The Probabilistic Complexity of the Voronoi Diagram of Points on a Polyhedron*

Mordecai J. Golin
Dept. of Computer Science
Hong Kong UST
golin@cs.ust.hk

Hyeon-Suk Na
Inria-Lorraine
Hyeonsuk.Na@loria.fr

December 8, 2001

Abstract

It is well known that the complexity, i.e., the number of vertices, edges and faces, of the 3-dimensional Voronoi diagram of n points can be as bad as $\Theta(n^2)$. Interest has recently arisen as to what happens, both in deterministic and probabilistic situations, when the three dimensional points are restricted to lie on the surface of a 2-dimensional object. In this paper we consider the situation when the points are drawn from a 2-dimensional Poisson distribution with rate n over a fixed union of triangles in $\mathbb{R}^3$. We show that with high probability the complexity of their Voronoi diagram is $\tilde{O}(n)$.

This implies, for example, that the complexity of the Voronoi diagram of points chosen from the surface of a general fixed polyhedron in $\mathbb{R}^3$ will also be $\tilde{O}(n)$ with high probability.

1 Introduction

Let P be a set of 3-dimensional points. The *Voronoi Diagram* of the points and its dual, the Delaunay triangulation, are extremely well studied structures. The *complexity*, $|VD(P)|$ of the Voronoi diagram is the number of lower dimensional pieces of which it is composed, i.e., the total number of vertices, edges and faces and regions that it contains. It is well known that, in the worst case, the complexity can be as high as $\Theta(n^2)$ [6]. It has also been observed that, if the points are sampled from some types of restricted point sets, the complexity tends, in practice, to be much lower.

The problem of understanding the structure of the 3-dimensional Voronoi diagram of point sets from 2-dimensional surfaces has begun to be of interest in recent years. This is because, as described in [1] and [3], Voronoi diagrams and Delaunay triangulations are of use in several geometric problems, e.g., surface reconstruction, mesh generation and surface modeling. In these problems a 2-dimensional surface is often sampled and then modeled, at least initially, by the Delaunay triangulation of the sample. Many parameters of such algorithms such as their running times and the complexity of their representations, then depend upon the complexity of the Delaunay triangulation (which is the same as that of the Voronoi diagram).

*The work of both authors was partially supported by HK RGC CERG grants 6082/97E and 6137/98E.

The two results [1] and [3] mentioned above seem to be the first to try and formally analyze the complexity of such Voronoi diagrams. In [1] Attali and Boissonnat prove that if n "well-sampled" points are chosen from a polyhedral surface then the complexity of their Voronoi diagram is $O(n^{7/4})$ where "well-sampled" is defined using the concept of local feature size; if the points are drawn in the same way from the surface of a *convex* polytope this reduces down to $O(n^{3/2})$.[1] In [3] Erikson proves that there is a set of n "well-sampled" points from the cylinder with Voronoi diagram complexity $\Omega(n^{3/2})$.

Working from a probabilistic perspective the authors of this work showed [4] that if points are drawn from a 2-dimensional Poisson distribution with rate n from the surface of a fixed *convex* polytope then the expected complexity of the Voronoi diagram of the points would be $O(n)$ (with the same result also holding if n points were chosen IID uniformly from the surface of the polytope).

The major result of this paper is to prove a high probability theorem when points are drawn from the surface of a collection of triangles. More specifically

Theorem 1 *Let $\mathcal{F} = \{F_1, \ldots, F_k\}$ be a collection of k triangles in $\mathbb{R}^3$. Let P_n be a set of points drawn from a 2-dimensional Poisson distribution on $\mathcal{F}$ with rate n. Then* $\Pr\left(|\mathrm{VD}(P_n)| = \tilde{O}(n)\right) = 1 - n^{-\Omega(\log n)}$.

Note that since a polyhedron can be decomposed into a finite set of triangles this immediately implies that if P_n is a set of points drawn from the standard 2-dimensional Poisson distribution with rate n on the surface of a fixed polyhedron then $\Pr\left(|\mathrm{VD}(P_n)| = \tilde{O}(n)\right) = 1 - n^{-\Omega(\log n)}$.

We start by defining some of the terms used in the theorem. A *triangle* will denote a *closed* triangle that contains its interior and edges.

A *Poisson distribution on $\mathcal{F}$ with rate n* is the distribution in which, for every measurable $M \subset \cup_i F_i$, we have that $N(M)$, the number of points in M, satisfies $\Pr(N(M) = k) = \frac{(n\mathrm{Area}(M))^k e^{-(n\mathrm{Area}(M))}}{k!}$ and for $M, M' \subset F$ with $M \cap M' = \emptyset$, $N(M)$ and $N(M')$ are independent. The expected number of points in P_n will be $\mathrm{Area}(\mathcal{F}) \cdot n$.

$g(n) = \tilde{O}(f(n))$ denotes that $g(n) = O(f(n) \log^i n)$ for some fixed i., i.e., it "hides" $\log n$ factors. In Theorem 1 as proved in this paper the $\tilde{O}(n)$ is actually a $O(n \log^{16} n)$ term. The reason for using the tilde notation is that it vastly simplifies the proofs by allowing us to lump many cases together and makes them more readible. We point out that a more careful analysis that splits the world into more cases can reduce the $\tilde{O}(n)$ term down to $O(n \log^4 n)$.

Also, with slight modifications to the proofs, all of the probabilistic results given in this paper will hold if we change the distribution so that P_n, instead of being drawn from a Poisson distribution, is a set of n points independently identically distributed uniformly from $\mathcal{F}$.

In the next section we discuss how to reduce the proof of Theorem 1 to a case-by-case analysis of a simpler problem. In Section 3 we introduce some tools that we will need in our proof and in Section 4 we perform the case by case analysis.

Before ending this section we review some notation that we will be using: For a point $p \in \mathbb{R}^3$ and any closed or finite set $X \subseteq \mathbb{R}^3$ extend the Euclidean distance function so that $d(p, X) = \min_{q \in X} d(p, q)$. Also set $NN(p, X)$ to be a nearest neighbor to p in X, i.e., a $q \in X$ such that

[1] Just prior to submission we learnt of new work [2] by Attali and Boissonnat that proves a linear bound on the complexity of the Delaunay Triangulation of n points well-sampled from a polyhedral surface (using a slightly revised definition of well-sampled). This seems to be the deterministic analogue of the result in this current paper.

$d(p,q) = d(p,X)$. In this paper X will always be a closed polygonal piece so $NN(p,X)$ will be unique.

For $r > 0$ define

$$S(p,r) = \{q \in \mathbb{R}^3 \,:\, d(q,p) \leq r\}$$

to be the closed sphere of radius r around p.

Let Π be a plane and $p \in \Pi$. Then define $C_\Pi(p,r) = \{q \in \Pi \,:\, d(q,p) \leq r\}$ to be the closed disc of radius r on Π centered at p.

Finally, for F, F' triangles and Π, Π' their supporting planes we say that $F \| F'$ if Π is parallel to Π' and $F \not\| F'$ otherwise.

Important Note: In this extended abstract we only provide the upper level description of some of the proofs and omit some details.

2 Reductions

Instead of calculating the complexity $|VD(P_n)|$ directly we will instead bound the number of *Voronoi Spheres* of P_n. A Voronoi sphere is a sphere that has at least four points of P_n on its boundary and no other points of P_n in its interior. It is well known that there is a one-to-one correspondence between Voronoi Vertices and Voronoi Spheres [6]. Also, every edge in the Voronoi diagram corresponds to some triple of vertices that are on the same Voronoi Sphere and every face of the Voronoi diagram corresponds to some pair on the same Voronoi Sphere. Since the event of 5 points in a P_n chosen from the Poisson distribution being cospherical has probability zero we can assume that each Voronoi Sphere has *exactly* 4 points on its surface so the complexity of $|VD(P_n)|$ is proportional to the number of Voronoi Spheres. To simplify matters, in the rest of this paper we will therefore assume that $|VD(P_n)|$ is the number of Voronoi Spheres and not the full complexity.

Bounding the probabilistic complexity of this number *directly* can be quite difficult since the conditionality of assuming that certain spheres are Voronoi skews the rest of the distribution. In this section we show how to reduce the problem to a more manageable one. This will require introducing new definitions and utility lemmas. In everything that follows it is implictly assumed that n is fixed.

Definition 1 *Let $F \subset \mathbb{R}^3$ be a triangle, $p \in \mathbb{R}^3$ and $r \geq 0$. Then $S(p,r)$ is an F-*good sphere *if $Area(S(p,r) \cap F) \leq \frac{\log^2 n}{n}$.*

*If $\mathcal{F} = \{F_1, \ldots, F_k\}$ is a collection of triangles then $S(p,r)$ is an $\mathcal{F}$-*good sphere *if $S(p,r)$ is an F_i-good sphere for every triangle F_i.*

The important observation is

Lemma 1 *Let $\mathcal{F} = \{F_1, \ldots, F_k\}$ be a collection of k triangles in $\mathbb{R}^3$. Let P_n be a set of points drawn from a 2-dimensional Poisson distribution on $\mathcal{F}$ with rate n. Then*

$$\Pr(\text{ all Voronoi spheres in } VD(P_n) \text{ are } \mathcal{F}\text{-good}) = 1 - n^{-\Omega(\log n)}. \tag{1}$$

The inituition behind this proof is that if a given sphere is not $\mathcal{F}$-good then the probability that it contains no points of the Poisson distributed P_n is no more than $e^{-\frac{\log^2 n}{n} n} = n^{-\Omega(\log n)}$. This intuition can be formalized into a rigorous proof (a similar lemma was proved in [4] for $\mathcal{F}$ the boundary of a convex polygon.)

We now define something easier to bound than Voronoi spheres:

Definition 2 *Let $\mathcal{F} = \{F_1, \dots, F_k\}$ be a collection of triangles, $X \subseteq \mathbb{R}^3$ and $P \subseteq \cup_i F_i$. Now define*

$$\begin{aligned} D(P) &= \{\{p_1, p_2, p_3, p_4\} : \{p_1, p_2, p_3, p_4\} \subseteq P \cap S(p,r) \\ &\qquad \text{for some } \mathcal{F}\text{-good sphere } S(p,r)\} \\ D_X(P) &= \{\{p_1, p_2, p_3, p_4\} : \{p_1, p_2, p_3, p_4\} \subseteq P \cap S(p,r) \\ &\qquad \text{for some } \mathcal{F}\text{-good sphere } S(p,r) \text{ with } p \in X\} \end{aligned}$$

Note that if some Voronoi Sphere $S = S(p,r)$ containing 4 vertices $\{p_1, p_2, p_3, p_4\}$ on its boundary is also a $\mathcal{F}$-good sphere then $\{p_1, p_2, p_3, p_4\} \in D(P_n)$. So, if *all* Voronoi spheres in $VD(P_n)$ are $\mathcal{F}$-good spheres then $|VD(P_n)| \leq |D(P_n)|$. Combining this with Lemma 1 gives that

$$\Pr\left(|D(P_n)| = \tilde{O}(n)\right) = 1 - n^{-\Omega(\log n)} \text{ implies } \Pr\left(|\mathrm{VD}(P_n)| = \tilde{O}(n)\right) = 1 - n^{-\Omega(\log n)}.$$

We will now devote ourselves to proving

Theorem 2 *Let $\mathcal{F} = \{F_1, \dots, F_k\}$ be a collection of k triangles in $\mathbb{R}^3$. Let P_n be a set of points drawn from a 2-dimensional Poisson distribution on $\mathcal{F}$ with rate n. Then* $\Pr\left(|D(P_n)| = \tilde{O}(n)\right) = 1 - n^{-\Omega(\log n)}$.

From the discussion above, proving Theorem 2 implies Theorem 1.

We can actually go one step further and notice that, since (i) if $S(p,r)$ is a $\mathcal{F}$-good sphere with respect to P_n then $S(p,r)$ is a $\{F_{i_1}, F_{i_2}, F_{i_3}, F_{i_4}\}$-good sphere for any $1 \leq i_1, i_2, i_3, i_4 \leq k$ and (ii) $D(P_n)$ is defined by 4-tuples of points, we have

$$D(P_n) \subseteq \bigcup_{1 \leq i_1, i_2, i_3, i_4 \leq k} D(P_n \cap (F_{i_1} \cup F_{i_2} \cup F_{i_3} \cup F_{i_4})).$$

so

$$|D(P_n)| \leq \sum_{1 \leq i_1, i_2, i_3, i_4 \leq k} |D(P_n \cap (F_{i_1} \cup F_{i_2} \cup F_{i_3} \cup F_{i_4}))| \tag{2}$$

Suppose we had

Theorem 3 *Let $\mathcal{F} = \{F_1, F_2, F_3, F_4\}$ be a collection of 4 triangles in $\mathbb{R}^3$. Let P_n be a set of points drawn from a 2-dimensional Poisson distribution on $\mathcal{F}$ with rate n. Then* $\Pr\left(|D(P_n)| = \tilde{O}(n)\right) = 1 - n^{-\Omega(\log n)}$.

One of the properties of the Poisson distribution is that if P_n is a set of points drawn from a 2-dimensional Poisson distribution on $\mathcal{F}$ with rate n and $X \subset \mathcal{F}$ is measurable, then $P_n \cap X$ has the same distribution as a set of points drawn from a 2-dimensional Poisson distribution on X with rate n. Therefore, combining Theorem 3 and (2) would imply Theorem 2. Thus, to prove Theorem 2 it suffices to prove Theorem 3.

From now on we will therefore assume that $\mathcal{F} = \{F_1, F_2, F_3, F_4\}$ is composed of four triangles. Our approach to proving Theorem 3 will be to split $D(P_n)$ up into managable pieces. We do this as follows:

Definition 3 *Let $F \subset \mathbb{R}^3$ be a triangle. Set $V(F)$, $E(F)$ and $I(F)$, to be, respectively, the vertices, edges and interior of F. That is $V(F)$ are the three vertices of F, $E(F)$ the three edges minus the vertices and $I(F)$, the triangle without its edges.*

For any $p \in \mathbb{R}^3$ set

$$r_p = \max\{r \,:\, S(p,r) \text{ is a } \mathcal{F}\text{-good sphere}\}.$$

For $i = 1,2,3,4$, define $L_i(p) \in \{\emptyset, V, E, I\}$ such that

$$L_i(p) = \begin{cases} \emptyset & \text{if } S(p,r_p) \cap F_i = \emptyset \\ V & \text{if } S(p,r_p) \cap F_i \neq \emptyset \quad \text{and} \quad NN(p,F_i) \in V(F_i) \\ E & \text{if } S(p,r_p) \cap F_i \neq \emptyset \quad \text{and} \quad NN(p,F_i) \in E(F_i) \\ I & \text{if } S(p,r_p) \cap F_i \neq \emptyset \quad \text{and} \quad NN(p,F_i) \in I(F_i) \end{cases}$$

$L = (L_1, L_2, L_3, L_4)$ will be the *label* of p. Figure 1 illustrates the ways in which a sphere can intersect a triangle. Figure 2 shows a point with $L(p) = (E, E, I, V)$. For any given label $L = (L_1, L_2, L_3, L_4) \in \{\emptyset, V, E, I\}^4$ we can define

$$R_L = \left\{p \in \mathbb{R}^3 \,:\, L(p) = L\right\}.$$

Since every $p \in \mathbb{R}^3$ has a *unique* label $L(p)$ the R_L form a partition of $\mathbb{R}_3$ into $4^4 - 1 = 255$ regions (label $(\emptyset,\emptyset,\emptyset,\emptyset)$ can trivially not occur). Note that this partition depends upon n but not on the Poisson distribution.

Now, let P_n be any finite set of points in $\mathbb{R}^3$ (not necessarily a random one). Since every tuple $\{p_1, p_2, p_3, p_4\} \in D(P_n)$ is contained in at least one $\mathcal{F}$-good sphere $S(p,r)$ and every p is in exactly one R_L, we have $D(P_n) = \bigcup_L D_{R_L}(P_n)$ so $|D(P_n)| \le \sum_L |D_{R_L}(P_n)|$. Our major theorem is:

Theorem 4 *Let $\mathcal{F} = \{F_1, F_2, F_3, F_4\}$ be a collection of 4 triangles in $\mathbb{R}^3$ and P_n be a set of points drawn from a 2-dimensional Poisson distribution on $\mathcal{F}$ with rate n. Let $L \in \{\emptyset, V, E, I\}^4$ be an arbitrary fixed label. Then*

$$\Pr\left(|D_{R_L}(P_n)| = \tilde{O}(n)\right) = 1 - n^{-\Omega(\log n)}. \tag{3}$$

Since this is true for all labels L and there are "only" 255 different labels summing gives

$$\Pr\left(|D(P_n)| = \tilde{O}(n)\right) = 1 - n^{-\Omega(\log n)}$$

proving Theorem 3 which, as discussed before, implies Theorem 1.

We have therefore just demonstrated that to prove Theorem 1 it suffices to prove Theorem 4. The sequel of this paper will be devoted to proving this.

3 Tools

In this section we introduce the major tools that we will use in our proof. In what follows we always assume that $\mathcal{F} = \{F_1, F_2, F_3, F_4\}$ is a collection of four triangles.

Definition 4 *Let $F \subset \mathbb{R}^3$ be a triangle, $p' \in F$ and $X \subseteq F$. Define*

$$\begin{aligned}\mathcal{G}_F(p') &= \left\{S(p,r) \,:\, NN(p,F) = p' \text{ and } S(p,r) \text{ is an } F\text{-good sphere}\right\}\\ M_F(p') &= \bigcup_{S(p,r)\in\mathcal{G}_F(p')} (S(p,r)\cap F)\\ M_F(X) &= \bigcup_{p'\in X} M_F(p').\end{aligned}$$

Intuitively $M(p')$ is the set of all points in F that can belong to some F-good sphere such that if p is the center of the sphere then $NN(p,F) = p'$.

Definition 5 *Let $p \in \mathbb{R}^3$ be fixed. For $i = 1, 2, 3, 4$ set*

$$G_i(p) = \begin{cases} M_{F_i}(NN(p,F_i) & L_i(p) \neq \emptyset \\ \emptyset & L_i(p) = \emptyset \end{cases}$$

and define $G(p) = \cup_{i=1}^{4} G_i(p)$.

For $X \subseteq \mathbb{R}^3$ set $G_i(X) = \cup_{p\in X} G_i(p)$ and $G(X) = \cup_{i=1}^{4} G_i(X) = \cup_{p\in X} G(p)$.

The important thing to notice is that $G(p)$ contains *all* points that can be contained by some $\mathcal{F}$-good sphere centered at p. More formally, if $S(p,r)$ is an F_i-good sphere then, by definition, $S(p,r)\cap F_i \subseteq G_i(p)$; thus, if $\{p_1, p_2, p_3, p_4\} \subset S(p,r)$ for some $\mathcal{F}$-good sphere then $\{p_1, p_2, p_3, p_4\} \subset G(p)$. This in turn implies

Lemma 2 *Let $P_n \subset \mathbb{R}^3$ be finite and $X \subseteq \mathbb{R}^3$. Then*

$$|D_X(P_n)| \leq |P_n \cap G(X)|^4.$$

Noting the fact that if $X \subset \mathbb{R}^3$ is measurable in $\mathbb{R}^3$ then $G(X)$ is measurable in $\mathcal{F}$ and using the definition of the Poisson distribution we can then prove:

Corollary 3 *Let $X \subset \mathbb{R}^3$ be measurable and $f > 0$ such that $Area(G(X)) \leq f/n$. Let P_n be a set of points drawn from a 2-dimensional Poisson distribution on $\mathcal{F}$ with rate n. Then*

$$\Pr\left(|D_X(P_n)| \geq (f \log n)^4\right) = n^{-\Omega(\log n)}. \tag{4}$$

This Corollary will be our major tool in proving Theorem 4. To employ it properly we will need to be able to efficiently bound $G(X)$. We will do this using the following:

Lemma 4 *Let $F \subset \mathbb{R}^3$ be a triangle. There exists constant $c_F > 0$ such that:*

1. *Let $p' \in I(F)$. Then $Area(M_F(p')) = \frac{\log^2 n}{n}$.*
2. *Let $B \subset I(F)$ be an $\frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{n}}$ square. Then $Area(M_F(B)) \leq c_F \frac{\log^2 n}{n}$.*
3. *Let $s \subset E(F)$ be an edge segment[2] with $length(s) \leq \frac{\log n}{\sqrt{n}}$. Then $Area(M_F(s)) \leq c_F \frac{\log^3 n}{n}$.*

[2]an edge segment can be either closed, open, or half open, half closed.

4. Let $v \in V(F)$ be a vertex of F. Then $Area(M_F(v)) \leq c_F \frac{\log^3 n}{n}$.

In this extended abstract we do not give the full proof of this lemma. To provide the intuition as to why it is correct suppose that $p' \in I(F)$. Let Π be the supporting plane of F. Then $M_F(p')$ is *exactly* $C_\Pi(p', r) \cap F$ where r is the unique value such that $Area(C_\Pi(p', r) \cap F) = \frac{\log^2 n}{n}$. This proves (1). To prove (2) we examine the union of all such discs whose center can be in the small square B. The proofs of (3) and (4) are much more tedious and require a detailed case-by-case analysis. This analysis can be found in the proof of Lemma 3 in [5].

As a first consequence of Lemma 4 let $X = S(p', r)$ be any ball; we will find a general bound on $|D_X(P_n)|$. For all i let Π_i be the supporting plane of F_i and $p_i'' = NN(p', \Pi_i)$. Now let $p \in X$. If $NN(p, F_i) \in I(F)$ then $NN(p, F_i) \in C_{\Pi_i}(p_i'', r)$, i.e., the projection of X on Π_i. Such a circle can be covered by $4nr^2$ squares of size $\frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{n}}$. So, from part (2) of the Lemma,

$$\text{Area}\left(\bigcup \{G_i(p) \,:\, p \in X \text{ and } NN(p, F_i) \in I(F_i)\}\right) \leq 4nr^2 c_F \frac{\log^2 n}{n} = 4c_F r^2 \log^2 n.$$

If $NN(p, F_i) \in E(F)$ it is not hard to see that all such nearest neighbors must be contained in the projection of X on the three edges of F which can be composed of at most three segments, each of length at most $2r$. These can be partitioned into $\frac{6r\sqrt{n}}{\log n}$ segments of length $\leq \frac{\log n}{\sqrt{n}}$. From part (3) of the Lemma,

$$\text{Area}\left(\bigcup \{G_i(p) \,:\, p \in X \text{ and } NN(p, F_i) \in E(F_i)\}\right) \leq \frac{6r\sqrt{n}}{\log n} c_F \frac{\log^3 n}{n} = \frac{6rc_F \log^2 n}{\sqrt{n}}.$$

Finally, from part (4) of the Lemma and the fact that $V(F_i)$ contains only 3 vertices,

$$\text{Area}\left(\bigcup \{G_i(p) \,:\, p \in X \text{ and } NN(p, F_i) \in V(F_i)\}\right) \leq 3c_F \frac{\log^3 n}{n}.$$

Combining the above we have

$$\text{Area}(G_i(X)) = O\left(r^2 \log^2 n + \frac{r \log^2 n}{\sqrt{n}} + \frac{\log^3 n}{n}\right).$$

In particular, if $r = \tilde{O}\left(\frac{1}{\sqrt{n}}\right)$ then $\text{Area}(G_i(X)) = \tilde{O}\left(\frac{1}{n}\right)$ and $\text{Area}(G(X)) \leq \sum_{i=1}^{4} \text{Area}(G_i(X)) = \tilde{O}\left(\frac{1}{n}\right)$ as well. Applying Corollary 3 we have just proven:

Lemma 5 *If $X = S(p', r)$ with $r = \tilde{O}\left(\frac{1}{\sqrt{n}}\right)$ then*

$$\Pr\left(|D_X(P_n)| = \tilde{O}(1)\right) = 1 - n^{-\Omega(\log n)}.$$

4 Proof of Theorem 4

We will now prove Theorem 4 by doing a case by case analysis for the 255 different possible labels $L = (L_1, L_2, L_3, L_4)$ and proving (3) for each one. As we will see, after symmetry and other reductions, there will only be seven distinct cases.

In this extended abstract we do not have room to provide proofs of all seven; we therefore only prove the first five. The proof of the sixth and seventh are omitted; they essentially use a a combination of the techniques developed for the first five.

Case 1: $\forall i,\ L_i \in \{\emptyset, V\}$.

In this case $\forall p \in R_L,\ G_i(p) \subseteq M_{F_i}(V(F_i))$. Thus $G(p) \subseteq \cup_{i=1}^4 M_{F_i}(V(F_i))$. Since this is true for all $p \in R_L$ we also have $G(R_L) \subseteq \cup_{i=1}^4 M_{F_i}(V(F_i))$ so by Lemma 4, $\text{Area}(G(R_L)) \le 12 c_F \frac{\log^3 n}{n}$. Applying Corollary 3 gives

$$\Pr\left(|D_{R_L}(P_n)| = \tilde{O}(1)\right) = 1 - n^{-\Omega(\log n)}.$$

Case 2: $\exists i$ such that $L_i = E$ and $\forall j \neq i,\ L_j \in \{\emptyset, V\}$.

Without loss of generality we will assume that $L_1 = E$. Partition the edges of E up into $t = O\left(\frac{\sqrt{n}}{\log n}\right)$ smaller segments $s_1, s_2, \ldots, s_t$ each of length $\le \frac{\log n}{\sqrt{n}}$. $\forall m \le t$ set

$$X_m = \{p \in R_L \,:\, NN(p, F_1) \in s_m\}.$$

Since $R_L = \cup_{m=1}^t X_m$ we have $|D_{R_L}(P_n)| \le \sum_{m=1}^t |D_{X_m}(P_n)|$.

By definition $G_1(X_m) \subseteq M_{F_1}(s_m)$ and $\forall j > 1,\ G_j(X_m) \subseteq M_{F_j}(V(F_j))$. By Lemma 4

$$\text{Area}(G(R_L)) \le \sum_{i=1}^4 \text{Area}(G_i(R_L)) = O\left(\frac{\log^3 n}{n}\right)$$

so applying Corollary 3 gives

$$\Pr\left(|D_{X_m}(P_n)| = \tilde{O}(1)\right) = 1 - n^{-\Omega(\log n)}.$$

Summing over all m yields

$$\Pr\left(|D_{R_L}(P_n)| = \tilde{O}\left(\sqrt{n}\right)\right) = 1 - n^{-\Omega(\log n)}.$$

Case 3: $\exists i$ such that $L_i = I$ and $\forall j \neq i,\ L_j \in \{\emptyset, V\}$.

Without loss of generality we will assume that $L_1 = I$. Cover the interior of F_1 by $t = O(n)$ squares $b_1, b_2, \ldots, b_t$ each of size $\frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{n}}$.

$\forall m \le t$ set

$$X_m = \{p \in R_L \,:\, NN(p, F_1) \in b_m\}.$$

Since $R_L = \cup_{m=1}^t X_m$ we have $|D_{R_L}(P_n)| \le \sum_{m=1}^t |D_{X_m}(P_n)|$.

By definition $G_1(X_m) \subseteq M_{F_1}(b_m)$ and $\forall j > 1,\ G_j(X_m) \subseteq M_{F_j}(V(F_j))$. By Lemma 4

$$\text{Area}(G(R_L)) \le \sum_{i=1}^4 \text{Area}(G_i(R_L)) = O\left(\frac{\log^3 n}{n}\right)$$

Applying Corollary 3 as in the previous case and summing over all m will yield

$$\Pr\left(|D_{R_L}(P_n)| = \tilde{O}(n)\right) = 1 - n^{-\Omega(\log n)}.$$

Case 4: $\exists i \neq j$ such that $L_i = L_j = I$ and $F_i \not\parallel F_j$.

Without loss of generality we will assume that $L_1 = L_2 = I$ and $F_1 \not\parallel F_2$. Let Π_1, Π_2 be the supporting planes of F_1 and F_2. Define Π'_1 and Π'_2 to be the *bisecting planes* of Π_1, Π_2, i.e., the two planes that contain the line $\Pi_1 \cap \Pi_2$ and satsfy

$$\Pi'_1 \cup \Pi'_2 = \{p \,:\, d(p, \Pi_1) = d(p, \Pi_2)\}.$$

(Π'_1, Π'_2 exist because Π_1 and Π_2 are not parallel). For arbitrary plane Π define the *h-sandwich* around Π as

$$\Pi(h) = \{p \,:\, d(p, \Pi) \leq h\}.$$

Straightforward geometric arguments show that (for large enough n) if $p \notin \Pi'_1\left(\frac{\log^2 n}{\sqrt{n}}\right) \cup \Pi'_2\left(\frac{\log^2 n}{\sqrt{n}}\right)$ then $|d(p, \Pi_1) - d(p, \Pi_2)| = \Omega\left(\frac{\log^2 n}{\sqrt{n}}\right)$.

Now suppose that p is such that $L_1(p) = L_2(p) = I$ and $p \notin \Pi'_1\left(\frac{\log^2 n}{\sqrt{n}}\right) \cup \Pi'_2\left(\frac{\log^2 n}{\sqrt{n}}\right)$. If $S(p,r)$ is *any* sphere that intersects *both* F_1 and F_2 then $r \geq \max(d(p, \Pi_1),\, d(p, \Pi_2))$. This would imply that $S(p,r)$ would have to contain at least one of $C_{\Pi_1}(NN(p, F_1), r')$ or $C_{\Pi_2}(NN(p, F_2), r')$ for some $r' = \Omega\left(\frac{\log^2 n}{\sqrt{n}}\right)$. $S(p,r)$ can therefore not be *both* F_1-good *and* F_2-good and can therefore not be $\mathcal{F}$-good.

We have therefore just seen that if $S(p,r)$ is a $\mathcal{F}$-good sphere then $p \in \Pi'_1\left(\frac{\log^2 n}{\sqrt{n}}\right) \cup \Pi'_2\left(\frac{\log^2 n}{\sqrt{n}}\right)$. In particular, this implies

$$R_L \in \Pi'_1\left(\frac{\log^2 n}{\sqrt{n}}\right) \cup \Pi'_2\left(\frac{\log^2 n}{\sqrt{n}}\right).$$

As in case 3 we now cover the interior of F_1 by $t = O(n)$ squares $b_1, b_2, \ldots, b_t$, each of size $\frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{n}}$.

$\forall m \leq t$ set

$$X_m = \{p \in R_L \,:\, NN(p, F_1) \in b_m\}.$$

By definition, $R_L = \cup_{m=1}^{t} X_m$ so $|D_{R_L}(P_n)| \leq \sum_{m=1}^{t} |D_{X_m}(P_n)|$.

From the argument above we know that $X_m \subset \Pi'_1\left(\frac{\log^2 n}{\sqrt{n}}\right) \cup \Pi'_2\left(\frac{\log^2 n}{\sqrt{n}}\right)$. For $j = 1, 2$, set $X_m^j = X_m \cap \Pi'_j\left(\frac{\log^2 n}{\sqrt{n}}\right)$. Each X_m^j is contained in the intersection of an infinite square prism with $\frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{n}}$ crossection and a sandwich of width $\frac{\log^2 n}{\sqrt{n}}$ so there exists p_m^j, r_m^j with $r_m^j = O\left(\frac{\log^2 n}{\sqrt{n}}\right)$ such that $X_m^j \subset S(p_m^j, r_m^j)$. Applying Lemma 5 shows that $\Pr\left(|D_{X_m^j}(P_n)| = \tilde{O}(1)\right) = 1 - n^{-\Omega(\log n)}$. Since $|D_{R_L}(P_n)| \leq \sum_{m,j} |D_{X_m^j}(P_n)|$, summing over all m, j yields

$$\Pr\left(|D_{R_L}(P_n)| = \tilde{O}(n)\right) = 1 - n^{-\Omega(\log n)}.$$

Case 5: $\exists i \neq j$ such that $L_i = L_j = I$ and $F_i \| F_j$.

Without loss of generality we will assume that $L_1 = L_2 = I$ and $F_1 \| F_2$. As before, let Π_1, Π_2 be the supporting planes of F_1, F_2. Now, let Π_3 be the plane parallel to Π_1, Π_2 that is equidistant from both. If p has $L_1(p) = L_2(p) = I$ then a straightforward geometric argument similar to the one in case 4 show that if, for some $r \geq 0$, $S(p,r)$ is $\mathcal{F}$-good, then $p \in \Pi_3\left(\frac{\log^2 n}{\sqrt{n}}\right)$.

We now repeat the same procedure. Ccover the interior of F_1 by $t = O(n)$ squares $b_1, b_2, \ldots, b_t$ each of size $\frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{n}}$.

$\forall m \leq t$ set

$$X_m = \{p \in R_L \,:\, NN(p, F_1) \in b_m\}.$$

Since $R_L = \cup_{m=1}^{t} X_m$, we have $|D_{R_L}(P_n)| \leq \sum_{m=1}^{t} |D_{X_m}(P_n)|$.

Note that, from our discussion above, $X_m \in \Pi_3\left(\frac{\log^2 n}{\sqrt{n}}\right)$ so X_m is contained in the intersection of an infinite rectangular prism with $\frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{n}}$ crossection and a sandwich of width $\frac{\log^2 n}{\sqrt{n}}$ so it must be contained in some ball of radius $O\left(\frac{\log^2 n}{\sqrt{n}}\right)$. Applying Lemma 5 shows that $\Pr\left(|D_{X_m}(P_n)| = \tilde{O}\left(1\right)\right) = 1 - n^{-\Omega(\log n)}$. Summing over all m yields

$$\Pr\left(|D_{R_L}(P_n)| = \tilde{O}\left(n\right)\right) = 1 - n^{-\Omega(\log n)}.$$

Case 6: $\exists i \neq j$ such that $L_i = L_j = E$.

Omitted in this extended abstract.

Case 7: $\exists i, j, l, m$ such that $L_i = I$, $L_j = E$ and $L_l, L_m \in \{\emptyset, V\}$.

Omitted in this extended abstract.

References

[1] D. Attali and J-D. Boissonnat, "Complexity of the Delaunay Triangulation of Points on Polyhedral Surfaces," *INRIA Technical report 4232,* (July 2001).

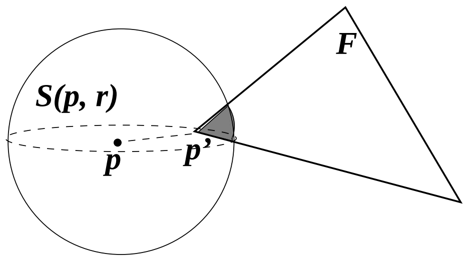
[2] D. Attali and J-D. Boissonnat, "A Linear Bound on the complexity ofthe Delaunay triangulation of points on polyhedral surfaces," Submitted to *Solid Modeling 2002.* Also available at *http://www.inria.fr/sophia/prisme/personnel/boissonnat/Articles/linearDT.ps.gz.*

[3] Jeff Erikson, "Nice Point Sets Can have Nasty Delaunay Triangulations," *Proceedings of the Seventeenth Annual ACM Symposium on Computational Geometry,* (June 3-5, 2001) 96-105.

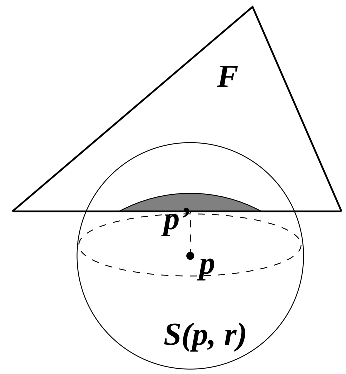
[4] M. J. Golin and HyeonSuk Na, "On the Average Complexity of 3D-Voronoi Diagrams of Random Points On Convex Polytopes," *Proceedings of the 12th Canadian Conference on Computational Geometry (CCCG00),* (2000) 127-135. Extended version submitted to *Computational Geometry: Theory and Applications.*

[5] M. J. Golin and Hyeon-Suk Na "On the Proofs of Two Lemmas Describing the Intersections of Spheres with the Boundary of a Convex Polytope," *Hong Kong UST Theoretical Computer Science Center Technical Report HKUST-TCSC-2001-09,* July 2001. Available from www.cs.ust.hk/tcsc/RR

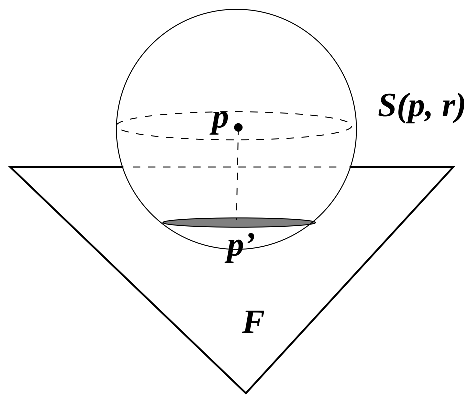
[6] Franco P. Preparata and Michael Ian Shamos, *Computational Geometry: An Introduction*, Springer-Verlag, New York.(1985).



(a) p' is a vertex (V)



(b) p' is on an edge (E)



(c) p' is in interior (I)

Figure 1: Different ways for spheres to intersect triangles. $p' = NN(p, F)$.

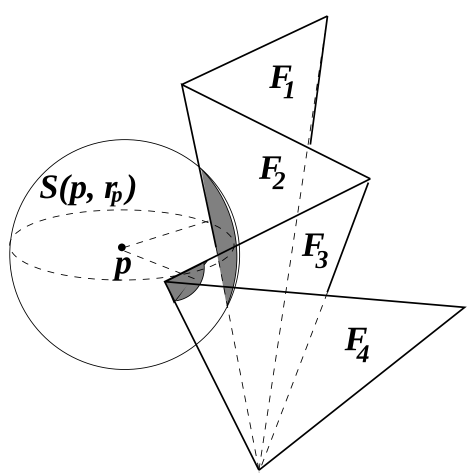


Figure 2: A point p with $L(p) = (E, E, I, V)$.