}length(qh) \le \frac{1}{\cos\frac{\theta}{2}}length(qh) \le \frac{C}{\cos\frac{\theta}{2}}\frac{\log n}{\sqrt{n}}.$$

As in the proof of part 1 we know that if $r \ge d(p,q) + c_0\frac{\log n}{\sqrt{n}}$ then $S(q, c_0\frac{\log n}{\sqrt{n}}) \subseteq S$. But, from Definition 6 the $\frac{\log n}{\sqrt{n}}$-goodness of S forces $S(q, c_0\frac{\log n}{\sqrt{n}}) \not\subseteq S$. Thus $r < d(p,q) + c_0\frac{\log n}{\sqrt{n}}$. This in turn implies, again as in the proof of part 1, that

$$\forall u \in S(p,r), \quad d(q,u) \le d(q,p) + d(p,u) \le 2r \le 2\left(\frac{C}{\cos\frac{\theta}{2}} + c_0\right)\frac{\log n}{\sqrt{n}}. \qquad (29)$$

Setting $c_4 = 2\left(\frac{C}{\cos\frac{\theta}{2}} + c_0\right)$, we will be done.

We now prove that $length(qh) \le C\frac{\log n}{\sqrt{n}}$. See Figure 13(b). Recall that $h \in \mathcal{F}_k$. There are two cases; either $h \in F_k$, or $h \notin F_k$.

If $h \in F_k$, then note that $q = NN(p,F) \in S \cap F_k$. By definition $ph \perp \Pi_k$, the supporting plane of F_k. Thus $h = NN(p, \Pi_k)$ so $S \cap \Pi_k = D_{\Pi_k}(h, r')$ for some $r' > 0$. Since $q \in \Pi_k \cap S$ we have $r' > length(qh)$. But, from Definition 6, the $\frac{\log n}{\sqrt{n}}$-goodness of S and the fact that $h \in F_k$, we find that $r' \le c_0\frac{\log n}{\sqrt{n}}$ proving that $length(qh) < c_0\frac{\log n}{\sqrt{n}}$.

If $h \notin F_k$, then we claim that $length(qh) \le \frac{c_0}{1-\sin\frac{\theta}{2}}\frac{\log n}{\sqrt{n}}$; Suppose that $length(qh) > \frac{c_0}{1-\sin\frac{\theta}{2}}\frac{\log n}{\sqrt{n}}$. Then we have

$$\begin{aligned}
length(pq) - length(ph) &= \tfrac{1}{\cos\theta'}length(qh) - (\tan\theta')length(qh) \\
&= \tfrac{1-\sin\theta'}{\cos\theta'}length(qh) \\
&> \tfrac{1-\sin\theta'}{\cos\theta'}\tfrac{c_0}{1-\sin\frac{\theta}{2}}\tfrac{\log n}{\sqrt{n}} \\
&\ge \tfrac{1-\sin\frac{\theta}{2}}{\cos\theta'}\tfrac{c_0}{1-\sin\frac{\theta}{2}}\tfrac{\log n}{\sqrt{n}} \quad \text{since } \sin\theta' \le \sin\tfrac{\theta}{2} \\
&\ge c_0\tfrac{\log n}{\sqrt{n}}.
\end{aligned} \qquad (30)$$

Since $h \in \mathcal{F}_k \setminus F_k$, h must be outside $\mathcal{P}$. Since p is inside $\mathcal{P}$ and h is outside $\mathcal{P}$, there must be a point h' such that $h' \in ph \cap \mathcal{P}$. For such h', (30) gives that

$$length(pq) - length(ph') > length(pq) - length(ph) > c_0\frac{\log n}{\sqrt{n}},$$

which means that the disc centered at h' with radius $c_0\frac{\log n}{\sqrt{n}}$ is in S. By Definition 6, this contradicts the goodness of S. Hence we have proven that if $h \in F_k$, then $length(qh) \le c_0\frac{\log n}{\sqrt{n}}$ and if $h \notin F_k$, then $length(qh) \le \frac{c_0}{1-\sin\frac{\theta}{2}}\frac{\log n}{\sqrt{n}}$. By letting $C = \max\{c_0, \frac{c_0}{1-\sin\frac{\theta}{2}}\}$, $length(qh) \le C\frac{\log n}{\sqrt{n}}$, so we are done with part 2.

In 1 we proved the lemma for the case that q *is not* a vertex of P; in 2 we proved it when q *is* a vertex of P. Thus, the proof of the lemma is complete. □

References

[1] `Mordecai Golin and Hyeon-Suk Na` "On the Average Complexity of 3D-Voronoi Diagrams of Random Points on Convex Polytopes," *Hong Kong UST Theoretical Computer Science Center Technical Report,* July 2001.