Chebyshev polynomials and spanning tree formulas for circulant and related graphs*

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Abstract

Kirchhoff's Matrix Tree Theorem permits the calculation of the number of spanning trees in any given graph G through the evaluation of the determinant of an associated matrix. In the case of some special graphs Boesch and Prodinger [9] have shown how to use properties of Chebyshev polynomials to evaluate the associated determinants and derive closed formulas for the number of spanning trees of graphs.

In this paper we extend this idea and describe how to use Chebyshev polynomials to evaluate the number of spanning trees in G when G belongs to one of three different classes of graphs: (i) when G is a circulant graph with fixed jumps (substantially simplifying earlier proofs), (ii) when G is a circulant graph with some *non-fixed* jumps and when (ii) $G = K_n \pm C$ where K_n is the complete graph on n vertices and C is a circulant graph.

1 Introduction

An undirected graph G is a pair (V, E), in which V is the vertex set and $E \subseteq V \times V$ is the edge set. In a graph, a (*self-*)*loop* is an edge joining a vertex to itself and *multiple edges* are several edges joining the same two vertices. All graphs considered in this paper are finite, and undirected with self-loops and multiple edges permitted.

For graph G, a spanning tree in G is a tree which has the same vertex set as G. The number of spanning trees in G denoted by $T(G)$, is a well studied quantity, being interesting both for its own sake and because it has practical implications for network reliability, e.g., [11, 12].

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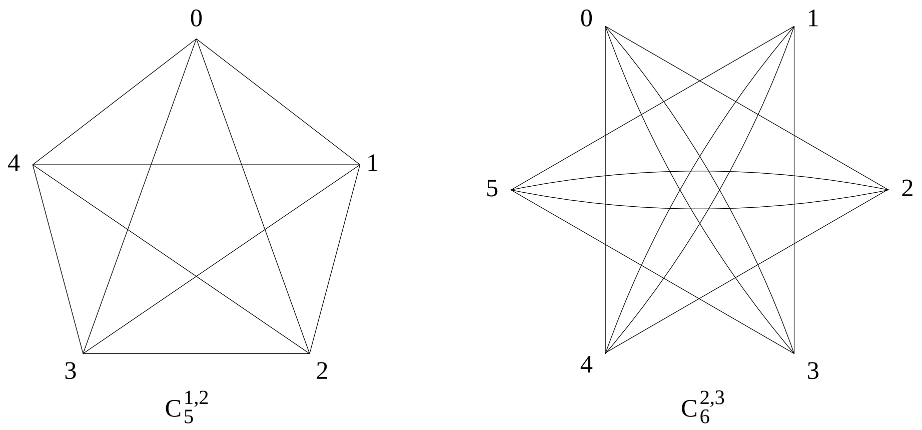

Figure 1: Two examples of circulant graphs. Note that $C_6^{2,3}$ has multiple edges.

In this paper we discuss how to derive closed formulas for $T(G)$ when G belongs to one of three graph classes: (i) when G is a circulant graph with fixed jumps (substantially simplifying earlier proofs), (i) when G is a circulant graph with some *non-fixed* jumps and (iii) when $G = K_n \pm C$ where K_n is the complete graph on n vertices and C is a circulant graph. In all three cases we start with the matrix-tree formulation of $T(G)$ which rewrites $T(G)$ as a cofactor of the Kirchoff matrix of the graph. We then describe how the special structure of the Kirchoff matrix permits rewriting the cofactor in terms of Chebyshev polynomials.

We start by providing some definitions and background.

Let $1 \le s_1 < s_2 < \cdots < s_k$, $s_1,\ s_2,\ \cdots,\ s_k$ integers. The *undirected circulant graph*, $C_n^{s_1,s_2,\cdots,s_k}$, has n vertices labeled $0,\ 1,\ 2,\ \cdots,\ n-1$, with each vertex i $(0 \le i \le n-1)$ adjacent to $2k$ vertices $i \pm s_1,\ i \pm s_2,\ \cdots,\ i \pm s_k \bmod n$. The simplest circulant graph is the n vertex cycle C_n^1 or C_n. More generally, if $(m,\ s) = 1$ then C_m^s is the m node cycle while if $(m,\ s) = d > 1$ then C_m^s is the disjoint union of d cycles $C_{m/d}^1$. Figure 1 illustrates two circulant graphs. We note that our definition here specifically forces the graph to be $2k$ regular so, if $i \pm s_i \equiv i \pm s_j \pmod n$ for some i, j then the graph would have repeated edges. See, for example, $C_6^{2,3}$ in Figure 1. Note too that in our definition, the s_i are *arbitrary*, they could be fixed or they could be functions of n. We will elaborate on this distinction further later.

K_n, the *complete graph* on n vertices, has one edge between each pair of distinct vertices. Let S be a subset of the edge set of K_n (or S be a subgraph of K_n). $K_n - S$, the graph remaining when all edges in S are removed from K_n, is the *complement* of S in K_n and also denoted as $\overline{S}$. For an edge set S, we denote by $K_n + S$ the graph K_n with all edges in S added to it; if S is nonempty then $K_n + S$ contains some multiple (repeated) edges. Figure 2 gives examples of $K_6 - C_4^1$ and $K_6 + C_4^1$, which are K_4 with, respectively, the four cycle deleted and added.

Let $V = \{v_1,\ v_2,\ \cdots,\ v_n\}$ be the vertex set of G and d_i denote the degree of v_i. Set $A(G)$, or simply A, to be the adjacency matrix of G. Let B denote

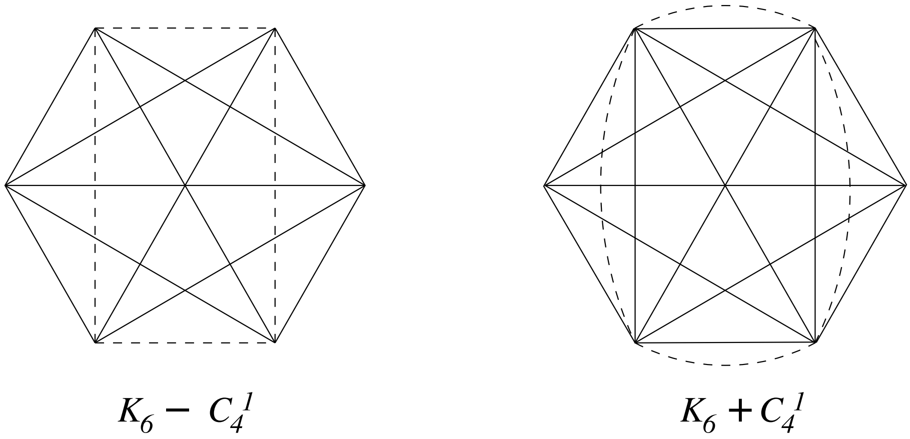

Figure 2: Two examples. In $K_6 - C_4^1$, the dashed lines are deleted edges; in $K_6 + C_4^1$ the dashed lines are added edges.

the $n \times n$ diagonal matrix with $\{d_1,\ d_2,\ \cdots,\ d_n\}$ as diagonal entries (and all other entries 0). The *Matrix Tree Theorem* [17] states that, the *Kirchhoff* matrix $H = B - A$ has all its co-factors[1] equal to $T(G)$, providing a method for calculating $T(G)$ for any particular given graph. For example, the Kirchhoff matrix of the graph $K_6 - C_4^1$ shown in Figure 2 is

$$H = \begin{pmatrix} 3 & 0 & -1 & -1 & 0 & -1 \\ 0 & 3 & -1 & 0 & -1 & -1 \\ -1 & -1 & 5 & -1 & -1 & -1 \\ -1 & 0 & -1 & 3 & 0 & -1 \\ 0 & -1 & -1 & 0 & 3 & -1 \\ -1 & -1 & -1 & -1 & -1 & 5 \end{pmatrix};$$

all its co-factors are 192 which is the number of spanning trees in $K_6 - C_4^1$.

The number of spanning trees in graph G also can be calculated from the eigenvalues of the Kirchhoff matrix H. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n (= 0)$ denote[2] all of H's eigenvalues. Kel'mans and Chelnokov [16] have shown that the Matrix Tree Theorem implies

$$T(G) = \frac{1}{n} \prod_{j=1}^{n-1} \mu_j. \tag{1}$$

For special classes of graphs it is possible to show that their Kirchhoff matrices have special structures and then bootstrap off of Kel'mans and Chelnokov's formula to get formulae for $T(G)$ when G is in those classes.

In [9] Boesch and Prodinger use this approach to derive closed formulae when G belong to the classes of wheels, fans, ladders, Moebius ladders, squares

[1] The (i, j)-th co-factor of A is the determinant of the $(n-1) \times (n-1)$ matrix that results from deleting the i-th row and j-th column from A, with symbol $(-1)^{i+j}$.

[2] Because H is symmetric it has all real eigenvalues. It is not difficult to see that all of the eigenvalues are, in fact, nonnegative, and that 0 is an eigenvalue

of cycles and complete prisms. Their main technique was to show that in these cases (1) can be rewritten in terms of Chebyshev polynomials and to then use properties of these polynomials to derive the closed formulae.

Separately, the class of circulant graphs have also been well studied. The $C_n^{1,2}$ graphs, in particular, deserve special mention. The formula $T(C_n^{1,2}) = nF_n^2$, F_n the Fibonacci numbers, was originally conjectured by Bedrosian [5] and subsequently proven by Kleitman and Golden [18]. The same formula was also conjectured by Boesch and Wang [8] (without the knowledge of [18]). Different proofs can been found in [1, 9, 27]. The $C_n^{1,2}$ graphs are actually the squares of cycles mentioned above and the formula for $T(C_n^{1,2})$ was also rederived using Chebyshev polynomials by Boesch and Prodinger [9] as described above.

Going further, formulae for $T(C_n^{1,3})$ and $T(C_n^{1,4})$ are provided in [26]. A connection between these formulae was given in [28] by showing that, for any *fixed* $s_1, s_2, \cdots, s_k$,

$$T(C_n^{s_1,s_2,\cdots,s_k}) = na_n^2,$$

where the a_n satisfy a recurrence relation of the form

$$\forall n > 2^{s_k-1},\ a_n = \sum_{i=1}^{2^{s_k-1}} b_i a_{n-i},$$

and the b_i are reals (but not necessarily nonnegative). Recall that the Matrix Tree Theorem gives us a method of calculating $T(C_n^{s_1,s_2,\cdots,s_k}) = na_n^2$ for any arbitrary n by building the Kirchhoff matrix and evaluating any of its cofactors. This means that we can find the b_i by calculating all of the a_i for $i \le 2^{s_k}$ and then solving for the b_i. The asymptotics of $T(C_n^{s_1,s_2,\cdots,s_k})$ could then be found by solving for the minimum modulus root of the characteristic polynomial of the recurrence relation. This was done in [28] for all circulant graphs with $s_k \le 5$.

In this paper we extend the ideas in [9] in three directions. In the first we show how to use the Chebyshev polynomial technique to derive a much simpler proof that $T(C_n^{s_1,s_2,\cdots,s_k}) = na_n^2$ where the a_n satisfy a linear recurrence relation of order 2^{s_k-1}.[3] This new proof will have the added advantage of providing a method of deriving the minimum modulus root of the characteristic polynomial of the recurrence relation *without having to construct the recurrence relation*, thus obviating the need to calculate the determinants (it will only require finding the roots of a particular polynomial of order $s_k - 1$).

In the second we describe how to use the Chebyshev polynomial technique for deriving closed formulae for some circulant graphs with *non-fixed* jumps, a problem which does not seem to have been generally attacked previously. More specifically, the technique will permit the derivation of formulae for circulant graphs of the form $C_n^{s1,\cdots,s_k,\frac{n}{a_1},\cdots,\frac{n}{a_l}}$ where $s_1, \cdots, s_k$ are constant integers, $a_1, \cdots, a_l \in \{2, 3, 4, 6\}$ and $\forall u \le l$, $a_u | n$, *i.e.*, n is a multiple of the least common multiple of the a_u. As examples, we will derive formulae for $T(C_{2n}^{1,n})$, $T(C_{3n}^{1,n})$, $T(C_{4n}^{1,n})$, $T(C_{6n}^{1,n})$ and $T(C_{6n}^{1,2n,3n})$.

[3] Note that this new proof only works for undirected circulant graphs as discussed in this paper. For *directed circulant graphs* the proof in [28] still seems to be the only general one.

In the third we describe how to use the *Chebyshev* polynomial technique to calculate $T(K_n \pm S)$ where S is a circulant graph.

The problem of calculating $T(K_n - S)$ has already been studied for many different types of S. The first work in this area seems to have been by Weinberg [24] who gave formulae for $T(K_n - S)$ when all edges in S are not adjacent or are adjacent at one vertex. Subsequently, in a series of papers [2, 3, 4, 6], Bedrosian extended this to show how to calculate $T(K_n - S)$ when all edges in S are not adjacent or adjacent at one vertex, or form a path, a cycle, a complete graph, or are some combination of these configurations. Weinberg's results have also been generalized in [21]. Closed formulae also exist for the cases where S is a star [20], a complete k-partite graph [22], a multi-star [19, 25], and so on. The number of spanning trees in the complement graph is investigated in [13, 16] when the graph with maximal number of spanning trees is studied. The formulae for the number of spanning trees in the complement graphs of a disjoint union of cycles or paths are given in generic forms in [13]. Not as much seems to be known about $T(K_n + S)$; Bedrosian [3] considered it for some simple configurations S, i.e., all edges in S form a cycle, complete graph, or $|S|$ is quite small but not much more seems to be known.

In the third part of this paper we add to this literature by deriving formulas for $K_n \pm S$ where S is a circulant graph with fixed jumps. Our technique is to first start by developing a new approach to deriving a closed form for $T(K_n - C_m^s)$, i.e., the cycle or union of cycles (a closed form for this was previously derived using different techniques in [13]). We then continue by showing that it is easy to generalize this approach to getting a formula for $T(K_n \pm C_m^{s_1,s_2,\cdots,s_k})$. In the case that all of the $s_i \le 4$ we will actually be able to derive a simple closed form function $g(n,m;s_1,s_2,\cdots,s_k) = T(K_n \pm C_m^{s_1,s_2,\cdots,s_k})$ of n, m. Even more, we derive that $T(K_n \pm C_m^{s_1,s_2,\cdots,s_k})$ satisfy a recurrence relation when n is fixed and m is changing.

The rest of the paper is structured as follows. In section 2 we briefly review the basic facts we will need. In section 3 we rederive $T(C_n^{s_1,s_2,\cdots,s_k}) = na_n^2$ and describe how to efficiently calculate its asymptotics. In section 4, we discuss non-constant jumps. In section 5, we derive $T(K_n \pm S)$ where S is a circulant graph. In Section 6 we conclude and present an open problem.

2 Basic concepts and lemmas

We start by reviewing some basic facts from [7] concerning circulant matrices and graphs. An $n \times n$ matrix C is said to be a *circulant matrix* if its entries satisfy $c_{ij} = c_{1,j-i+1}$, where the subscripts are reduced modulo n and lie in the set $\{1, 2, \cdots, n\}$. In other words, the i-th row of C is obtained from the first row of C by a cyclic shift of $i-1$ steps, and so any circulant matrix is determined by its first row. It is clear that the adjacency matrix of the circulant graph $C_n^{s_1,s_2,\cdots,s_k}$ is a circulant matrix. The first row $(c_1, c_2, \cdots, c_n)$ of the adjacency matrix is determined by the connection jumps $s_1, s_2, \cdots, s_k$. More specifically, an edge $(1,i)$ is in the graph if and only if $i \equiv (1 \pm s_j) \pmod n$ for some s_j, $1 \le j \le k$. (Note that it is possible for the $c_i > 1$. This happens if $(1 \pm s_j) \equiv (1 \pm s_{j'}) \pmod n$ for some $j \neq j'$. In this case the graph is a multigraph and c_i is the *number* of different edges connecting 1 and i. This can only happen when n is small, though.) From the adjacency matrix of $C_n^{s_1,s_2,\cdots,s_k}$ and the definition of the *Kirchhoff* matrix it is easy to see that the *Kirchhoff* matrix of $C_n^{s_1,s_2,\cdots,s_k}$ is also a circulant matrix.

The starting point of our calculations is the following lemma which is a direct application of of Proposition 3.5 of [7]:

Lemma 1 *The Kirchhoff matrix of the circulant graph $C_n^{s_1,s_2,\cdots,s_k}$ has n eigenvalues. They are 0 and, $\forall j$, $1 \leq j \leq n-1$ the values $2k - \varepsilon^{-s_1 j} - \cdots - \varepsilon^{-s_k j} - \varepsilon^{s_1 j} - \cdots - \varepsilon^{s_k j}$, where $\varepsilon = e^{\frac{2\pi i}{n}}$.*

Plugging this into (1) yields the following well known corollary, see, e.g., [28],

Corollary 1 *Set $\varepsilon = e^{\frac{2\pi i}{n}}$. Then*

$$\begin{aligned}
T(C_n^{s_1,s_2,\cdots,s_k}) &= \frac{1}{n}\prod_{j=1}^{n-1}(2k - \varepsilon^{-s_1 j} - \varepsilon^{-s_2 j} - \cdots - \varepsilon^{-s_k j} \\
&\qquad -\varepsilon^{s_1 j} - \varepsilon^{s_2 j} - \cdots - \varepsilon^{s_k j}), \\
&= \frac{1}{n}\prod_{j=1}^{n-1}\left(\sum_{i=1}^{k}(2 - 2\cos\frac{2js_i\pi}{n})\right).
\end{aligned}$$

An important special case of this occurs when we examine the cycle C_n^1. Clearly C_n^1 has exactly n spanning trees. Applying the corollary therefore yields [9] the non-obvious

$$n = T(C_n^1) = \frac{1}{n}\prod_{j=1}^{n-1}\left(2 - 2\cos\frac{2j\pi}{n}\right) = \frac{1}{n}\prod_{j=1}^{n-1}(4\sin^2\frac{j\pi}{n}), \tag{2}$$

which will be useful to us later.

The other main tools we use are various standard properties of *Chebyshev* polynomials of the second kind. For reference we quickly review them here. The following definitions and derivations (with the exception of (10)) follow [9].

For positive integer m, the *Chebyshev* polynomials of the first kind are defined by

$$T_m(x) = \cos(m \arccos x). \tag{3}$$

The *Chebyshev* polynomials of the second kind are defined by

$$U_{m-1}(x) = \frac{1}{m}\frac{d}{dx}T_m(x) = \frac{\sin(m\arccos x)}{\sin(\arccos x)}. \tag{4}$$

It is easily verified that

$$U_m(x) - 2xU_{m-1}(x) + U_{m-2}(x) = 0. \tag{5}$$

Solving this recursion by using standard methods yields

$$U_m(x) = \frac{1}{2\sqrt{x^2-1}}[(x+\sqrt{x^2-1})^{m+1} - (x-\sqrt{x^2-1})^{m+1}], \tag{6}$$

where the identity is true for all complex x (except at $x = \pm 1$ where the function can be taken as the limit).

The definition of $U_m(x)$ easily yields its zeros and it can therefore be verified that

$$U_{m-1}(x) = 2^{m-1} \prod_{j=1}^{m-1} (x - \cos\frac{j\pi}{m}). \tag{7}$$

One further notes that

$$U_{m-1}(-x) = (-1)^{m-1} U_{m-1}(x). \tag{8}$$

These two results yield another formula for $U_m(x)$

$$U_{m-1}^2(x) = 4^{m-1} \prod_{j=1}^{m-1} \left(x^2 - \cos^2\frac{j\pi}{m}\right). \tag{9}$$

Finally, simple manipulation of the above formula yields the following, which will also be extremely useful to us later:

$$U_{m-1}^2\left(\sqrt{\frac{x+2}{4}}\right) = \prod_{j=1}^{m-1} \left(x - 2\cos\frac{2\pi j}{m}\right). \tag{10}$$

3 Recurrence relations for fixed step circulant graphs

In this section we assume that $s_1, s_2, \cdots, s_k$ are *fixed* positive integers with $1 \le s_1 < s_2 < \cdots < s_k$ and use the properties of Chebyshev polynomials to reprove the main result in [28], i.e., that there exist $b_1, b_2, \ldots, b_{2^{s_k}-1}$ such that

$$T(C_n^{s_1, s_2, \cdots, s_k}) = n a_n^2, \quad \text{where} \quad \forall n > 2^{s_k - 1},\ a_n = \sum_{i=1}^{2^{s_k-1}} b_i a_{n-i}. \tag{11}$$

We start with a basic lemma on trigonometric polynomials; its proof is quite tedious but straightforward so we omit it here.

Lemma 2 *Let $k > 0$ be any integer. Then $2 - 2\cos(2kx)$ can be rewritten in the form $4^k f_k(\cos^2 x)\sin^2 x$, where $f_k(x)$ is a polynomial of order $k-1$ with leading coefficient 1 that does not have 1 as a root.*

Combining this with Corollary 1 and some manipulation yields

Lemma 3 *The number of spanning trees $T(C_n^{s_1,s_2,\cdots,s_k})$ satisfies*

$$T(C_n^{s_1,s_2,\cdots,s_k}) = \frac{1}{n}\prod_{j=1}^{n-1} 4^{s_k} f(\cos^2\frac{j\pi}{n})\sin^2\left(\frac{j\pi}{n}\right),$$

where $f(x)$ is a polynomial of order s_k-1 with leading coefficient 1 that does not have 1 as a root.

Now let $x_1, x_2, \cdots, x_{s_k-1}$ be the roots of $f(x)$. Then

$$f(x) = (-1)^{s_k-1}\prod_{i=1}^{s_k-1}(x_i - x).$$

Plugging this into Lemma 3 and using formulae (2) and (9) gives

$$\begin{aligned}
&T(C_n^{s_1,s_2,\cdots,s_k})\\
&= \frac{1}{n}\prod_{j=1}^{n-1} 4^{s_k}(-1)^{s_k-1}\left(\prod_{i=1}^{s_k-1}(x_i-\cos^2\frac{j\pi}{n})\right)\sin^2\left(\frac{j\pi}{n}\right)\\
&= (-1)^{(n-1)(s_k-1)}\frac{1}{n}\prod_{i=1}^{s_k-1}\left(4^{n-1}\prod_{j=1}^{n-1}(x_i-\cos^2\frac{j\pi}{n})\right)\cdot 4^{n-1}\prod_{j=1}^{n-1}\sin^2\frac{j\pi}{n}\\
&= (-1)^{(n-1)(s_k-1)} n \prod_{i=1}^{s_k-1} U_{n-1}^2(\sqrt{x_i}).
\end{aligned}$$

Using formula (6) to rewrite $U_{n-1}^2(\sqrt{x_i})$ gives

$$T(C_n^{s_1,s_2,\cdots,s_k}) = n\left[\prod_{i=1}^{s_k-1}\frac{1}{2\sqrt{1-x_i}}\left((\sqrt{-x_i}+\sqrt{1-x_i})^n - (\sqrt{-x_i}-\sqrt{1-x_i})^n\right)\right]^2.$$

This actually provides a 'closed formula' for $T(C_n^{s_1,s_2,\cdots,s_k})$, albeit, not a particularly satisfying one. We now continue by, for all i, $1 \le i \le s_k-1$, set $y_{i,0} = \sqrt{-x_i}+\sqrt{1-x_i}$ and $y_{i,1} = \sqrt{-x_i}-\sqrt{1-x_i}$. For $(\delta_1, \delta_2, \ldots, \delta_{s_k-1}) \in \{0,1\}^{s_k-1}$ set

$$R_{\delta_1,\delta_2,\ldots,\delta_{s_k-1}} = (-1)^{\sum_{i=1}^{s_k-1}\delta_i}\cdot\prod_{i=1}^{s_k-1} y_{i,\delta_i}.$$

Also set $c = \prod_{i=1}^{s_k-1}\frac{1}{2\sqrt{1-x_i}}$. If a_n is defined so that $T(C_n^{s_1,s_2,\cdots,s_k}) = na_n^2$, then

$$a_n = c\sum_{(\delta_1,\delta_2,\ldots,\delta_{s_k-1})\in\{0,1\}^{s_k-1}} R^n_{\delta_1,\delta_2,\ldots,\delta_{s_k-1}}.$$

Since there are at most 2^{s_k-1} different values $R_{\delta_1,\delta_2,\ldots,\delta_{s_k-1}}$ this immediately implies (11) and we have proved what we claimed.

As noted in [28] one way to find the b_i is to simply use the Matrix Tree Theorem to calculate the value of $T(C_n^{s_1,s_2,\cdots,s_k})$ for all $n \le 2^{s_k}$ yielding all of the values of a_n and then solve for the b_n. Once the b_n are known the asymptotics of a_n (and therefore $T(C_n^{s_1,s_2,\cdots,s_k})$) could be found by standard generating function techniques, i.e., by calculating the roots of the characteristic equation of the a_n. This is what was done in [28]. That paper actually proved a stronger result; that is, if $\gcd(s_1,s_2,\cdots,s_k)=1$, then ϕ, the smallest modulus root of the generating function of the a_n, is unique and real so $a_n \sim c\phi^n$ for some c, and $T(C_n^{s_1,s_2,\cdots,s_k}) \sim nc^2\phi^{2n}$. The asymptotics of $T(C_n^{s_1,s_2,\cdots,s_k})$ could therefore be found by calculating the smallest modulus root of the generating function.[4]

The difficulty with this technique is that, in order to derive the generating function, it was necessary to apply the *Matrix Tree Theorem* 2^{s_k} times, evaluating a determinant each time.

Our new proof of (11) immediately yields a much more efficient method of deriving the asymptotics. Note that the roots of the generating function are exactly $\frac{1}{R_{\delta_1,\delta_2,\ldots,\delta_{s_k-1}}}$. Finding the smallest modulus root is therefore the same as finding $R_{\max}$, the $R_{\delta_1,\delta_2,\ldots,\delta_{s_k-1}}$ with maximum modulus; since the smallest modulus root is real, $R_{\max}$ is real as well. We can therefore easily find[5] $R_{\max}$ by setting $y_i=\max(|y_{i,0}|,|y_{i,1}|)$ for all $i\le s_k-1$ and then noting that $|R_{\max}|=\prod_{i=1}^{s_k-1} y_i$. This technique yields the asymptotics of $T(C_n^{s_1,s_2,\cdots,s_k})$ without requiring the evaluation of any determinants; all that is needed is the calculation of all of the roots of a degree s_k-1 polynomial.

As an example we work through the process for $T(C_n^{1,2,3})$:

$$\begin{aligned}
T(C_n^{1,2,3}) &= \frac{1}{n}\prod_{j=1}^{n-1}(6-e^{\frac{2\pi j}{n}}-e^{\frac{4\pi j}{n}}-e^{\frac{6\pi j}{n}}-e^{\frac{-2\pi j}{n}}-e^{\frac{-4\pi j}{n}}-e^{\frac{-6\pi j}{n}})\\
&= \frac{1}{n}\prod_{j=1}^{n-1}(6-2\cos\frac{2\pi j}{n}-2\cos\frac{4\pi j}{n}-2\cos\frac{6\pi j}{n})\\
&= \frac{1}{n}\prod_{j=1}^{n-1}64(\cos^4\frac{\pi j}{n}-\frac{1}{4}\cos^2\frac{\pi j}{n}+\frac{1}{8})\sin^2\frac{\pi j}{n}\\
&= n\prod_{j=1}^{n-1}16(\cos^4\frac{\pi j}{n}-\frac{1}{4}\cos^2\frac{\pi j}{n}+\frac{1}{8}).
\end{aligned}$$

The roots of the polynomial $x^2-\frac{1}{4}x+\frac{1}{8}$ are

$$x_1=\frac{1}{8}-\frac{\sqrt{7}}{8}i \quad \text{and} \quad x_2=\frac{1}{8}+\frac{\sqrt{7}}{8}i.$$

Thus

$$y_{1,0}=\sqrt{-x_1}+\sqrt{1-x_1}=\frac{1}{4}\sqrt{-2+2\sqrt{7}i}+\frac{1}{4}\sqrt{14+2\sqrt{7}i},$$

[4]If $\gcd(s_1,s_2,\cdots,s_k)=d\neq 1$ it is described in [28] how this case can be reduced down to evaluating $T(C_n^{s_1/d,s_2/d,\cdots,s_k/d})$. Since $\gcd(s_1/d,s_2/d,\cdots,s_k/d)=1$ we may always restrict ourselves to assuming that $\gcd(s_1,s_2,\cdots,s_k)=1$.

[5]It is not a-priori obvious that $R_{\max}$ is positive but, since we are only interested in na_n^2 and not a_n, knowing $|R_{\max}|$ suffices.

$$y_{1,1} = \sqrt{-x_1} - \sqrt{1-x_1} = \frac{1}{4}\sqrt{-2+2\sqrt{7}i} - \frac{1}{4}\sqrt{14+2\sqrt{7}i},$$
$$y_{2,0} = \sqrt{-x_2} + \sqrt{1-x_2} = \frac{1}{4}\sqrt{-2-2\sqrt{7}i} + \frac{1}{4}\sqrt{14-2\sqrt{7}i},$$
$$y_{2,1} = \sqrt{-x_2} - \sqrt{1-x_2} = \frac{1}{4}\sqrt{-2-2\sqrt{7}i} - \frac{1}{4}\sqrt{14-2\sqrt{7}i}.$$

Therefore, $T(C_n^{1,2,3}) = na_n^2$, $a_n \sim c\phi^n$ where $c = \frac{1}{2\sqrt{1-x_1}} \cdot \frac{1}{2\sqrt{1-x_2}} = \frac{1}{\sqrt{14}} \approx 0.2672612$ and $\phi = y_{1,0} \cdot y_{2,0} = \frac{1}{16}(\sqrt{32} + \sqrt{224} + \sqrt{64\sqrt{7}}) \approx 2.102256$. These are exactly the same values c and ϕ derived in [28] using the longer method.

4 The number of spanning trees in some non fixed-jump circulant graphs

In the previous section we examined the spanning tree numbers for circulant graphs in which the *steps* or *jumps*, i.e., the s_i, were fixed and the number of nodes, i.e., n, changing. In this section we derive formulae for some graphs in which the step size can be functions of n. Figure 3 illustrates two examples of such graphs. Our approach is, as before, to expand $T()$ as a product of trigonometric polynomials and then express it in terms of *Chebyshev* polynomials, in this case, ratios of such polynomials. We will see though, that this technique is not totally general and only works for particular values of jumps.

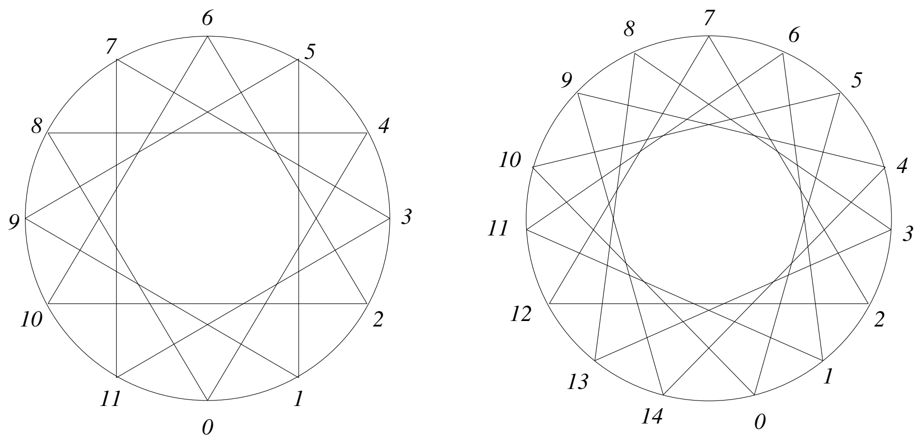

Figure 3: Examples of non fixed-jump circulant graph $C_{3n}^{1,n}$ with $n = 4$ and $n = 5$.

We illustrate the technique via three examples; starting from a easy one, $T(C_{2n}^{1,n})$, that illustrates the core ideas, continuing on to $T(C_{3n}^{1,n})$, which is more complicated, and ending at $T(C_{4n}^{1,n})$ which reveals where the difficulties lie in extending the technique further.

We start by calculating $T(C_{2n}^{1,n})$. Recall that, according to our definition of circulant graphs, $C_{2n}^{1,n}$ is the *four*-regular graph[6] with $2n$ vertices $0, 1, \ldots, 2n-1$

[6]We should note that this is not the same graph as the *Moebius* ladder which is a *three-*

such that node i has one edge connecting it to $(i+1) \pmod{2n}$ one edge connecting it to $(i-1) \pmod{2n}$ and *two* edges connecting it to $(i+n) \pmod{2n}$.

Theorem 4

$$T(C_{2n}^{1,n}) = \frac{n}{2}[(\sqrt{2}+1)^n + (\sqrt{2}-1)^n]^2.$$

Proof. Let $\varepsilon_2 = e^{\frac{2\pi i}{2n}}$. By Lemma 1, we have

$$\begin{aligned}
T(C_{2n}^{1,n}) &= \frac{1}{2n}\prod_{j=1}^{2n-1}(4-\varepsilon_2^j-\varepsilon_2^{-j}-\varepsilon_2^{nj}-\varepsilon_2^{-nj}) \\
&= \frac{1}{2n}\prod_{j=1}^{2n-1}\left(4-2\cos\frac{2\pi j}{2n}-2\cos(\pi j)\right) \\
&= \frac{1}{2n}\prod_{\substack{j=1 \\ 2\nmid j}}^{2n-1}\left(6-2\cos\frac{2\pi j}{2n}\right)\prod_{\substack{j=1 \\ 2|j}}^{2n-1}\left(2-2\cos\frac{2\pi j}{2n}\right).
\end{aligned}$$

Noting that if $j = 2j'$ for some integer j' then $\cos\frac{2\pi j}{2n} = \cos\frac{2\pi j'}{n}$ gives

$$\begin{aligned}
T(C_{2n}^{1,n}) &= \frac{1}{2n}\prod_{j=1}^{2n-1}\left(6-2\cos\frac{2\pi j}{2n}\right)\prod_{j=1}^{n-1}\frac{2-2\cos\frac{2\pi j}{n}}{6-2\cos\frac{2\pi j}{n}} \\
&= \frac{1}{2n}U_{2n-1}^2(\sqrt{2})\cdot\frac{n^2}{U_{n-1}^2(\sqrt{2})} \\
&= \frac{n}{2}[(\sqrt{2}+1)^n+(\sqrt{2}-1)^n]^2.
\end{aligned}$$

where (2), (6) and (10) are used to derive the last two steps. □

We now continue to

Theorem 5

$$T(C_{3n}^{1,n}) = \frac{n}{3}\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{2n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{2n}+1\right]^2.$$

Proof. The proof starts similarly to the previous one. Let $\varepsilon_3 = e^{\frac{2\pi i}{3n}}$. By Lemma 1, we have

$$T(C_{3n}^{1,n}) = \frac{1}{3n}\prod_{j=1}^{3n-1}(4-\varepsilon_3^j-\varepsilon_3^{-j}-\varepsilon_3^{nj}-\varepsilon_3^{-nj})$$

regular graph on the same vertex set in which node i has one edge connecting it to each of $(i+1) \pmod{2n}$, $(i-1) \pmod{2n}$ and $(i+n) \pmod{2n}$. The techniques described here, though, could be used to rederive closed formulae for the spanning tree numbers of *Moebius* ladders and similar graphs (see [9] for such a derivation).

$$
\begin{aligned}
&= \frac{1}{3n}\prod_{j=1}^{3n-1}\left(4-2\cos\frac{2\pi j}{3n}-2\cos\frac{2\pi j}{3}\right)\\
&= \frac{1}{3n}\prod_{\substack{j=1\\3\nmid j}}^{3n-1}\left(5-2\cos\frac{2\pi j}{3n}\right)\prod_{\substack{j=1\\3|j}}^{3n-1}\left(2-2\cos\frac{2\pi j}{3n}\right).
\end{aligned}
$$

Note that if $j=3j'$ for some integer j' then $\cos\frac{2\pi j}{3n}=\cos\frac{2\pi j'}{n}$. Note too that if $3\nmid j$ then $\cos\frac{2\pi j}{3}=-\frac{1}{2}$. This gives

$$
\begin{aligned}
T(C_{3n}^{1,n}) &= \frac{1}{3n}\prod_{j=1}^{3n-1}\left(5-2\cos\frac{2\pi j}{3n}\right)\prod_{j=1}^{n-1}\frac{2-2\cos\frac{2\pi j}{n}}{5-2\cos\frac{2\pi j}{n}}\\
&= \frac{1}{3n}U_{3n-1}^2\left(\sqrt{\frac{7}{4}}\right)\cdot\frac{n^2}{U_{n-1}^2\left(\sqrt{\frac{7}{4}}\right)}\\
&= \frac{n}{3}\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{2n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{2n}+1\right]^2.
\end{aligned}
$$

□

We next see

Theorem 6

$$
T(C_{4n}^{1,n})=\frac{n}{4}\left[\left(\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}}\right)^{2n}+\left(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}}\right)^{2n}\right]^2\left[\left(\sqrt{2}+1\right)^n+\left(\sqrt{2}-1\right)^n\right]^2.
$$

Proof. The proof again starts similarly to the previous ones. Let $\varepsilon_4=e^{\frac{2\pi i}{4n}}$. We have

$$
\begin{aligned}
T(C_{4n}^{1,n}) &= \frac{1}{4n}\prod_{j=1}^{4n-1}(4-\varepsilon_4^j-\varepsilon_4^{-j}-\varepsilon_4^{nj}-\varepsilon_4^{-nj})\\
&= \frac{1}{4n}\prod_{j=1}^{4n-1}\left(4-2\cos\frac{2\pi j}{4n}-2\cos\frac{\pi j}{2}\right)\\
&= \frac{1}{4n}\prod_{\substack{j=1\\2\nmid j}}^{4n-1}\left(4-2\cos\frac{2\pi j}{4n}\right)\prod_{\substack{j=1\\2|j}}^{4n-1}\left(4-2\cos\frac{2\pi j}{4n}-2\cos\frac{\pi j}{2}\right),
\end{aligned}
$$

where the last derivation follows from the fact that if $2\nmid j$ then $\cos\frac{2\pi j}{4n}=0$. Unlike in the previous proofs, though, if $2|j$ it is not true that $\cos\frac{2\pi j}{4n}$ equals some constant, so we will have to derive further. We use the fact that if $j = 2j'$ then $\cos\frac{2\pi j}{4n} = \cos\frac{2\pi j}{2n}$ to get

$$T(C_{4n}^{1,n}) \;=\; \frac{1}{4n}\prod_{j=1}^{4n-1}\left(4-2\cos\frac{2\pi j}{4n}\right)\prod_{j=1}^{2n-1}\frac{4-2\cos\frac{2\pi j}{2n}-2\cos(\pi j)}{4-2\cos\frac{2\pi j}{2n}}.$$

At this point we can evaluate both the leftmost product and the denominator of the rightmost product in terms of Chebyshev polynomials. To evaluate the numerator of the rightmost product we will need to split it into two cases depending upon whether j is odd or even, and apply the same type of procedure *again*. This yields

$$\begin{aligned}
T(C_{4n}^{1,n}) &= \frac{1}{4n}\cdot\frac{U_{4n-1}^2\left(\sqrt{\frac{3}{2}}\right)}{U_{2n-1}^2\left(\sqrt{\frac{3}{2}}\right)}\prod_{j=1}^{2n-1}\left(6-2\cos\frac{2\pi j}{2n}\right)\prod_{j=1}^{n-1}\frac{2-2\cos\frac{2\pi j}{n}}{6-2\cos\frac{2\pi j}{n}}\\
&= \frac{1}{4n}\cdot\frac{U_{4n-1}^2\left(\sqrt{\frac{3}{2}}\right)}{U_{2n-1}^2\left(\sqrt{\frac{3}{2}}\right)}\cdot\frac{U_{2n-1}^2(\sqrt{2})}{U_{n-1}^2(\sqrt{2})}\cdot n^2\\
&= \frac{n}{4}\left[\left(\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}}\right)^{2n}+\left(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}}\right)^{2n}\right]^2\left[\left(\sqrt{2}+1\right)^n+\left(\sqrt{2}-1\right)^n\right]^2.
\end{aligned}$$

□

The proofs of Theorem 4, 5 and 6 depend on certain symmetry properties of the *cosine* functions, e.g., if $3 \nmid j$ then $\cos\frac{2\pi j}{3} = -\frac{1}{2}$ that permitted us to write products out as ratios that were in the proper form to express as *Chebyshev* polynomials. Unfortunately, this can not always be done. For example, we do not seem to be able to use this technique to derive a formula for $T(C_{5n}^{1,n})$. The furthest that we are currently able to push this technique is to derive closed formulae for the number of spanning trees (as a function of n) for all circulant graphs of $C_n^{s1,\cdots,s_k,\frac{n}{a_1},\cdots,\frac{n}{a_l}}$ where $s_1,\cdots,s_k$ are constant integers and all $a_1,\cdots,a_l$ are in the set $\{2,3,4,6\}$ with $a_u|n$ for any u, $1 \le u \le l$.

We conclude with a few more applications (proofs omitted):

Theorem 7

$$\begin{aligned}
T(C_{6n}^{1,n}) &= \frac{n}{6}\left[\left(\sqrt{\frac{5}{4}}+\sqrt{\frac{1}{4}}\right)^{3n}+\left(\sqrt{\frac{5}{4}}-\sqrt{\frac{1}{4}}\right)^{3n}\right]^2\left[\left(\sqrt{2}+1\right)^n+\left(\sqrt{2}-1\right)^n\right]^2\cdot\\
&\quad\left[\left(\sqrt{\frac{5}{4}}+\sqrt{\frac{1}{4}}\right)^{n}+\left(\sqrt{\frac{5}{4}}-\sqrt{\frac{1}{4}}\right)^{n}\right]^2\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{2n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{2n}+1\right]^2.
\end{aligned}$$

Theorem 8

$$T(C_{6n}^{1,2n,3n}) \;=\; \frac{n}{6}\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{2n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{2n}-1\right]^2.$$

$$\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{2n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{2n}+1\right]^2\left[\left(\sqrt{2}+1\right)^n+\left(\sqrt{2}-1\right)^n\right]^2.$$

5 The number of spanning trees in $T(K_n \pm S)$ with S a circulant graph

In this section we derive methods to calculate $T(K_n \pm S)$ when S is a circulant graph. We first review some notation and basic results.

Lemma 9 ([16]) *Let G be a graph with n vertices and $\overline{G}$ the complement graph of G in K_n. If the* Kirchhoff *matrix of G has eigenvalues $\mu_1,\ \mu_2,\ \cdots,\ \mu_{n-1}$ and 0, then the* Kirchhoff *matrix of $\overline{G}$ has eigenvalues $n-\mu_1,\ n-\mu_2,\ \cdots,\ n-\mu_{n-1}$ and 0.*

Following the proof of Lemma 9, we can easily prove the next lemma:

Lemma 10 *Let G be a graph with the same vertex set as K_n. If the* Kirchhoff *matrix of G has eigenvalues $\mu_1,\ \mu_2,\ \cdots,\ \mu_{n-1}$ and 0, then the* Kirchhoff *matrix of K_n+G has eigenvalues $n+\mu_1,\ n+\mu_2,\ \cdots,\ n+\mu_{n-1}$ and 0.*

Let $G_1=(V_1,\ E_1)$ and $G_2=(V_2,\ E_2)$ be two graphs with disjoint vertex sets. The join $G=G_1\bigoplus G_2$ is defined as the graph with vertex set $V=V_1\cup V_2$ and edge set $E=E_1\cup E_2\cup\{uv|u\in V_1,\ v\in V_2\}$ [10]. (Please note that in this paper we use "$\bigoplus$" to denote the join graph instead of "+" as used in some other references. This is because we are already using "+" to denote the graph that is resulted by adding edges to some other graph.) The following lemma describes the relation of the eigenvalues of the *Kirchhoff* matrix of join graph and the eigenvalues of *Kirchhoff* matrices of the original graphs.

Lemma 11 ([15, 16]) *If the* Kirchhoff *matrix of graph G_1 with n vertices has eigenvalues $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n(=0)$ and that of graph G_2 with m vertices has eigenvalues $\mu_1,\ \mu_2,\ \cdots,\ \mu_m(=0)$, then the* Kirchhoff *matrix of the join $G_1\bigoplus G_2$ has eigenvalues $m+n,\ \lambda_1+m,\ \cdots,\ \lambda_{n-1}+m$ and $\mu_1+n,\ \cdots,\ \mu_{m-1}+n,\ 0$.*

Let $C_{m_1}^{s_{1_1},s_{2_1},\cdots,s_{k_1}},\ C_{m_2}^{s_{1_2},s_{2_2},\cdots,s_{k_2}},\ \cdots,\ C_{m_l}^{s_{1_l},s_{2_l},\cdots,s_{k_l}}$ be a collection of circulant graphs, and $\bigcup_{u=1}^{l} C_{m_u}^{s_{1_u},s_{2_u},\cdots,s_{k_u}}$ be their disjoint union. For each u, $1\le u\le l$, suppose $m_u>2s_{k_u}$ and let $\overline{C_{m_u}^{s_{1_u},s_{2_u},\cdots,s_{k_u}}}$ be the complement graph of $C_{m_u}^{s_{1_u},s_{2_u},\cdots,s_{k_u}}$ in K_{m_u}. Note that, for any n, $n\ge\sum_{u=1}^{l} m_u$,

$$\begin{aligned}
&K_n-\bigcup_{u=1}^{l} C_{m_u}^{s_{1_u},s_{2_u},\cdots,s_{k_u}}\\
&=\left(K_{n-\sum_{u=1}^{l} m_u}\right)\bigoplus\left(K_{m_1}-C_{m_1}^{s_{1_1},s_{2_1},\cdots,s_{k_1}}\right)\bigoplus\cdots\bigoplus\left(K_{m_l}-C_{m_l}^{s_{1_l},s_{2_l},\cdots,s_{k_l}}\right)\\
&=\left(K_{n-\sum_{u=1}^{l} m_u}\right)\bigoplus\overline{C_{m_1}^{s_{1_1},s_{2_1},\cdots,s_{k_1}}}\bigoplus\cdots\bigoplus\overline{C_{m_l}^{s_{1_l},s_{2_l},\cdots,s_{k_l}}}.
\end{aligned}$$

So, by Lemma 1, Lemma 9, Lemma 11 and (1), we have the following result.

Corollary 2 *For $n \geq \sum_{u=1}^{l} m_u$ and for each u, $1 \leq u \leq l$, $m_u > 2s_{k_u}$,*

$$
\begin{aligned}
&T(K_n - \bigcup_{u=1}^{l} C_{m_u}^{s_{1u}, s_{2u}, \cdots, s_{ku}}) \\
&= n^{n-\sum_{u=1}^{l} m_u + l - 2} \prod_{u=1}^{l} \prod_{j=1}^{m_u - 1} (n - 2k_u + \varepsilon_u^{-s_{1u}j} + \cdots + \varepsilon_u^{-s_{ku}j} + \varepsilon_u^{s_{1u}j} + \cdots + \varepsilon_u^{s_{ku}j}),
\end{aligned}
$$

where $\varepsilon_u = e^{\frac{2\pi i}{m_u}}$, for each u, $1 \leq u \leq l$.

In a similar fashion, the following corollary can be derived from Lemma 1, Lemma 10, Lemma 11 and (1):

Corollary 3 *For $n \geq \sum_{u=1}^{l} m_u$,*

$$
\begin{aligned}
&T(K_n + \bigcup_{u=1}^{l} C_{m_u}^{s_{1u}, s_{2u}, \cdots, s_{ku}}) \\
&= n^{n-\sum_{u=1}^{l} m_u + l - 2} \prod_{u=1}^{l} \prod_{j=1}^{m_u - 1} (n + 2k_u - \varepsilon_u^{-s_{1u}j} - \cdots - \varepsilon_u^{-s_{ku}j} - \varepsilon_u^{s_{1u}j} - \cdots - \varepsilon_u^{s_{ku}j}),
\end{aligned}
$$

where $\varepsilon_u = e^{\frac{2\pi i}{m_u}}$, for each u, $1 \leq u \leq l$.

Now we start to calculate $T(K_n - S)$ by assuming that $S = C_m^s$. As previously noted, if $(m, s) = 1$ this is just the m-cycle and if $(m, s) = d > 1$ this is the disjoint union of d cycles, each of length m/d.

Before proceeding we note that *Gilbert* and *Myrvold* [13] already gave a formula for the number of spanning trees in the graph $K_n - S$ where S is the disjoint union of cycles. The following theorem can actually be derived from *Gilbert* and *Myrvold*'s formula. The proof here is new, though; we derive it since it provides a 'pure' way of illustrating the techniques we will use later.

Theorem 12 *For $n \geq m > 2s$, if $(m,\ s) = d$, then*

$$
T(K_n - C_m^s) = n^{n-m-2} \left[\left(\sqrt{\frac{n}{4}} + \sqrt{\frac{n-4}{4}} \right)^{m/d} - \left(-\sqrt{\frac{n}{4}} + \sqrt{\frac{n-4}{4}} \right)^{m/d} \right]^{2d}.
$$

Proof. Let $\varepsilon_1 = e^{\frac{2d\pi i}{m}}$. If $(m,\ s) = d$, then C_m^s is the disjoint union of d cycles $C_{m/d}^1$. So, by Corollary 2, we have

$$
\begin{aligned}
T(K_n - C_m^s) &= T(K_n - \bigcup_{u=1}^{d} C_{m/d}^1) \\
&= n^{n-m+d-2} \prod_{u=1}^{d} \prod_{j=1}^{\frac{m}{d}-1} (n - 2 + \varepsilon_1^{-j} + \varepsilon_1^{j})
\end{aligned}
$$

$$
\begin{aligned}
&= n^{n-m+d-2}\prod_{u=1}^{d}\prod_{j=1}^{\frac{m}{d}-1}\left(n-2+2\cos\frac{2dj\pi}{m}\right)\\
&= n^{n-m+d-2}\prod_{u=1}^{d}\left[(-4)^{\frac{m}{d}-1}\prod_{j=1}^{\frac{m}{d}-1}\left(\frac{-n+4}{4}-\cos^2\frac{dj\pi}{m}\right)\right],
\end{aligned}
$$

where we are using the fact that $1+\cos(2x)=\cos^2 x$.

Applying the formulas (9) and then (6) yields the required

$$
\begin{aligned}
T(K_n-C_m^s) &= n^{n-m+d-2}\prod_{u=1}^{d}\left[(-1)^{\frac{m}{d}-1}U^2_{\frac{m}{d}-1}\left(\sqrt{\frac{-n+4}{4}}\right)\right]\\
&= n^{n-m-2}\left[\left(\sqrt{\frac{n}{4}}+\sqrt{\frac{n-4}{4}}\right)^{m/d}-\left(-\sqrt{\frac{n}{4}}+\sqrt{\frac{n-4}{4}}\right)^{m/d}\right]^{2d}.
\end{aligned}
$$

□

As a first consequence of Theorem 12 we can easily derive:

Corollary 4

$$
\begin{aligned}
T(K_n-C_3^1) &= n^{n-4}(n-3)^2,\ n\geq 3;\\
T(K_n-C_4^1) &= n^{n-5}(n-2)^2(n-4),\ n\geq 4;\\
T(K_n-C_5^1) &= n^{n-6}(n^2-5n+5)^2,\ n\geq 5;\\
T(K_n-C_6^1) &= n^{n-7}(n-1)^2(n-3)^2(n-4),\ n\geq 6;\\
T(K_n-C_6^2) &= n^{n-6}(n-3)^4,\ n\geq 6.
\end{aligned}
$$

The first four formulae of the above corollary already appear in [6] where they are given in generic forms and derived from *Kel'mans* and *Chelnokov*'s result (1) by direct computation.

The proof above illustrates our general tools. We now see how to apply them when looking at the complement of a more complicated circulant graph.

Theorem 13 *For $n\geq m>4$,*

$$
\begin{aligned}
T(K_n-C_m^{1,2}) &= n^{n-m-2}\left[\left(x_1+\sqrt{x_1^2-1}\right)^m-\left(x_1-\sqrt{x_1^2-1}\right)^m\right]^2\cdot\\
&\quad\left[\left(x_2+\sqrt{x_2^2-1}\right)^m-\left(x_2-\sqrt{x_2^2-1}\right)^m\right]^2,
\end{aligned}
$$

where $x_1=\sqrt{\frac{3}{8}+\frac{1}{8}\sqrt{25-4n}}$, $x_2=\sqrt{\frac{3}{8}-\frac{1}{8}\sqrt{25-4n}}$.

Proof. We use a very similar technique to the proof of Theorem 12. In this proof let $\varepsilon_1=e^{\frac{2\pi i}{m}}$, and x_1, x_2 be defined as above. Then

$$
T(K_n-C_m^{1,2}) = n^{n-m-1}\prod_{j=1}^{m-1}(n-4+\varepsilon_1^{-j}+\varepsilon_1^{-2j}+\varepsilon_1^{j}+\varepsilon_1^{2j})
$$

$$
\begin{aligned}
&= n^{n-m-1} \prod_{j=1}^{m-1} \left(n - 4 - 12\cos^2 \frac{j\pi}{m} + 16\cos^4 \frac{j\pi}{m}\right) \\
&= n^{n-m-1} 16^{m-1} \prod_{j=1}^{m-1} \left(x_1^2 - \cos^2 \frac{j\pi}{m}\right)\left(x_2^2 - \cos^2 \frac{j\pi}{m}\right) \\
&= n^{n-m-1} U_{m-1}^2(x_1) U_{m-1}^2(x_2).
\end{aligned}
$$

The closed formula in the theorem statement follows from (9) and then (6). $\square$

As a simple application, Theorem 13 can directly imply the following formulae:

Corollary 5

$$
\begin{aligned}
T(K_n - C_5^{1,2}) &= n^{n-6}(n-5)^4,\ n \geq 5; \\
T(K_n - C_6^{1,2}) &= n^{n-7}(n-6)^2(n-4)^3,\ n \geq 6; \\
T(K_n - C_7^{1,2}) &= n^{n-8}(n^3 - 14n^2 + 63n - 91)^2,\ n \geq 7.
\end{aligned}
$$

We now examine the complement of a slightly more complicated circulant graph.

Theorem 14 *For $n \geq m > 8$, if m is odd, then*

$$
T(K_n - C_m^{2,4}) = T(K_n - C_m^{1,2});
$$

Otherwise, if m is even, then

$$
\begin{aligned}
T(K_n - C_m^{2,4}) &= n^{n-m-2} \left[\left(x_1 + \sqrt{x_1^2 - 1}\right)^{m/2} - \left(x_1 - \sqrt{x_1^2 - 1}\right)^{m/2}\right]^4 \cdot \\
&\quad \left[\left(x_2 + \sqrt{x_2^2 - 1}\right)^{m/2} - \left(x_2 - \sqrt{x_2^2 - 1}\right)^{m/2}\right]^4,
\end{aligned}
$$

where x_1 and x_2 are as defined in Theorem 13.

Proof.

If m is odd then $C_m^{2,4}$ is isomorphic to $C_m^{1,2}$ (see, e.g., [28], the note after Lemma 7), so the result of Theorem 13 applies. If m is even $C_m^{2,4}$ is the disjoint union of 2 circulant graphs $C_{m/2}^{1,2}$. The proof in this case is just to combine Corollary 2 and the proof of Theorem 13. When m is even then let $\varepsilon_2 = e^{\frac{4\pi i}{m}}$, we have

$$
\begin{aligned}
T(K_n - C_m^{2,4}) &= T(K_n - C_{m/2}^{1,2} \bigcup C_{m/2}^{1,2}) \\
&= n^{n-m} \left(\prod_{j=1}^{\frac{m}{2}-1} (n - 4 + \varepsilon_2^{-j} + \varepsilon_2^{-2j} + \varepsilon_2^{j} + \varepsilon_2^{2j})\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&= n^{n-m}\left(\prod_{j=1}^{\frac{m}{2}-1}\left(n-4-12\cos^2\frac{2j\pi}{m}+16\cos^4\frac{2j\pi}{m}\right)\right)^2\\
&= n^{n-m}\left(16^{\frac{m}{2}-1}\prod_{j=1}^{\frac{m}{2}-1}\left(x_1^2-\cos^2\frac{2j\pi}{m}\right)\left(x_2^2-\cos^2\frac{2j\pi}{m}\right)\right)^2\\
&= n^{n-m}U^4_{\frac{m}{2}-1}(x_1)U^4_{\frac{m}{2}-1}(x_2).
\end{aligned}
$$

□

Corollary 6

$$
\begin{aligned}
T(K_n-C_9^{2,4}) &= n^{n-10}(n-6)^2(n^3-12n^2+45n-51)^2,\ n\geq 9;\\
T(K_n-C_{10}^{2,4}) &= n^{n-10}(n-5)^8,\ n\geq 10;\\
T(K_n-C_{11}^{2,4}) &= n^{n-12}(n^5-22n^4+187n^3-759n^2+1441n-979)^2,\ n\geq 11.
\end{aligned}
$$

We now discuss the general technique for calculating $T(K_n - C_m^{s_1,s_2,\cdots,s_k})$ when $\gcd(s_1,s_2,\cdots,s_k,m)=1$ (the case $\gcd(s_1,s_2,\cdots,s_k,m)\neq 1$ can then be dealt with similarly to the case "m is even" in the proof of Theorem 14). In the following paragraphs let $\varepsilon = e^{\frac{2\pi i}{m}}$. From Corollary 2,

$$
\begin{aligned}
&T(K_n-C_m^{s_1,s_2,\cdots,s_k})\\
&= n^{n-m-1}\prod_{j=1}^{m-1}(n-2k+\varepsilon^{-s_1j}+\varepsilon^{-s_2j}+\cdots+\varepsilon^{-s_kj}+\varepsilon^{s_1j}+\varepsilon^{s_2j}+\cdots+\varepsilon^{s_kj})\\
&= n^{n-m-1}\prod_{j=1}^{m-1}\left(n-2k+2\cos\frac{2s_1\pi}{m}+2\cos\frac{2s_2\pi}{m}+\cdots+2\cos\frac{2s_k\pi}{m}\right).
\end{aligned}
$$

Similarly to the situation in Lemma 2 it is easy to prove by induction that $\cos(kx)$ can be expressed as a polynomial in $\cos x$ of order k. Using this fact, for any integer s, $\cos\frac{2sj\pi}{m}$ can be written as a polynomial in $\cos^2\frac{j\pi}{m}$ of order s. So,

$$
T(K_n-C_m^{s_1,s_2,\cdots,s_k}) = n^{n-m-1}\prod_{j=1}^{m-1}\left(n-2k+g(\cos^2\frac{j\pi}{m})\right),
$$

where $g(x)$ is a polynomial of order s_k (dependent only upon $s_1,s_2,\ldots,s_k$, and not on m). Thus,

$$
T(K_n-C_m^{s_1,s_2,\cdots,s_k}) = n^{n-m-1}\prod_{j=1}^{m-1}h(\cos^2\frac{\pi j}{m}),
$$

where $h(x)$ is a polynomial of degree s_k whose constant term is a linear function of n. Even more, by explicit calculation we can see that the coefficient of x^{s_k} in $h(x)$ is 4^{s_k}. We can therefore write

$$
h(x) = (-4)^{s_k}\prod_{i=1}^{s_k}(x_i-\cos^2\frac{\pi j}{m}),
$$

where $x_1,\ x_2,\ \cdots,\ x_k$ are zeros of $h(x)$. Then, combining formula (9) with the last two equations we have

$$T(K_n - C_m^{s_1,s_2,\cdots,s_k}) = (-1)^{s_k} n^{n-m-1} \prod_{i=1}^{s_k} U_{m-1}^2(\sqrt{x_i}). \tag{12}$$

Plugging in (6) gives

$$\begin{aligned} T(K_n - C_m^{s_1,s_2,\cdots,s_k}) &= (-1)^{s_k} n^{n-m-1} \left(\prod_{i=1}^{s_k} \frac{1}{4(x_i^2-1)} \right) \cdot \\ & \prod_{i=1}^{s_k} \left[\left(x_i + \sqrt{x_i^2-1} \right)^m - \left(x_i - \sqrt{x_i^2-1} \right)^m \right]^2, \end{aligned} \tag{13}$$

an exact formula for $T(K_n - C_m^{s_1,s_2,\cdots,s_k})$ in terms of the x_i, which are the roots of polynomial $h(x)$ which, in turn, is only dependent upon the s_i and n.

In the special case $s_k \leq 4$, the polynomial $h(x)$ can be explicitly factored so we can find an explicit formula for the x_i as a function of n and therefore an exact formula for the number spanning trees in $K_n - C_m^{s_1,s_2,\cdots,s_k}$ as a function of n. In the appendix we illustrate this by listing the formulas for all $T\left(K_n - C_m^{s_1,s_2,\cdots,s_k}\right)$ with distinct s_i such that $s_k \leq 4$.

From Corollary 3 and the properties of Chebyshev polynomials, we also can derive the following closed formulae for the numbers of spanning trees in complete graphs with circulant graphs added. As before we start by adding C_m^s.

Theorem 15 *For $n \geq m$, if $(m,\ s) = d$, then*

$$T(K_n + C_m^s) = n^{n-m-2} \left[\left(\sqrt{\frac{n+4}{4}} + \sqrt{\frac{n}{4}} \right)^{m/d} - \left(\sqrt{\frac{n+4}{4}} - \sqrt{\frac{n}{4}} \right)^{m/d} \right]^{2d}.$$

Proof. This is similar to the proof of Theorem 12. By Corollary 3, we have

$$\begin{aligned} T(K_n + C_m^s) &= T(K_n + \bigcup_{u=1}^{d} C_{m/d}^1) \\ &= n^{n-m+d-2} \prod_{u=1}^{d} \prod_{j=1}^{\frac{m}{d}-1} (n + 2 - \varepsilon_1^{-j} - \varepsilon_1^j) \\ &= n^{n-m+d-2} \prod_{u=1}^{d} \prod_{j=1}^{\frac{m}{d}-1} \left(n + 2 - 2\cos\frac{2dj\pi}{m} \right) \\ &= n^{n-m+d-2} \prod_{u=1}^{d} \left[4^{\frac{m}{d}-1} \prod_{j=1}^{\frac{m}{d}-1} \left(\frac{n+4}{4} - \cos^2\frac{dj\pi}{m} \right) \right]. \end{aligned}$$

By using the formulae (9) and then (6),

$$
\begin{aligned}
T(K_n + C_m^s) &= n^{n-m+d-2} \prod_{u=1}^{d} \left[U^2_{\frac{m}{d}-1}\left(\sqrt{\frac{n+4}{4}}\right)\right] \\
&= n^{n-m-2}\left[\left(\sqrt{\frac{n+4}{4}}+\sqrt{\frac{n}{4}}\right)^{m/d} - \left(\sqrt{\frac{n+4}{4}}-\sqrt{\frac{n}{4}}\right)^{m/d}\right]^{2d}.
\end{aligned}
$$

□

This then immediately gives us, for example,

Corollary 7

$$
\begin{aligned}
T(K_n + C_2^1) &= n^{n-3}(n+4),\ n \geq 2; \\
T(K_n + C_3^1) &= n^{n-4}(n+3)^2,\ n \geq 3; \\
T(K_n + C_4^1) &= n^{n-5}(n+4)(n+2)^2,\ n \geq 4; \\
T(K_n + C_4^2) &= n^{n-4}(n+4)^2,\ n \geq 4; \\
T(K_n + C_5^1) &= n^{n-6}(n^2+5n+5)^2,\ n \geq 5.
\end{aligned}
$$

We can also derive results for general $K_n + C_m^{s_1,s_2,\cdots,s_k}$ that are analogous to the ones previously derived for $K_n - C_m^{s_1,s_2,\cdots,s_k}$. Since the proofs are so similar, we omit them.

Theorem 16 *For $n \geq m$,*

$$
\begin{aligned}
T(K_n + C_m^{1,2}) = {} & (-1)^m n^{n-m-2}\left[\left(x_1+\sqrt{x_1^2-1}\right)^m - \left(x_1-\sqrt{x_1^2-1}\right)^m\right]^2 \cdot \\
& \left[\left(x_2+\sqrt{x_2^2-1}\right)^m - \left(x_2-\sqrt{x_2^2-1}\right)^m\right]^2,
\end{aligned}
$$

where $x_1 = \sqrt{\frac{3}{8}+\frac{1}{8}\sqrt{25+4n}}$, $x_2 = \sqrt{\frac{3}{8}-\frac{1}{8}\sqrt{25+4n}}$.

Corollary 8

$$
\begin{aligned}
T(K_n + C_3^{1,2}) &= n^{n-4}(n+6)^2,\ n \geq 3; \\
T(K_n + C_4^{1,2}) &= n^{n-5}(n+4)(n+6)^2,\ n \geq 4; \\
T(K_n + C_5^{1,2}) &= n^{n-6}(n+5)^2,\ n \geq 5; \\
T(K_n + C_6^{1,2}) &= n^{n-7}(n+6)^2(n+4)^3,\ n \geq 6; \\
T(K_n + C_7^{1,2}) &= n^{n-8}(n^3+14n^2+63n+91)^2,\ n \geq 7.
\end{aligned}
$$

Theorem 17 *For $n \geq m$, if m is odd, then*

$$
T(K_n + C_m^{2,4}) = T(K_n + C_m^{1,2});
$$

Otherwise m is even, then

$$\begin{aligned} T(K_n + C_m^{2,4}) &= n^{n-m-2}\left[\left(x_1+\sqrt{x_1^2-1}\right)^{m/2} - \left(x_1-\sqrt{x_1^2-1}\right)^{m/2}\right]^4 \cdot \\ & \left[\left(x_2+\sqrt{x_2^2-1}\right)^{m/2} - \left(x_2-\sqrt{x_2^2-1}\right)^{m/2}\right]^4, \end{aligned}$$

where x_1 and x_2 are defined as in Theorem 16.

Corollary 9

$$\begin{aligned} T(K_n + C_6^{2,4}) &= n^{n-6}(n+6)^4,\ n \geq 6; \\ T(K_n + C_7^{2,4}) &= n^{n-8}(n^3+14n^2+63n+91)^2,\ n \geq 7; \\ T(K_n + C_8^{2,4}) &= n^{n-8}(n+4)^2(n+6)^4,\ n \geq 8; \\ T(K_n + C_9^{2,4}) &= n^{n-10}(n+6)^2(n^3+12n^2+45n+51)^2,\ n \geq 9. \end{aligned}$$

We conclude this discussion by quickly pointing out that $T(K_n - C_m^{s_1,s_2,\cdots,s_k})$ can be shown to satisfy recurrence relations in m. (Recurrence relations for $T(K_n + C_m^{s_1,s_2,\cdots,s_k})$ can be derived similarly).

We already know from (12) that

$$T(K_n - C_m^{s_1,s_2,\cdots,s_k}) = (-1)^{s_k} n^{n-m-1} \prod_{i=1}^{s_k} U_{m-1}^2(\sqrt{x_i})$$

where the x_i depend only upon the s_i and n. Writing $T(K_n - C_m^{s_1,s_2,\cdots,s_k}) = n^{n-m-1}a_m^2$, from the formula (6) it is seen that $a_m = r\sum_{i=1}^{2^{s_k}} r_i^m$, where r, r_i, $1 \leq i \leq 2^{s_k}$ are functions of n. So, the a_m satisfy a recurrence relation of the form

$$\forall m > 2^{s_k} + 2s_k,\ a_m = \sum_{i=2s_k+1}^{2^{2s_k+s_k}} b_i a_{m-i}.$$

To derive the b_i for specific cases we can use the Matrix-Tree Theorem to calculate a_i for $2s_k + 1 \leq i \leq 2^{s_k+1} + 2s_k$ and then solve for b_i. Two examples (without proof) are given below:

Theorem 18 *For $n \geq m \geq 3$,*

$$T(K_n - C_m^1) = n^{n-m-1}a_m^2,$$

where a_m satisfies the recurrence relation:

$$a_m = \sqrt{n-4}\,a_{m-1} + a_{m-2}$$

with initial conditions $a_3 = n-3$, $a_4 = \sqrt{n-4}(n-2)$.

Theorem 19 *For $n \geq m \geq 5$,*

$$T(K_n - C_m^{1,2}) = n^{n-m-1} a_m^2,$$

where a_m satisfies the recurrence relation:

$$a_m = \sqrt{n-4}\,a_{m-1} - a_{m-2} + \sqrt{n-4}\,a_{m-3} - a_{m-4}$$

with initial conditions $a_5 = (n-5)^2$, $a_6 = \sqrt{n-4}(n-4)(n-6)$, $a_7 = n^3 - 14n^2 + 63n - 91$, $a_8 = \sqrt{n-4}(n-6)(n^2 - 8n - 14)$.

6 Conclusion and open problems

In this paper we used properties of Chebyshev polynomials to derive closed formulas for the number of spanning trees in graphs belonging to classes related to circulant graphs. The first problem we described was to rederive that the number of spanning trees in the circulant graph $C_n^{s_1,s_2,\cdots,s_k}$ with fixed step sizes has the form $T(C_n^{s_1,s_2,\cdots,s_k}) = na_n^2$ where a_n satisfies a recurrence relation of order 2^{s_k-1}. This theorem had previously been proven in [28]; the method provided here is simpler though and also provides a new more efficient technique, for deriving asymptotics.

We then discussed how to use a similar approach to derive closed formulas for some $T(C_n^{s1,\cdots,s_k,\frac{n}{a_1},\cdots,\frac{n}{a_l}})$ where the step sizes are not constant. More specifically, the technique is applicable whenever $s_1, \cdots, s_k$ are constant integers and all $a_1, \cdots, a_l$ are in the set $\{2, 3, 4, 6\}$ with $a_u | n$ for any u, $1 \leq u \leq l$.

We concluded by deriving closed formulas for the number of spanning trees in $K_n \pm S$ where $S = C_m^{s_1,s_2,\cdots,s_k}$ is a circulant graph. Our key step was to factorize a polynomial of order s_k and then express the number of spanning trees in terms of Chebyshev polynomials evaluated at functions of the roots of the polynomial. In particular, when $s_k \leq 4$, we could explicitly factorize the polynomial and derive a"closed" form for the number of spanning trees.

One thing that we should point out is that, in all the formulas we derived, we assumed that $s_1 < s_2 < \cdots < s_k$. This was just for the sake of convenience, though, and was not necessary for our proofs. The techniques above still work for repeated s_i values, e.g., we could use them to evaluate $T(K_n + C_m^{1,1})$ $(m \leq n)$ where $C_m^{1,1}$ is the doubly-linked cycle.

A major open problem still remaining is to devise a technique that would work to derive closed formulae for $T\left(C_n^{s1,\cdots,s_k,\frac{n}{a_1},\cdots,\frac{n}{a_l}}\right)$ where the a_i could be arbitrary.

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Appendix

In Section 5 we discussed a general method for calculating $T(K_n - C_m^{s_1, s_2, \cdots, s_k})$ where $\gcd(s_1, s_2, \cdots, s_k, m) = 1$ For fixed jumps $s_{1,2}, \ldots, s_k$ (13) tells us that

$$\begin{aligned} T(K_n - C_m^{s_1, s_2, \cdots, s_k}) &= (-1)^{s_k} n^{n-m-1} \left(\prod_{i=1}^{s_k} \frac{1}{4(x_i^2 - 1)} \right) \cdot \\ & \prod_{i=1}^{s_k} \left[\left(x_i + \sqrt{x_i^2 - 1} \right)^m - \left(x_i - \sqrt{x_i^2 - 1} \right)^m \right]^2, \end{aligned}$$

where the x_i are the roots of a degree s_k polynomial defined in terms of the s_i and n. In what follows, for $k > 1$ and $s_k \leq 4$, we give all of these $h(x)$s and their roots.

1. For the graph $K_n - C_m^{1,2}$, the corresponding polynomial $h(x)$ is

$$n - 4 - 12x + 16x^2 = 16(x_1 - x)(x_2 - x),$$

where x_1, x_2 are as follows:

$$\frac{3}{8} + \frac{1}{8}\sqrt{25 - 4\,n},\ \frac{3}{8} - \frac{1}{8}\sqrt{25 - 4\,n}.$$

2. For the graph $K_n - C_m^{1,3}$, the corresponding polynomial $h(x)$ is

$$n - 8 + 40x - 96x^2 + 64x^3 = -64(x_1 - x)(x_2 - x)(x_3 - x),$$

where x_1, x_2, x_3 are as follows:

$$\begin{aligned}
&\frac{1}{24}\,\alpha^{(1/3)} + \frac{1}{\alpha^{(1/3)}} + \frac{1}{2},\\
&-\frac{1}{48}\,\alpha^{(1/3)} - \frac{1}{2}\,\frac{1}{\alpha^{(1/3)}} + \frac{1}{2} + \frac{1}{8}\,i\,\sqrt{3}\,(\frac{1}{6}\,\alpha^{(1/3)} - 4\,\frac{1}{\alpha^{(1/3)}}),\\
&-\frac{1}{48}\,\alpha^{(1/3)} - \frac{1}{2}\,\frac{1}{\alpha^{(1/3)}} + \frac{1}{2} - \frac{1}{8}\,i\,\sqrt{3}\,(\frac{1}{6}\,\alpha^{(1/3)} - 4\,\frac{1}{\alpha^{(1/3)}}),\\
&\alpha := 432 - 108\,n + 12\,\sqrt{1200 - 648\,n + 81\,n^2}.
\end{aligned}$$

3. For the graph $K_n - C_m^{1,4}$, the corresponding polynomial $h(x)$ is

$$n - 4 - 60x + 320x^2 - 512x^3 + 256x^4 = 256(x_1 - x)(x_2 - x)(x_3 - x)(x_4 - x),$$

where x_1, x_2, x_3, x_4 are as follows:

$$\begin{aligned}
&\frac{1}{2} + \frac{1}{48}\,\sqrt{6}\,\sqrt{\alpha_2} + \frac{1}{48}\,\sqrt{-\frac{-192\,\alpha_1{}^{(1/3)}\,\sqrt{\alpha_2} + 6\,\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)} - 192\,\sqrt{\alpha_2} + 288\,\sqrt{\alpha_2}\,n + 72\,\sqrt{6}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\,\sqrt{\alpha_2}}},\\
&\frac{1}{2} + \frac{1}{48}\,\sqrt{6}\,\sqrt{\alpha_2} - \frac{1}{48}\,\sqrt{-\frac{-192\,\alpha_1{}^{(1/3)}\,\sqrt{\alpha_2} + 6\,\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)} - 192\,\sqrt{\alpha_2} + 288\,\sqrt{\alpha_2}\,n + 72\,\sqrt{6}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\,\sqrt{\alpha_2}}},\\
&\frac{1}{2} - \frac{1}{48}\,\sqrt{6}\,\sqrt{\alpha_2} + \frac{1}{48}\,\sqrt{-\frac{-192\,\alpha_1{}^{(1/3)}\,\sqrt{\alpha_2} + 6\,\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)} - 192\,\sqrt{\alpha_2} + 288\,\sqrt{\alpha_2}\,n - 72\,\sqrt{6}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\,\sqrt{\alpha_2}}},\\
&\frac{1}{2} - \frac{1}{48}\,\sqrt{6}\,\sqrt{\alpha_2} - \frac{1}{48}\,\sqrt{-\frac{-192\,\alpha_1{}^{(1/3)}\,\sqrt{\alpha_2} + 6\,\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)} - 192\,\sqrt{\alpha_2} + 288\,\sqrt{\alpha_2}\,n - 72\,\sqrt{6}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\,\sqrt{\alpha_2}}},\\
&\alpha_1 := -2708 + 1152\,n + 12\,\sqrt{51153 - 44352\,n + 10752\,n^2 - 768\,n^3},\\
&\alpha_2 := \frac{16\,\alpha_1{}^{(1/3)} + \alpha_1{}^{(2/3)} - 32 + 48\,n}{\alpha_1{}^{(1/3)}}.
\end{aligned}$$

4. For the graph $K_n - C_m^{2,3}$, the corresponding polynomial $h(x)$ is

$$n - 4 + 20x - 80x^2 + 64x^3 = -64(x_1 - x)(x_2 - x)(x_3 - x),$$

where x_1, x_2, x_3 are as follows:

$$\begin{aligned}
&\frac{1}{24}\,\alpha^{(1/3)} + \frac{5}{3}\,\frac{1}{\alpha^{(1/3)}} + \frac{5}{12},\\
&-\frac{1}{48}\,\alpha^{(1/3)} - \frac{5}{6}\,\frac{1}{\alpha^{(1/3)}} + \frac{5}{12} + \frac{1}{8}\,i\,\sqrt{3}\,(\frac{1}{6}\,\alpha^{(1/3)} - \frac{20}{3}\,\frac{1}{\alpha^{(1/3)}}),\\
&-\frac{1}{48}\,\alpha^{(1/3)} - \frac{5}{6}\,\frac{1}{\alpha^{(1/3)}} + \frac{5}{12} - \frac{1}{8}\,i\,\sqrt{3}\,(\frac{1}{6}\,\alpha^{(1/3)} - \frac{20}{3}\,\frac{1}{\alpha^{(1/3)}}),\\
&\alpha := 532 - 108\,n + 12\,\sqrt{1521 - 798\,n + 81\,n^2}.
\end{aligned}$$

5. For the graph $K_n - C_m^{3,4}$, the corresponding polynomial $h(x)$ is

$$n - 4 - 28x + 224x^2 - 448x^3 + 256x^4 = 256(x_1 - x)(x_2 - x)(x_3 - x)(x_4 - x),$$

where x_1, x_2, x_3, x_4 are as follows:

$$\frac{7}{16}+\frac{1}{48}\sqrt{3}\sqrt{\alpha_2}+\frac{1}{48}\sqrt{-\frac{-210\,\alpha_1{}^{(1/3)}\sqrt{\alpha_2}+6\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)}+24\sqrt{\alpha_2}+288\sqrt{\alpha_2}\,n-126\sqrt{3}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\sqrt{\alpha_2}}},$$

$$\frac{7}{16}+\frac{1}{48}\sqrt{3}\sqrt{\alpha_2}-\frac{1}{48}\sqrt{-\frac{-210\,\alpha_1{}^{(1/3)}\sqrt{\alpha_2}+6\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)}+24\sqrt{\alpha_2}+288\sqrt{\alpha_2}\,n-126\sqrt{3}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\sqrt{\alpha_2}}},$$

$$\frac{7}{16}-\frac{1}{48}\sqrt{3}\sqrt{\alpha_2}+\frac{1}{48}\sqrt{-\frac{-210\,\alpha_1{}^{(1/3)}\sqrt{\alpha_2}+6\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)}+24\sqrt{\alpha_2}+288\sqrt{\alpha_2}\,n+126\sqrt{3}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\sqrt{\alpha_2}}},$$

$$\frac{7}{16}-\frac{1}{48}\sqrt{3}\sqrt{\alpha_2}-\frac{1}{48}\sqrt{-\frac{-210\,\alpha_1{}^{(1/3)}\sqrt{\alpha_2}+6\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)}+24\sqrt{\alpha_2}+288\sqrt{\alpha_2}\,n+126\sqrt{3}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\sqrt{\alpha_2}}},$$

$$\alpha_1 := -2492+1260\,n+12\sqrt{43125-43626\,n+10833\,n^2-768\,n^3},$$

$$\alpha_2 := \frac{35\,\alpha_1{}^{(1/3)}+2\,\alpha_1{}^{(2/3)}+8+96\,n}{\alpha_1{}^{(1/3)}}.$$

6. For the graph $K_n - C_m^{1,2,3}$, the corresponding polynomial $h(x)$ is

$$n-8+24x-80x^2+64x^3 = -64(x_1-x)(x_2-x)(x_3-x),$$

where x_1, x_2, x_3 are as follows:

$$\frac{1}{24}\,\alpha^{(1/3)}+\frac{7}{6}\,\frac{1}{\alpha^{(1/3)}}+\frac{5}{12},$$

$$-\frac{1}{48}\,\alpha^{(1/3)}-\frac{7}{12}\,\frac{1}{\alpha^{(1/3)}}+\frac{5}{12}+\frac{1}{8}\,i\sqrt{3}\,(\frac{1}{6}\,\alpha^{(1/3)}-\frac{14}{3}\,\frac{1}{\alpha^{(1/3)}}),$$

$$-\frac{1}{48}\,\alpha^{(1/3)}-\frac{7}{12}\,\frac{1}{\alpha^{(1/3)}}+\frac{5}{12}-\frac{1}{8}\,i\sqrt{3}\,(\frac{1}{6}\,\alpha^{(1/3)}-\frac{14}{3}\,\frac{1}{\alpha^{(1/3)}}),$$

$$\alpha := 784-108\,n+12\sqrt{4116-1176\,n+81\,n^2}.$$

7. For the graph $K_n - C_m^{1,2,4}$, the corresponding polynomial $h(x)$ is

$$n-4-76x+336x^2-512x^3+256x^4 = 256(x_1-x)(x_2-x)(x_3-x)(x_4-x),$$

where x_1, x_2, x_3, x_4 are as follows:

$$\frac{1}{2}+\frac{1}{48}\sqrt{6}\sqrt{\alpha_2}+\frac{1}{48}\sqrt{-\frac{-144\,\alpha_1{}^{(1/3)}\sqrt{\alpha_2}+6\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)}-1512\sqrt{\alpha_2}+288\sqrt{\alpha_2}\,n+72\sqrt{6}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\sqrt{\alpha_2}}},$$

$$\frac{1}{2}+\frac{1}{48}\sqrt{6}\sqrt{\alpha_2}-\frac{1}{48}\sqrt{-\frac{-144\,\alpha_1{}^{(1/3)}\sqrt{\alpha_2}+6\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)}-1512\sqrt{\alpha_2}+288\sqrt{\alpha_2}\,n+72\sqrt{6}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\sqrt{\alpha_2}}},$$

$$\frac{1}{2}-\frac{1}{48}\sqrt{6}\sqrt{\alpha_2}+\frac{1}{48}\sqrt{-\frac{-144\,\alpha_1{}^{(1/3)}\sqrt{\alpha_2}+6\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)}-1512\sqrt{\alpha_2}+288\sqrt{\alpha_2}\,n-72\sqrt{6}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\sqrt{\alpha_2}}},$$

$$\frac{1}{2}-\frac{1}{48}\sqrt{6}\sqrt{\alpha_2}-\frac{1}{48}\sqrt{-\frac{-144\,\alpha_1{}^{(1/3)}\sqrt{\alpha_2}+6\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)}-1512\sqrt{\alpha_2}+288\sqrt{\alpha_2}\,n-72\sqrt{6}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\sqrt{\alpha_2}}},$$

$$\alpha_1 := -5292+864\,n+12\sqrt{305613-127008\,n+17280\,n^2-768\,n^3},$$

$$\alpha_2 := \frac{12\,\alpha_1{}^{(1/3)}+\alpha_1{}^{(2/3)}-252+48\,n}{\alpha_1{}^{(1/3)}}.$$

8. For the graph $K_n - C_m^{1,3,4}$, the corresponding polynomial $h(x)$ is

$$n-8-24x+224x^2-448x^3+256x^4 = 256(x_1-x)(x_2-x)(x_3-x)(x_4-x),$$

where x_1, x_2, x_3, x_4 are as follows:

$$\frac{7}{16}+\frac{1}{48}\sqrt{3}\sqrt{\alpha_2}+\frac{1}{48}\sqrt{-\frac{-210\,\alpha_1{}^{(1/3)}\sqrt{\alpha_2}+6\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)}-624\sqrt{\alpha_2}+288\sqrt{\alpha_2}\,n+18\sqrt{3}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\sqrt{\alpha_2}}},$$

$$\frac{7}{16}+\frac{1}{48}\sqrt{3}\sqrt{\alpha_2}-\frac{1}{48}\sqrt{-\frac{-210\,\alpha_1{}^{(1/3)}\sqrt{\alpha_2}+6\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)}-624\sqrt{\alpha_2}+288\sqrt{\alpha_2}\,n+18\sqrt{3}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\sqrt{\alpha_2}}},$$

$$\frac{7}{16}-\frac{1}{48}\sqrt{3}\sqrt{\alpha_2}+\frac{1}{48}\sqrt{-\frac{-210\,\alpha_1{}^{(1/3)}\sqrt{\alpha_2}+6\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)}-624\sqrt{\alpha_2}+288\sqrt{\alpha_2}\,n-18\sqrt{3}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\sqrt{\alpha_2}}},$$
$$\frac{7}{16}-\frac{1}{48}\sqrt{3}\sqrt{\alpha_2}-\frac{1}{48}\sqrt{-\frac{-210\,\alpha_1{}^{(1/3)}\sqrt{\alpha_2}+6\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)}-624\sqrt{\alpha_2}+288\sqrt{\alpha_2}\,n-18\sqrt{3}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\sqrt{\alpha_2}}},$$
$$\alpha_1:=-5408+1260\,n+12\sqrt{210912-105456\,n+16017\,n^2-768\,n^3},$$
$$\alpha_2:=\frac{35\,\alpha_1{}^{(1/3)}+2\,\alpha_1{}^{(2/3)}-208+96\,n}{\alpha_1{}^{(1/3)}}.$$

9. For the graph $K_n - C_m^{2,3,4}$, the corresponding polynomial $h(x)$ is

$$n-4-44x+240x^2-448x^3+256x^4=256(x_1-x)(x_2-x)(x_3-x)(x_4-x),$$

where x_1, x_2, x_3, x_4 are as follows:

$$\frac{7}{16}+\frac{1}{48}\sqrt{3}\sqrt{\alpha_2}+\frac{1}{48}\sqrt{-\frac{-162\,\alpha_1{}^{(1/3)}\sqrt{\alpha_2}+6\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)}-1296\sqrt{\alpha_2}+288\sqrt{\alpha_2}\,n-198\sqrt{3}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\sqrt{\alpha_2}}},$$
$$\frac{7}{16}+\frac{1}{48}\sqrt{3}\sqrt{\alpha_2}-\frac{1}{48}\sqrt{-\frac{-162\,\alpha_1{}^{(1/3)}\sqrt{\alpha_2}+6\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)}-1296\sqrt{\alpha_2}+288\sqrt{\alpha_2}\,n-198\sqrt{3}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\sqrt{\alpha_2}}},$$
$$\frac{7}{16}-\frac{1}{48}\sqrt{3}\sqrt{\alpha_2}+\frac{1}{48}\sqrt{-\frac{-162\,\alpha_1{}^{(1/3)}\sqrt{\alpha_2}+6\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)}-1296\sqrt{\alpha_2}+288\sqrt{\alpha_2}\,n+198\sqrt{3}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\sqrt{\alpha_2}}},$$
$$\frac{7}{16}-\frac{1}{48}\sqrt{3}\sqrt{\alpha_2}-\frac{1}{48}\sqrt{-\frac{-162\,\alpha_1{}^{(1/3)}\sqrt{\alpha_2}+6\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)}-1296\sqrt{\alpha_2}+288\sqrt{\alpha_2}\,n+198\sqrt{3}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\sqrt{\alpha_2}}},$$
$$\alpha_1:=-5400+972\,n+12\sqrt{272484-119556\,n+16929\,n^2-768\,n^3},$$
$$\alpha_2:=\frac{27\,\alpha_1{}^{(1/3)}+2\,\alpha_1{}^{(2/3)}-432+96\,n}{\alpha_1{}^{(1/3)}}.$$

10. For the graph $K_n - C_m^{1,2,3,4}$, the corresponding polynomial $h(x)$ is

$$n-8-40x+240x^2-448x^3+256x^4=256(x_1-x)(x_2-x)(x_3-x)(x_4-x),$$

where x_1, x_2, x_3, x_4 are as follows:

$$\frac{7}{16}+\frac{1}{48}\sqrt{3}\sqrt{\alpha_2}+\frac{1}{48}\sqrt{-\frac{-162\,\alpha_1{}^{(1/3)}\sqrt{\alpha_2}+6\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)}-1944\sqrt{\alpha_2}+288\sqrt{\alpha_2}\,n-54\sqrt{3}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\sqrt{\alpha_2}}},$$
$$\frac{7}{16}+\frac{1}{48}\sqrt{3}\sqrt{\alpha_2}-\frac{1}{48}\sqrt{-\frac{-162\,\alpha_1{}^{(1/3)}\sqrt{\alpha_2}+6\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)}-1944\sqrt{\alpha_2}+288\sqrt{\alpha_2}\,n-54\sqrt{3}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\sqrt{\alpha_2}}},$$
$$\frac{7}{16}-\frac{1}{48}\sqrt{3}\sqrt{\alpha_2}+\frac{1}{48}\sqrt{-\frac{-162\,\alpha_1{}^{(1/3)}\sqrt{\alpha_2}+6\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)}-1944\sqrt{\alpha_2}+288\sqrt{\alpha_2}\,n+54\sqrt{3}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\sqrt{\alpha_2}}},$$
$$\frac{7}{16}-\frac{1}{48}\sqrt{3}\sqrt{\alpha_2}-\frac{1}{48}\sqrt{-\frac{-162\,\alpha_1{}^{(1/3)}\sqrt{\alpha_2}+6\sqrt{\alpha_2}\,\alpha_1{}^{(2/3)}-1944\sqrt{\alpha_2}+288\sqrt{\alpha_2}\,n+54\sqrt{3}\,\alpha_1{}^{(1/3)}}{\alpha_1{}^{(1/3)}\sqrt{\alpha_2}}},$$
$$\alpha_1:=-7776+972\,n+12\sqrt{656100-209952\,n+22113\,n^2-768\,n^3},$$
$$\alpha_2:=\frac{27\,\alpha_1{}^{(1/3)}+2\,\alpha_1{}^{(2/3)}-648+96\,n}{\alpha_1{}^{(1/3)}}.$$