Approximating a Voronoi Cell

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Abstract

Given a set S of n points in $\mathbb{R}^d$, called *sites*, we consider the problem of approximating the Voronoi cell of a site p by a convex polyhedron with a small number of facets or, equivalently, of finding a small set of approximate Voronoi neighbors of p. More precisely, we define an *ϵ-approximate Voronoi neighborhood* of p, denoted $AVN_\epsilon(p, S)$, to be a subset of S satisfying the following property: p is an ϵ-approximate nearest neighbor for any point q inside the convex polyhedron defined by the bisectors between p and the sites in $AVN_\epsilon(p, S)$.

We show that there exists a set of ϵ-approximate Voronoi neighbors with cardinality $O(1/\sqrt{\epsilon})$ for $d = 2$ and cardinality $O((1/\epsilon)^{(d-1)/2} \log(1/\epsilon))$ for any fixed $d \geq 3$. We also provide a worst-case lower bound of $\Omega((1/\epsilon)^{(d-1)/2})$ on the number of approximate Voronoi neighbors. Thus, our bound is tight in the plane and within a factor of $O(\log(1/\epsilon))$ from optimal in dimension $d \geq 3$. Finally, based on our existence proofs, we design efficient algorithms for computing approximate Voronoi neighborhoods.

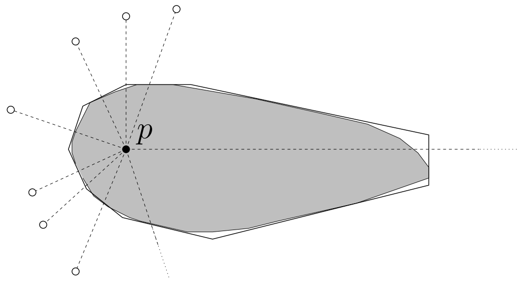

Figure 1: The Voronoi cell of p is shaded. Its approximation is the outer polygon; the approximate Voronoi neighborhood is represented by the white dots. Note that the approximation is better when we are close to p.

1 Introduction

Voronoi diagrams are among the most important structures in computational geometry and have far reaching applications in computer science and in other disciplines [4, 12]. Hence, it is important to study their combinatorial and computational complexity. Recently, the focus has been on approximating them [1, 2, 15].

In this paper, we address the problem of approximating the cells of a Voronoi diagram of points in d-dimensional Euclidean space. Our goal is to approximate the Voronoi cell of a given point p by a convex polyhedron that contains it, has a small number of facets, and such that p is an ϵ-approximate nearest neighbor for all the points in this polyhedron. (A detailed problem statement is in section 1.1 and an illustration can be found in Figure 1.)

Since Voronoi cells are convex, a first attempt to solve this problem would be to apply the well-known construction of Dudley [13] for approximating convex shapes. We would obtain an approximate cell with $O((1/\epsilon)^{(d-1)/2})$ facets, whose Hausdorff distance from the Voronoi cell is $O(D\epsilon)$, where D is the diameter of the cell. However, this approach does not work directly since our approximation measure must take into account the distance to the site p, while Dudley's method is uniform (see Fig. 2). A second attempt that

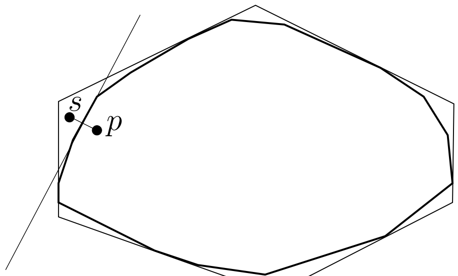

Figure 2: An example where Dudley's method does not yield a good approximate Voronoi cell. The inner polygon is the Voronoi cell of site p; the outer polygon is obtained by approximating it using Dudley's method. Site s is a Voronoi neighbor of p and it lies inside the outer polygon. The approximation factor here is infinite.

takes into account this non-uniformity would be to look at approximations with respect to scaling. For a polytope P and a point p, we say that polytope Q is a (p, ϵ)-approximation of P by scaling if $P \subseteq Q \subseteq p + (1 + \epsilon)(P - p)$. It is easy to see that such an approximation yields an $O(\epsilon)$-approximate Voronoi cell. Unfortunately, approximation with respect to scaling requires an unbounded number of facets in the worst case, even in the plane when $\epsilon = 1$ (the proof can be found in the appendix).

This major difficulty led us to believe that a compact approximation that is sensitive to the distance to the site p may not be possible at all. The surprising result of our work is that this can, in fact, be achieved and the number of required facets is at most a logarithmic factor away from Dudley's result. It shows that our concept of approximate Voronoi cells is fundamentally different from approximation by scaling. The difference between these two concepts may also help to explain why our proofs are considerably more involved than Dudley's.

1.1 Problem statement

Given a set S of n points in $\mathbb{R}^d$, called *sites*, the Voronoi diagram of S is a partition of space into cells, such that each cell is the region of space consisting of all points that are at least as close to a particular site as to any other. Here the distances are measured in the Euclidean metric. For a real parameter $\epsilon > 0$, we say that a site $p \in S$ is an *ϵ-approximate nearest neighbor* (*ϵ-NN*) of a point $q \in \mathbb{R}^d$ if the distance between q and p is at most $(1 + \epsilon)$ times the distance between q and its nearest neighbor in S. For any site p and any subset $R \subseteq S$, let $P(p, R)$ denote the convex polyhedron consisting of points that are at least as close to p as to any site in $R \setminus \{p\}$. We define an *ϵ-approximate Voronoi neighborhood* of p, denoted $AVN_\epsilon(p, S)$, to be a subset of S such that for any point q in $P(p, AVN_\epsilon(p, S))$, p is an ϵ-NN of q. We refer to the polyhedron defined by the bisectors between site p and its approximate Voronoi neighbors (i.e., $P(p, AVN_\epsilon(p, S))$) as an *ϵ-approximate Voronoi cell* of p. (See Fig. 1.) The goal is to find an approximate Voronoi cell with a small number of facets or, equivalently, to find a small set of approximate Voronoi neighbors.

1.2 Our results

Our main result is that there exists a set of ϵ-approximate Voronoi neighbors with cardinality $O(1/\sqrt{\epsilon})$ in the plane and cardinality $O((1/\epsilon)^{(d-1)/2} \log(1/\epsilon))$ in any fixed dimension d. We also show that, in the worst case, we need $\Omega(1/\epsilon^{(d-1)/2})$ neighbors in dimension d, which proves that our result is optimal in the plane and nearly optimal in higher dimension

Our existence proofs lend themselves to efficient constructions. We consider two scenarios depending on whether we need to compute approximate Voronoi neighborhoods for all the sites or for a single site. In the plane, our algorithm computes approximate Voronoi neighborhoods of size $O(1/\sqrt{\epsilon})$ for all the sites in time $O(n \log n)$ and for a single site in time $O(n + 1/\epsilon)$. In any fixed dimension d, our algorithm computes approximate Voronoi neighborhoods of size $O((1/\epsilon)^{(d-1)/2} \log(1/\epsilon))$ for all the sites in time $O(n \log n + n(1/\epsilon)^{3(d-1)/2} \log(1/\epsilon))$ and for a single site in time $O(n + (1/\epsilon)^{3(d-1)/2} \log(1/\epsilon))$.

1.3 Comparison with previous work

Perhaps the closest result to the work presented here was published by Clarkson [11]. He uses a different definition of approximate Voronoi cells that guarantees that for any site p and query point q, if p is not an ϵ-NN then one of its neighbors is closer to q by a factor $1 + \Omega(\epsilon)$ (in contrast, note that our definition only requires that if p is not an ϵ-NN then one of its neighbors be closer to q). He obtains a bound of $O((1/\epsilon)^{d/2}\log(\rho/\epsilon))$ on the number of facets, where ρ is the ratio between the diameter and the closest pair distance in the input set. Therefore, the bound on the number of facets in our case is smaller by a factor of $\Omega(1/\sqrt{\epsilon})$. In addition, we do not have a dependency on ρ, which is a significant improvement when ρ is large.

Recent work on approximate Voronoi diagrams(AVDs) [1, 2, 15] has focused on approximating the diagrams globally. The proposed structures form a partition of space into cells, where each cell c is associated with a site $p \in S$ such that, for any point in c, p is an ϵ-NN. A cell in this subdivision is either a hypercube or the difference of two hypercubes and a site may be associated with several cells.

Our approach is different in that we approximate Voronoi cells individually. In contrast to the best result on global AVDs that yields $O(n/\epsilon^{d-1})$ cells of constant complexity [2], we obtain n cells each with $O((1/\epsilon)^{(d-1)/2}\log(1/\epsilon))$ facets. A drawback of our work is that we do not obtain a space partition since the cells are not necessarily disjoint. (An interesting open question is whether we can obtain a convex partition having properties similar to ours.) On the upside, our work may be more suitable for applications that make use of the notion of Voronoi neighbors (for instance, natural neighbors interpolation [7, 17, 18]).

Both global AVDs and the concept of approximate Voronoi neighbors have proved useful in the design of approximate nearest neighbor algorithms. For global AVDs, the algorithm is based on descending a partition tree whose leaves correspond to the cells in the AVD. In contrast, in the new approach, the algorithm works by searching a hierarchy of graphs, whose bottom level consists of a graph in which each site is connected by edges to its approximate Voronoi neighbors [3]. Even though the known theoretical bounds using this method are inferior to those obtained using global AVDs, it is easy to implement and preliminary results indicate that its practical performance is very good.

Since Voronoi cells are convex, our work on approximating Voronoi cells is related to the well-studied problem of convex shape approximation. Clarkson [10] and Brönnimann and Goodrich [5] gave approximation algorithms for the problem of finding a convex polytope with the minimum number of facets that is contained between two given convex polytopes. Dudley [13] showed that, for any convex body of diameter at most unity, there exists a convex polytope with $O(1/\epsilon^{(d-1)/2})$ facets that is within Hausdorff distance ϵ of it. Har-Peled and Varadarajan [16] recently defined a new approximation method, which has proved very useful in many shape fitting problems: they require that the projection of the convex body and the approximating polytope onto any line be the same to within a factor of $1+\epsilon$. They showed how to adapt Dudley's technique to handle this case and obtain an approximating polytope with the same number of facets. Unfortunately, it seems that none of these techniques is applicable to our problem.

1.4 Overview

Our method in the plane is inspired by Dudley's approach for approximating convex shapes [13] (though our technical development is very different). We begin by briefly describing certain features of his approach that are relevant to us. Given a convex polytope P of diameter D, Dudley showed that it is possible to construct a polytope $P' \supseteq P$ consisting of $O(1/\epsilon^{(d-1)/2})$ facets such that the Hausdorff distance between P and P' is $O(D\epsilon)$. His method is based on sampling ∂P at $O(1/\epsilon^{(d-1)/2})$ points such that, for any point $q \in \partial P$, there is a sample point x satisfying (i) $|qx| = O(D\sqrt{\epsilon})$ (*distance criterion*), and (ii) the normals to ∂P at q and x make an angle of $O(\sqrt{\epsilon})$ (*normal angle criterion*). The approximating polytope P' is defined by the tangent planes at the sample points. Unfortunately, his method can sometimes yield a very poor approximation of a Voronoi cell, according to our definition. For example, if a site p is located arbitrarily close to the boundary of its Voronoi cell, then some of its Voronoi neighbors may actually be contained within the approximate cell, implying that the approximation error is infinity. (See Fig. 2.) Obviously, the reason why Dudley's method is unsuitable for our problem is that the sampling rule used is not sensitive to the location of the site inside the Voronoi cell. Intuitively, we need to sample boundary points more densely when we are close to site p. Indeed, we show that a simple modification to Dudley's sampling rule that incorporates this intuition performs very well. To be precise, we require that, for any point $q \in \partial P$, there is a sample point x satisfying the normal angle criterion given above and the following *polar angle criterion*: the angle $\angle qpx$ is $O(\sqrt{\epsilon})$. A fairly intricate geometric argument (Lemma 1 and Theorem 1) then establishes the correctness of our approach in the plane.

There seem to be several technical difficulties in generalizing the above method to dimension greater than two. Therefore, we devise a completely different strategy to attack the problem in higher dimensions. Let $p \in S$ be the site whose Voronoi cell we want to approximate. For any site $r \in S \setminus \{p\}$, let Σ_r be the sphere consisting of points x such that $|xp|/|xr| = 1+\epsilon$. Consider the cell in the arrangement of the spheres Σ_r that contains p and let Q denote the part of the boundary of this cell that is visible from p. Now, since we require that p be an ϵ-NN for all the points within the approximate Voronoi cell P', this means that Q must lie outside P'. In order to construct P' satisfying this property and having few facets, observe first that, for any point on Q, the distance to its nearest neighbor is less than its distance to p by a factor of $(1+\epsilon)$. Next suppose that, for some $\epsilon' < \epsilon$, we construct a set $R \subseteq S$ of *representatives* such that, for any point on Q, one of the sites in R is an ϵ'-NN . Then clearly Q would lie outside the cell P' defined by the bisectors between p and the sites in R (because for any point on Q, some site in R would be closer than p). Since the number of facets in P' is the same as the cardinality of R, the goal is to find a set R with small cardinality. Intuitively, the reason why one might hope that there exists a sparse set R is that there is an "empty region" surrounding Q (i.e., this region contains no sites of S). However, the separation between Q and the sites of S can be non-uniform (for example, when P is skinny or when p is close to the boundary of P), which makes it very difficult to find a suitably sparse R. We overcome this problem by partitioning Q carefully into smaller groups and finding representatives separately for each group. Finally, we succeed in constructing an approximate Voronoi cell with $O((1/\epsilon)^{(d-1)/2}\log(1/\epsilon))$ facets, which is suboptimal by a factor of at most $\log(1/\epsilon)$.

2 Conventions

Let a and b be two points in $\mathbb{R}^d$. We use $|ab|$ to denote the Euclidean distance between a and b, $\overline{ab}$ to denote the segment joining a and b, and $\overrightarrow{ab}$ to denote the vector from a to b. We let $B(a,r)$ denote the closed Euclidean ball of radius r centered at a and, for any ball B, we let $\overline{B}$ denote its complement, $\mathbb{R}^d \setminus B$. For any three points a, b and c in $\mathbb{R}^d$, we denote by $\triangle abc$ the triangle with vertices a,b and c, and we denote by $\angle abc$ the angle between $\overrightarrow{ba}$ and $\overrightarrow{bc}$. In Section 3, where we work in the plane, these angles will be directed and take value in $(-\pi, \pi]$. To be more precise, in Section 3, $\angle abc$ will be the signed angle of rotation that maps the ray ba into the ray bc. In the remainder of this article, we will work in arbitrary dimension d, so the angles won't be directed and will take value in $[0, \pi)$.

For any angle $\alpha \in [0, \pi/2]$, we let $C(ab, \alpha)$ denote the cone consisting of points x satisfying $\angle xab \leq \alpha$. We say that α is the *angular radius* of $C(ab, \alpha)$. Assuming a and b are distinct, we let $H(a,b)$ denote the half-space consisting of points that are at least as close to a as to b.

Let S be a set of n sites. Given a set $X \subseteq S$ of sites, and a point $q \in \mathbb{R}^d$, we define $n_q(X)$ to be a nearest neighbor of q in X and $d_q(X)$ to be the distance between q and $n_q(X)$. If there are no sites in X, then $n_q(X)$ is undefined and $d_q(X)$ is infinity. If $X = S$, we may replace $n_q(X)$ by n_q and $d_q(X)$ by d_q.

We will use the following well-known bounds on the trigonometric functions $\sin(\cdot)$ and $\cos(\cdot)$:

$$\forall t > 0, \;\; t - t^3/6 < \sin(t) < t \;\; \text{and} \;\; 1 - t^2/2 < \cos(t) \leq 1. \tag{1}$$

3 Approximate Voronoi Neighborhoods in the Plane

3.1 Size

In this section we show how to approximate a Voronoi cell in the plane. We obtain an approximate Voronoi neighborhood of size $O(1/\sqrt{\epsilon})$, which is optimal in the worst case. Our strategy is the following. The bisector between p and each site picked in the approximate Voronoi neighborhood is a line tangent to the exact Voronoi cell. We make sure that any two such lines that are consecutive along the boundary of the cell differ by $O(\sqrt{\epsilon})$ in terms of both the polar angle of the points where they are tangent and the angle that they make with each other. As we will show in Theorem 1, the result then follows by applying the following lemma (recall that angles are directed in this section).

Lemma 1 *Let p, b, q and q' be four points in $\mathbb{R}^2$. We assume that the triangle $\triangle bqq'$ is isoscele at b (that is, $|bq| = |bq'|$) and we denote $\theta' = \angle qbq'$. Let $\theta = \angle bpq$, $\alpha = \angle qbp$ and let δ be a real number in the interval $(0, \frac{1}{2}]$. (See Fig. 3.) We assume that θ and θ' are in $[0, \delta]$ and $\alpha \in [\delta, \pi]$. Then*

$$1 \leq \frac{|pq|}{|pq'|} \leq 1 + \frac{5}{2}\delta^2 + O(\delta^4).$$

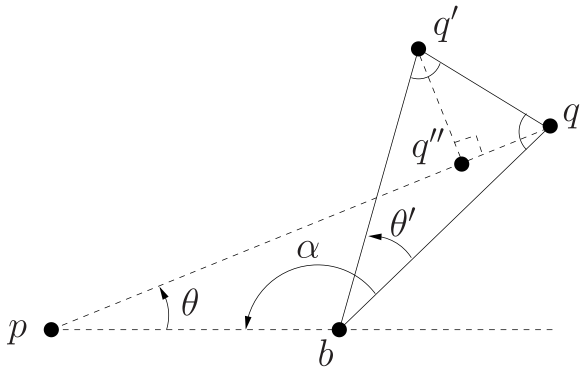

Figure 3: (Lemma 1) $\theta \leq \delta$, $\theta' \leq \delta$ and $\alpha \geq \delta$. Then $|pq|$ is an $O(\delta^2)$ approximation of $|pq'|$.

Proof bp makes an angle $\alpha - \theta'/2 \in (0, \pi]$ with the bisector of q and q', so p and q' are on the same side of this bisector and q is on the opposite side. Thus $|pq| \geq |pq'|$, which proves the left hand side of the inequality. We now prove the upper bound on $|pq|/|pq'|$.

Note that $\theta + \alpha + \angle pqb = \pi$, hence $\alpha \leq \pi - \theta$. It yields $\theta \leq \delta \leq \alpha \leq \pi - \theta$, and thus

$$\sin(\theta) \leq \sin(\alpha). \tag{2}$$

Applying the law of sines to $\triangle bqq'$ and $\triangle pbq$, we obtain

$$\frac{|qq'|}{|pq|} = \frac{|qq'|}{|bq|}\frac{|bq|}{|pq|} = \frac{\sin(\theta')\sin(\theta)}{\sin(\pi/2-\theta'/2)\sin(\alpha)} = \frac{\sin(\theta')\sin(\theta)}{\cos(\theta'/2)\sin(\alpha)}.$$

Note that $\sin(\theta') \leq \theta' \leq \delta$ and

$$\frac{1}{\cos(\theta'/2)} \leq \frac{1}{\cos(\delta/2)} = 1 + O(\delta^2).$$

Thus

$$\frac{|qq'|}{|pq|} \leq \frac{\sin(\theta)}{\sin(\alpha)}\delta(1+O(\delta^2)). \tag{3}$$

Let q'' denote the orthogonal projection of q' onto pq. (See Fig. 3.) We now estimate $|pq''|$. Note that $\angle pqb = \pi - \alpha - \theta$, hence $\angle q'qp = \pi/2 - \theta'/2 - (\pi - \alpha - \theta)$. Therefore

$$\cos(\angle q'qp) = \sin(\alpha + \theta - \theta'/2).$$

Hence

$$|pq''| = |pq| - |qq'|\cos(\angle q'qp) = |pq| - |qq'|\sin(\alpha+\theta-\theta'/2),$$

so

$$\frac{|pq''|}{|pq|} = 1 - \frac{|qq'|}{|pq|}\sin(\alpha+\theta-\theta'/2).$$

Combining with Equation 3, we get

$$\frac{|pq''|}{|pq|} \geq 1 - \frac{\sin(\theta)\sin(\alpha+\theta-\theta'/2)}{\sin(\alpha)}\delta(1+O(\delta^2)). \tag{4}$$

We now study the quantity $f(\alpha,\theta,\theta') = \sin(\theta)\sin(\alpha+\theta-\theta'/2)/\sin(\alpha)$. We distinguish between two cases according to the value of α.

- First case: $\alpha \geq \pi - \delta$. By Equation 2 we have $\sin(\theta)/\sin(\alpha) \leq 1$. Since $\{\pi-\alpha,\theta,\theta'\} \subset [0,\delta]$, we have $\pi - \alpha - \theta + \theta'/2 \leq 3\delta/2$. So by Equation 1 we get $\sin(\pi-\alpha-\theta+\theta'/2) \leq 3\delta/2$. Since $\sin(\pi-\alpha-\theta+\theta'/2) = \sin(\alpha+\theta-\theta'/2)$, it follows that $f(\alpha,\theta,\theta') \leq 3\delta/2$.
- Second case: $\delta \leq \alpha \leq \pi - \delta$. Then we have $\sin(\alpha) \geq \sin(\delta) = \delta(1+O(\delta^2))$. Moreover, $\sin(\cdot)$ is 1-Lipschitz, so

  $$\sin(\alpha+\theta-\theta'/2) \leq \sin(\alpha) + \theta + \theta'/2 \leq \sin(\alpha) + 3\delta/2.$$

  Using the fact that $\sin(\theta) \leq \delta$, we get

  $$f(\alpha,\theta,\theta') \leq \delta\left(1 + \frac{3\delta}{2\sin(\alpha)}\right) \leq \delta\left(1+\frac{3\delta}{2\sin(\delta)}\right) = \frac{5}{2}\delta(1+O(\delta^2)).$$

Thus, in both these cases, $f(\alpha,\theta,\theta') \leq 5\delta/2 + O(\delta^3)$. By Equation 4, we obtain

$$\frac{|pq''|}{|pq|} \geq 1 - \frac{5}{2}\delta^2(1+O(\delta^2)).$$

Therefore

$$\frac{|pq|}{|pq'|} \le \frac{|pq|}{|pq''|} \le 1 + \frac{5}{2}\delta^2 + O(\delta^4).$$

$\square$

Theorem 1 *Let $S \subset \mathbb{R}^2$ be a set of n sites, p be any site in S, and $\epsilon > 0$ be a real parameter. There exists a subset $AVN_\epsilon(p, S)$ of size $O(1/\sqrt{\epsilon})$.*

Proof We associate $\mathbb{R}^2$ with a polar coordinate system centered at p. For any point $m \in \mathbb{R}^2$, we denote by $r(m)$ (resp. $\theta(m)$) its distance to p (resp. polar angle). We first assume that $P(p, S)$ is bounded, hence it is a convex polygon containing p in its interior.

Let $S' \subset S$ be the set of all the sites s' such that the bisector of $\overline{ps'}$ contains an edge of $P(p, S)$; we denote this edge by $e(s')$. Edges are oriented counterclockwise around $P(p, S)$; with this convention we denote $e(s') = \overline{a(s')b(s')}$. (See Fig. 4.) We will say that two elements of S' are consecutive if they are

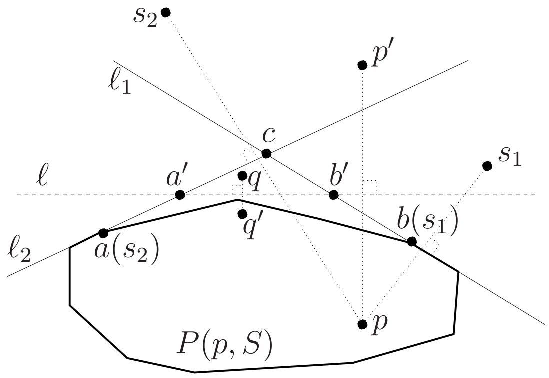

Figure 4: Proof of Theorem 1.

consecutive in counterclockwise order.

Let δ be a real number in $(0, 1/2)$. We choose $\delta = \Theta(\sqrt{\epsilon})$ such that we can apply Lemma 1 appropriately; in other words, with all the hypothesis of Lemma 1 met, we have that $|pq| \le (1 + \epsilon)|pq'|$.

We construct a subset $S_1 \subseteq S'$ in the following way. For all integers $k \in [0, 8\pi/\delta]$, if there is exactly one site $s' \in S$ such that $k\delta/4 \le \theta(s') < (k + 1)\delta/4$, then $s' \in S_1$. If there are two or more such sites, then we choose the one with smallest polar angle and the one with largest polar angle to be in S_1 (and no other site from this polar angle interval). Clearly the cardinality of S_1 is $O(1/\delta)$. The set S_1 has the following property that can be inferred directly from its construction: for any two consecutive sites in S_1, their polar angles differ by less than $\delta/2$ or they are also consecutive in S'.

We now construct another subset $S_2 \subseteq S'$. For all integers $k \in [0, 2\pi/\delta]$, if there is only one site $s' \in S'$ such that $k\delta \le \theta(a(s')) < (k + 1)\delta$, then we choose this site to be in S_2. If there are two or more such sites then we choose the one with smallest polar angle and the one with largest polar angle. Clearly the cardinality of S_2 is $O(1/\delta)$. Observe that for any two consecutive sites s_2 and s_2' in S_2, $\theta(a(s_2')) - \theta(b(s_2)) \le \delta$.

We combine the sets S_1 and S_2 into a set $R = S_1 \cup S_2$. It enjoys the properties of both:

1. R has cardinality $O(1/\delta) = O(1/\sqrt{\epsilon})$.

2. For any two consecutive sites in R, their polar angles differ by less than $\delta/2$ or they are also consecutive in S'.

3. For any two consecutive sites s and s' in R, we have $\theta(a(s')) - \theta(b(s)) \leq \delta$.

Now we prove that R is an $AVN_\epsilon(p, S)$. That is, for any point q in $P(p, R) \setminus P(p, S)$, we need to prove that p is an ϵ-NN of q. We prove this by contradiction, so assume that there is a site $p' \in S$ such that $|pq| > (1+\epsilon)|p'q|$. By construction, q lies in an area delimited by $P(p, S)$ and the bisectors of two segments $\overline{ps_1}$ and $\overline{ps_2}$ where s_1 and s_2 are two consecutive sites in R. Let ℓ_1 and ℓ_2 be these two bisectors, and let $c = \ell_1 \cap \ell_2$. (See Fig. 4.) Since the area delimited by ℓ_1, ℓ_2 and $P(p, S)$ is not empty, s_1 and s_2 cannot be consecutive in S'. It implies that $\theta(s_2) - \theta(s_1) \leq \delta/2$.

Let ℓ be the bisector of $\overline{pp'}$. Note that it cannot cross the interior of $P(p, S)$ and must cross $\overline{pq}$. It follows that ℓ intersects $\overline{a(s_2)c}$ and $\overline{b(s_1)c}$; we denote by a' and b' these two intersection points. Without loss of generality, we assume that $|pb'| \leq |pa'|$. The angle between ℓ_1 and ℓ is at most equal to the angle between ℓ_1 and ℓ_2, which is equal to $\theta(s_2) - \theta(s_1) \leq \delta/2$. In other words, $\angle cb'a' \leq \delta/2$. We denote by q' the point symmetric to q with respect to ℓ, so $|pq'| = |p'q|$. The angle $\angle qb'a'$ is at most $\delta/2$, thus $\angle qb'q' \leq \delta$. The polar angles of $a(s_2)$ and $b(s_1)$ differ by at most δ, which implies $\angle b'pq \leq \delta$. We assumed $|pb'| \leq |pa'|$ so $\angle a'b'p \geq (\pi - \angle b'pa')/2$ (there is equality when $\triangle a'b'p$ is isoscele at p). It follows that

$$\angle qb'p \geq \angle a'b'p \geq \frac{\pi - \angle b'pa'}{2} \geq \frac{\pi - \delta}{2} \geq \frac{\pi - 1/2}{2} \geq 1 \geq \delta.$$

Therefore, by Lemma 1, we have $|pq| \leq (1+\epsilon)|pq'| = (1+\epsilon)|p'q|$, a contradiction.

This proof can be easily adapted to handle the case where $P(p, S)$ is unbounded. In this case, there are two infinite edges on $P(p, S)$. The two corresponding sites are chosen to be in R, the other sites in R are chosen in the same way as in the bounded case. □

3.2 Construction time

Using the ideas in Theorem 1, it is easy to design efficient algorithms for computing approximate Voronoi cells in the plane. Our first algorithm computes approximate Voronoi cells for all the sites. Its running time is $O(n \log n)$, which is not better than the computation of an exact Voronoi diagram, but it has the advantage that the complexity of each cell is independent of n. Our second algorithm computes a single cell in time $O(n+1/\epsilon)$. This is linear in n, which is interesting in the sense that a single cell of an exact Voronoi diagram cannot be computed in less than $\Omega(n \log n)$ time.

Theorem 2 *Let S be a set of n sites in $\mathbb{R}^2$. We can compute ϵ-approximate Voronoi cells of size $O(1/\sqrt{\epsilon})$ for all the sites in S in time $O(n \log n)$. Alternatively, for a single site $p \in S$, we can compute an ϵ-approximate Voronoi cell of size $O(1/\sqrt{\epsilon})$ in time $O(n + 1/\epsilon)$.*

Proof The first algorithm is standard. We first compute the Voronoi diagram of S, which can be done in $O(n \log n)$ time. Then we implement the method described in the proof of Theorem 1 in $O(n)$ time by a linear scan of the edges of each cell in counterclockwise order.

If we are only interested in one Voronoi cell (say the Voronoi cell of a site $p \in S$), we can improve the time bound to $O(n + 1/\epsilon)$. The idea is a simplified version of the algorithm in higher dimension, so we only sketch the approach here. (See Theorem 4 for more details.) We first pick a subset of $S \setminus \{p\}$ with cardinality $O(1/\epsilon)$ using a cone based decomposition; this can be done in time $O(n + 1/\epsilon)$ using the floor function. The corresponding cell is then approximated using the same method as in the first algorithm. □

4 Approximate Voronoi Neighborhoods in Higher Dimensions

4.1 Size

In this section we prove the following result.

Theorem 3 *Let S be a set of n sites in $\mathbb{R}^d$, p be any site in S, and $\epsilon > 0$ be a real parameter. There exists a subset $AVN_\epsilon(p, S)$ of size $O((1/\epsilon)^{(d-1)/2} \log(1/\epsilon))$.*

Briefly, our proof works as follows. We first define a surface Q that encloses the Voronoi cell P of site p. Intuitively, Q delimits the region where an approximate Voronoi cell can possibly lie. Our goal is thus to fit a polyhedron between the Voronoi cell and Q. In other words, the bisector corresponding to each site we pick for our approximate Voronoi neighborhood cuts off part of Q; we will choose a subset of sites such that Q is completely cut off from the Voronoi cell. Our method will work in several rounds where, in each round, we cut off a layer Q_i of Q (to be defined later). We handle Q_i by subdividing it according to cones with apex p and handling each cone separately.

4.1.1 Preliminaries

Before presenting the proof of Theorem 3, we need to introduce some technical lemmas that will be useful to us. We first state a lemma proved by Arya and Malamatos [1]. Given two concentric balls, it provides a bound on the number of nearest-neighbor representatives needed so that given a query point in the inner ball its approximate nearest neighbor outside the outer ball will be one of these representatives. The lemma indicates how the number of representatives needed varies as a function of the separation between the two balls. Recall that, for any set $X \subseteq S$ of sites, $d_q(X)$ is the distance between a point q and its nearest neighbor in X.

Lemma 2 *Let B_1 and B_2 be two concentric balls of radius r and γr, respectively, where $\gamma \geq 2$. There exists a set $R \subseteq S$ consisting of $O\left(\left(1 + \frac{1}{\sqrt{\epsilon\gamma}}\right)^{d-1}\right)$ points such that for any point $q \in B_1$, $d_q(R) \leq (1+\epsilon) d_q(S \cap \overline{B_2})$. Moreover, R can be computed in time $O\left(\left(1 + \frac{1}{\sqrt{\epsilon\gamma}}\right)^{d-1} |S|\right)$.*

In addition, we will need Lemma 4 that bounds the number of representatives in a different geometric context. Its proof requires the following lemma.

Lemma 3 (Chan and Snoeyink [9]) *Let $\triangle xyz$ be a triangle with $\angle yzx = \theta, \angle zxy = \phi$, and $\angle xyz \geq \pi/2$. Then*

$$|xy| + |yz| \leq (1 + \sin\theta \sin\phi)|xz|.$$

Lemma 4 *Let C be a cone of angular radius α with apex at a point p. Let B_1 and B_2 be two balls of radius r_1 and r_2, respectively, centered at p, such that $\alpha r_2 \le r_1 \le r_2/4$. Then there exists a set $R \subseteq S \cap B_1$ consisting of $O\left(\left(1+\frac{r_1}{r_2\sqrt{\epsilon}}\right)^{d-1}\right)$ points such that for any point q in the closure of $C \cap \overline{B_2}$, $d_q(R) \le (1+\epsilon)\cdot d_q(S\cap B_1)$. Moreover, R can be computed in time $O\left(\left(1+\frac{r_1}{r_2\sqrt{\epsilon}}\right)^{d-1}|S|\right)$.*

Proof In order to describe the set R, we need some definitions. Let a denote the point of intersection of the axis of cone C with the ball B_2. Let o denote the point on the line pa such that the tangent from o to the boundary of ball B_1 makes an angle of α with $\overrightarrow{oa}$. (See Fig. 5.) Let $C' = C(oa, \alpha)$, and let h be the

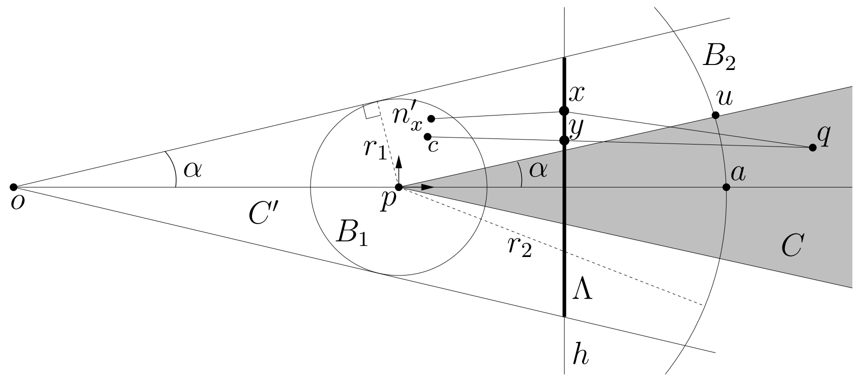

Figure 5: Proof of Lemma 4.

hyperplane orthogonal to $\overline{oa}$ at distance $|op|+r_2/2$ from o (towards a). Let Λ denote the $(d-1)$-dimensional ball $C' \cap h$.

Let r_Λ denote the radius of Λ and let $\delta = r_2\sqrt{\epsilon}/4$. We construct a set $R' \subset \Lambda$ that is δ-dense for Λ; that is, for all points $x \in \Lambda$, there is a point $y \in R'$ such that $|xy| \le \delta$. By a straightforward packing argument, we can construct such a set R' with cardinality $O((1 + r_\Lambda/(r_2\sqrt{\epsilon}))^{d-1})$. Finally, we set $R = \{n'_x : x \in R'\}$, where n'_x denotes the nearest point to x in $S \cap B_1$. Clearly, R can be computed in time $O(|S|\cdot|R'|)$ and its size is at most $|R'|$.

To bound r_Λ observe that, by elementary trigonometry, $r_\Lambda = (|op| + r_2/2)\tan\alpha$ and $|op| = r_1/\sin\alpha$. Thus $r_\Lambda = r_1/\cos\alpha + (r_2/2)\tan\alpha$. By the statement of the lemma, α cannot exceed $1/4$, so by Equation 1 we obtain $r_\Lambda \le 32(r_1 + r_2\alpha/2)/31$. Noting that $\alpha r_2 \le r_1$, it follows that $r_\Lambda \le 96r_1/62$, which yields the desired bound on $|R|$ and on the time for computing it.

It remains to show that R satisfies the property given in the statement of the lemma. Let q be any point in $C \cap \overline{B_2}$ and let c be its nearest neighbor among the points of $S \cap B_1$. It is easy to see that segment $\overline{qc}$ must intersect Λ. Let y denote this point of intersection. Since R' is δ-dense for Λ, there must be a point $x \in R'$ such that $|xy| \le \delta$. We will show that $|qn'_x| \le (1+\epsilon)|qc|$.

First we prove that $|qy| > r_2/4$. We associate a coordinate frame with the 2-flat through p, q and a, with the origin at p and with the x-axis directed towards a. (If p, q and a are collinear, take any 2-flat through these points.) Consider u, one of the two points on this flat that are contained in $C \cap \overline{B_2}$ and that have the smallest x-coordinate. (See Fig. 5.) The x-coordinate of u is $r_2\cos(\alpha)$, and so $|qy| \ge |x_q - x_y| \ge |x_u - x_y| = r_2\cos(\alpha) - r_2/2$. Using Equation 1 and the fact that $\alpha \le 1/4$, we conclude that $|qy| \ge 31r_2/32 - r_2/2 >$

$r_2/4$.

By the triangle inequality, we have $|qn'_x| \le |qx| + |xn'_x|$. Since n'_x is the nearest neighbor of x in $S \cap B_1$, we have $|xn'_x| \le |cx|$. Thus $|qn'_x| \le |qx| + |cx|$. In $\triangle qcx$, let θ denote $\angle xqc$ and ϕ denote $\angle qcx$. Note that $|qy| \ge r_2/4, |cy| \ge r_2/4$, and $|xy| \le r_2\sqrt{\epsilon}/4$. Applying the law of sines to $\triangle qxy$ and $\triangle cxy$, respectively, we obtain $\sin\theta/|xy| \le 1/|qy|$ and $\sin\phi/|xy| \le 1/|cy|$. Simplifying we get $\sin\theta \le \sqrt{\epsilon}, \sin\phi \le \sqrt{\epsilon}$, and $\angle cxq \ge \pi/2$ (assuming $\epsilon \le 1/2$). Applying Lemma 3, we now obtain $|qx| + |cx| \le (1+\epsilon)|qc|$. Thus $|qn'_x| \le (1+\epsilon)|qc|$, as desired. □

4.1.2 Proof of Theorem 3

We now describe our method for finding the ϵ-approximate Voronoi neighbors of site $p \in S$. Throughout this section we assume that $0 < \epsilon < 1/2$. Let Q denote the set of points x with the following property: starting from p, imagine moving along the infinite ray that passes through x; then p is an ϵ-NN for all the points until we reach x, and fails to be an ϵ-NN infinitesimally beyond x. Note that for any point $x \in Q$, its nearest neighbor n_x satisfies $|xp|/|xn_x| = 1+\epsilon$.

We define a function $\lambda(u) : Q \to [0, \pi/2], \lambda(u) = \pi/2 - \angle n_u p u$, where n_u denotes a nearest neighbor of u. (See Fig. 6.) The following lemma gives a useful lower bound on $\lambda(u)$.

Lemma 5 *For all $u \in Q$ we have $\lambda(u) > \sqrt{\epsilon}$.*

Proof Consider the 2-flat through p, u and n_u. (In case these three points are collinear, it is easy to see that $\lambda(u) = \pi/2$.) Let C be the circle that consists of all the points x on this 2-flat such that $|up|/|ux| = 1+\epsilon$. (See Fig. 6.) Let ℓ denote the line on this flat that contains p and is orthogonal to pu. The point n_u lies

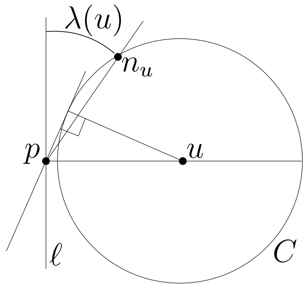

Figure 6: Proof of Lemma 5.

on C, hence $\lambda(u)$ is at least as large as the angle between ℓ and the tangents to C through p. Therefore $\cos(\lambda(u)) \le 1/(1+\epsilon)$. By Equation 1 we obtain $1 - (\lambda(u))^2/2 < 1/(1+\epsilon)$. Since $\epsilon < 1$, it follows immediately that $\lambda(u) > \sqrt{2\epsilon/(1+\epsilon)} \ge \sqrt{\epsilon}$. □

We partition the points in Q into $K+1$ groups based on their λ-value. Let $\lambda_0 = \sqrt{\epsilon}$ and, for $1 \le i \le K$, let $\lambda_i = 2^i\lambda_0$. For $1 \le i \le K$, the ith group, denoted Q_i, contains all points $x \in Q$ such that $\lambda(x)$ lies in the interval $(\lambda_{i-1}, \lambda_i]$. We choose K such that $1/20 \le \lambda_K < 1/10$. We set λ_{K+1} to $\pi/2$. The final group Q_{K+1} contains all points $x \in Q$ such that $\lambda(x) \in (\lambda_K, \lambda_{K+1}]$. Note that the number of groups is $K+1 = O(\log(1/\epsilon))$.

The method works in $K+1$ rounds. In the ith round, we find a set of neighbors $R_i \subseteq S$ such that (i) $|R_i| = O((1/\epsilon)^{(d-1)/2})$ and (ii) all the points in Q_i are outside the cell defined by p and the neighbors in R_i (in other words, the points in Q_i are outside the region $\cap_{r \in R_i} H(p,r)$). It is clear that $R = \cup_i R_i$ is an ϵ-approximate Voronoi neighborhood of p with cardinality $O((1/\epsilon)^{(d-1)/2} \log(1/\epsilon))$.

For $1 \leq i \leq K+1$, let $\mathcal{C}_i$ be a set of cones covering $\mathbb{R}^d$, such that each cone has its apex at p and angular radius at most $\alpha_i = \lambda_{i-1}/16$. Yao [19] has given a method for constructing such a set of cones having cardinality $O(1/\alpha_i^{d-1})$. We will show that, for any cone $C \in \mathcal{C}_i$, we can find a set of neighbors $R_{i,C} \subseteq S$, such that (i) $|R_{i,C}| = O((\alpha_i/\sqrt{\epsilon})^{d-1})$ and (ii) all the points in $Q_i \cap C$ lie outside the region $\cap_{r \in R_{i,C}} H(p,r)$. We form the set R_i by taking the union of the neighbor sets $R_{i,C}$ over all the cones $C \in \mathcal{C}_i$. Since the number of cones in $\mathcal{C}_i$ is $O(1/\alpha_i^{d-1})$, it follows that $|R_i| = O((1/\epsilon)^{(d-1)/2})$, as desired.

It remains only to show the existence of $R_{i,C}$ satisfying properties (i) and (ii) given above. We focus on the non-trivial case when $Q_i \cap C \neq \emptyset$ (otherwise we can set $R_{i,C} = \emptyset$). Our strategy for choosing $R_{i,C}$ relies on the following lemma.

Lemma 6 *Let ϵ' be any real number less than ϵ. Let C be any cone in $\mathcal{C}_i$ and let t be any site in $S \setminus \{p\}$. Let $A = Q_i \cap C \cap H(p,t)$. Let $R \subseteq S$ be a set of sites such that, for any point $u \in A$, there is a site in R that is an ϵ'-NN of u. Then the set $R_{i,C} = R \cup \{t\} \setminus \{p\}$ satisfies property (ii).*

Proof In other words, we need to show that, for any point $x \in Q_i \cap C$, $x \notin \cap_{r \in R_{i,C}} H(p,r)$. If x is outside $H(p,t)$ then, since $t \in R_{i,C}$, the claim is obvious. So suppose that $x \in H(p,t)$. It then follows from the statement of the lemma that there is a site $r' \in R$ such that $|xr'|/|xn_x| \leq 1+\epsilon' < 1+\epsilon$. Further, since $x \in Q$, its nearest neighbor n_x satisfies $|xp|/|xn_x| = 1+\epsilon$. Thus $|xr'| < |xp|$. That is, $x \notin H(p,r')$ and so $x \notin \cap_{r \in R_{i,C}} H(p,r)$. $\square$

Before continuing further, we need to introduce some more notation. Let $S' = S \setminus \{p\}$. For each site $r \in S'$, let Σ'_r denote the portion of the sphere $\Sigma_r = \{x : |xp|/|xr| = 1+\epsilon\}$ that is visible from p (i.e., the set of points $y \in \Sigma_r$ such that the open segment $\overline{py}$ does not intersect Σ_r). Consider the cell in the arrangement of the spheres $\Sigma_r, r \in S'$, that contains p. It is easy to see that Q represents the boundary of this cell that is visible from p. Clearly Q is a subset of $Q' = \cup_{r \in S'} \Sigma'_r$.

Next, with each site $r \in S'$, associate a function $\lambda_r : \Sigma'_r \to [0, \pi/2], \lambda_r(u) = \pi/2 - \angle rpu$. For $1 \leq i \leq K+1$, let Q'_i be the set of points $x \in Q'$ such that for some $r \in S'$, $\lambda_r(x) \in (\lambda_{i-1}, \lambda_i]$. (Note that a point may belong to more than one set Q'_i.) It is easy to see that $Q_i \subseteq Q'_i$ because, for any point $u \in Q, \lambda(u) = \lambda_{n_u}(u)$.

Let q be a point in $Q'_i \cap C$ that is closest to p, let t be a site that ensures $\lambda_t(q) \in (\lambda_{i-1}, \lambda_i]$, and let $\epsilon' = 9\epsilon/10$. We will show that there exists a set R of small cardinality satisfying the condition given in the statement of Lemma 6. To this end, in the following lemma, we first establish certain properties that hold for the points in set $A = Q_i \cap C \cap H(p,t)$. These properties will enable us to find a small set R by applying either Lemma 2 or Lemma 4.

Lemma 7 *Let C be any cone in $\mathcal{C}_i$. Let q, t, and ϵ' be as defined above. Let u be any point in $A = Q_i \cap C \cap H(p,t)$. Then* *(i)* $|pu| \geq |pq|$,

(ii) $|pu| < 7|pq|/6$, *and*

(iii) *if* $1 \leq i \leq K$, *then* $|pn_u| < 75\alpha_i|pq|$.

Proof Since $Q_i \subseteq Q'_i$, (i) follows from our choice of q. To prove (ii), let P be the 2-flat through p, q and u. (If p, q and u are collinear, let P be any 2-flat through these points.) Let ℓ_1 and ℓ_2 be the infinite rays from p that delimit the region $C \cap P$. Let ℓ be the intersection of the boundary of $H(p,t)$ with P. (See Fig. 7.) Since

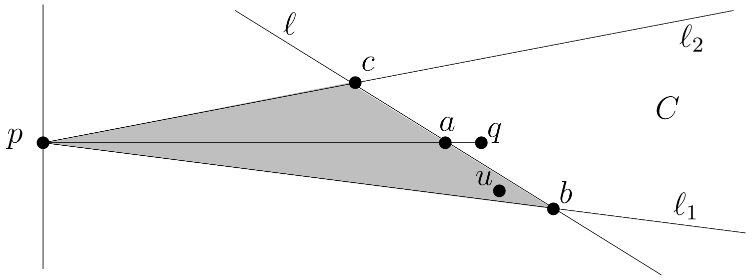

Figure 7: Proof of Lemma 7 (i). The shaded region is $C \cap H(p,t) \cap P$.

$q \in \Sigma'_t$, it follows that $|tq| < |pq|$ and thus ℓ intersects the interior of $\overline{pq}$; we let a denote this intersection point.

Note that the smaller angle that ℓ makes with pq must exceed λ_{i-1} (because $\lambda_t(q) > \lambda_{i-1}$). Since λ_{i-1} exceeds $2\alpha_i$, the angular diameter of the cone, ℓ must intersect both ℓ_1 and ℓ_2. Let b denote the point on $\ell \cap C$ that is farthest from p; without loss of generality we assume that $b = \ell_1 \cap \ell$, and we denote $c = \ell_2 \cap \ell$. In $\triangle pbc$, we know that $|pc| \leq |pb|$, so $\angle abp \leq \angle acp$. Thus $\angle abp \leq \pi/2$. Also, $\angle abp = \angle baq - \angle bpa \geq \lambda_{i-1} - 2\alpha_i$. Since $\alpha_i = \lambda_{i-1}/16$, we get $\angle abp \geq 7\lambda_{i-1}/8$. It now follows from the law of sines that

$$\frac{|ab|}{|ap|} = \frac{\sin(\angle bpa)}{\sin(\angle abp)} \leq \frac{\sin(2\alpha_i)}{\sin(7\lambda_{i-1}/8)} = \frac{\sin(\lambda_{i-1}/8)}{\sin(7\lambda_{i-1}/8)}.$$

Using Equation 1 and the fact that $\lambda_{i-1} < 1/10$ we obtain that $|ab|/|ap| < 1/6$. By the triangle inequality, $|pb|/|pa| \leq (|pa| + |ab|)/|pa| < 7/6$. Since $|pu| \leq |pb|$ and $|pq| > |pa|$, (i) follows.

To prove (iii), let m be the midpoint of $\overline{pn_u}$ and let u' be the intersection of the bisector of $\overline{pn_u}$ with the line pu. Since u is closer to n_u than to p, we know that u' is in the interior of $\overline{pu}$. Therefore $|pu'| < |pu|$. Note that $\angle pu'm = \lambda(u)$. (See Fig. 8.) So $|pm| = |pu'| \sin(\lambda(u))$. Since $\lambda(u) \leq \lambda_i = 2\lambda_{i-1} = 32\alpha_i$ we obtain $|pm| \leq 32\alpha_i|pu'| < 32\alpha_i|pu|$. We conclude using the facts that $|pm| = |pn_u|/2$ and $|pu| < 7|pq|/6$. $\square$

Let q, t and A be as in Lemma 7. We distinguish between two cases: (i) $1 \leq i \leq K$ and (ii) $i = K+1$. In Case (i), Lemma 7(iii) implies that the nearest neighbor of any point in A is contained in a ball of radius $75\alpha_i|pq|$ centered at p. Since $\lambda_{i-1} < 1/20$, $\alpha_i < 1/320$, which yields $75\alpha_i|pq| < |pq|/4$. Also, by Lemma 7(i), any point in A is at distance at least $|pq|$ from p. So we can apply Lemma 4 with $75\alpha_i|pq|$ playing the role of r_1 and $|pq|$ playing the role of r_2. It follows that we can find a set R of $O((\alpha_i/\sqrt{\epsilon})^{d-1})$ representatives such that, for any point in A, one of these representatives is an ϵ'-NN. In Case (ii), recall that

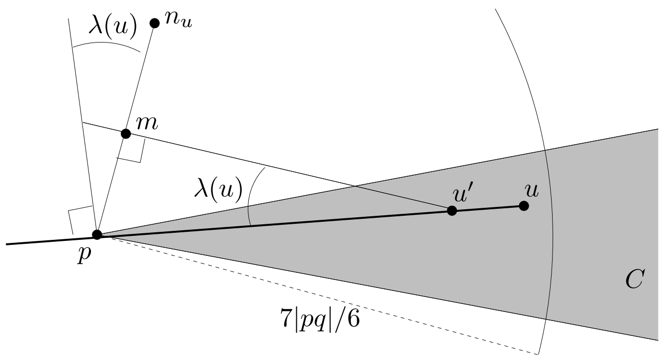

Figure 8: Proof of Lemma 7 (ii) .

the ball $\hat{B}_q$ centered at q and having radius $|pq|/(1+\epsilon)$ is free of any sites in S. Also, applying Lemma 7(i) and (ii), it is easy to see that the distance of any point in A from q is less than $(1/6 + (7/6)(2\alpha_{K+1}))|pq|$. Since $\alpha_{K+1} < 1/160$ and $\epsilon < 1/2$, it follows that A is contained in the ball centered at q and having half the radius of $\hat{B}_q$. Setting $\gamma = 2$ and applying Lemma 2, it now follows that we can find a set R of $O((1/\sqrt{\epsilon})^{d-1}) = O((\alpha_{K+1}/\sqrt{\epsilon})^{d-1})$ representatives such that, for any point in A, one of these representatives is an ϵ'-NN.

In both cases, we let $R_{i,C} = R \cup \{t\} \setminus \{p\}$. It follows from Lemma 6 that the points in $Q_i \cap C$ lie outside the region $\cap_{r \in R_{i,C}} H(p, r)$. Also, the cardinality of $R_{i,C}$ is $O((\alpha_i/\sqrt{\epsilon})^{d-1})$, as desired. This completes the proof.

4.2 Construction time

By building on the ideas contained in the proof of Theorem 3, we can efficiently construct approximate Voronoi neighborhoods in any fixed dimension. We have the following result, analogous to Theorem 2 in the plane.

Theorem 4 *Let S be a set of n sites in $\mathbb{R}^d$. We can compute ϵ-approximate Voronoi neighborhoods of size $O((1/\epsilon)^{(d-1)/2} \log(1/\epsilon))$ for all the sites in S in time $O(n \log n + n(1/\epsilon)^{3(d-1)/2} \log(1/\epsilon))$. Alternatively, for a single site $p \in S$, we can compute an ϵ-approximate Voronoi neighborhood of size $O((1/\epsilon)^{(d-1)/2} \log(1/\epsilon))$ in time $O(n + (1/\epsilon)^{3(d-1)/2} \log(1/\epsilon))$.*

We first consider the case when we need to construct approximate Voronoi neighborhoods for all the sites. Our construction utilizes the notion of approximate neighborhood as defined by Funke and Ramos [14]. Given a real parameter $\delta > 0$, they define a δ-*approximate neighborhood* of $p \in S$, denoted $AN_\delta(p, S)$, to be a subset of S such that, for any site $r \in S \setminus \{p\}$, there is a site $r' \in AN_\delta(p, S)$ satisfying $\angle rpr' \leq \delta$ and $|pr'| \leq |pr|/(1 - \sin \delta)$. Assuming the use of the floor function, their work implies the following lemma (without the floor function, the second term in the construction time goes up by a factor of $\log(1/\delta)$).

Lemma 8 *(Funke and Ramos [14], Callahan and Kosaraju [8]) It is possible to compute δ-approximate neighborhoods of size $O(1/\delta^{d-1})$ for all the sites in S in time $O(n \log n + n(1/\delta^d))$.*

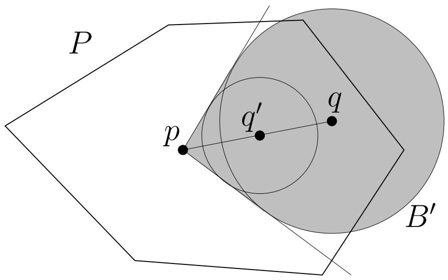

Figure 9: Proof of Lemma 9. X is the shaded region.

Our strategy is to first set $\delta = \epsilon/16$ and use the above lemma to find a set of $O(1/\epsilon^{d-1})$ neighbors for each site in S. This step takes time $O(n \log n + n(1/\epsilon^d))$. As we will show, for each site a small subset of the neighbors computed can work as its ϵ-approximate Voronoi neighbors. This idea is made precise in the following lemma.

Lemma 9 *Let $0 < \epsilon < 1, \delta_1 = \epsilon/16$, and $\delta_2 = \epsilon/4$. Let p be a site in S, $R_p = AN_{\delta_1}(p, S)$, and $N_p = AVN_{\delta_2}(p, R_p)$. Then N_p is an ϵ-approximate Voronoi neighborhood of p with respect to S.*

Proof Let P denote the polyhedron $P(p, N_p)$ and let q be any point in P. Since P is convex, any point q' on $\overline{pq}$ lies within P and so, by definition of P and N_p, p is a δ_2-NN for all $q' \in \overline{pq}$ with respect to R_p. In other words, the interior of $X = \cup_{q' \in \overline{pq}} B(q', |pq'|/(1 + \delta_2))$ contains no site of R_p. Letting $B' = B(q, |pq|/(1 + \delta_2))$, observe that X is the union of B' and Y, where Y is the region delimited by the rays from p that are tangential to B' and the boundary of B' that is visible from p. (See Fig. 9.)

We will now show that p is an ϵ-NN for q with respect to S, which will prove the lemma. For the sake of contradiction, suppose there is a site $r \in S$ that is contained in the interior of the ball $B'' = B(q, |pq|/(1+\epsilon))$. By definition of R_p, there must be a site $r' \in R_p$ such that $\angle rpr' \leq \delta_1$ and $|pr'| \leq |pr|/(1 - \sin \delta_1)$. Extend the segment $\overline{pr}$ (at r) to a point u and the segment $\overline{pr'}$ (at r') to a point v such that $|pu| = |pv| = |pr|/(1 - \sin \delta_1)$. By elementary geometry, we have $|uv| \leq (2 \sin(\delta_1/2))|pu|$ and $|ru| = (1/(1 - \sin \delta_1) - 1)|pr|$. Applying the triangle inequality, we obtain

$$|rv| \leq |ru| + |uv| \leq \left(\frac{1}{1 - \sin \delta_1} - 1 + 2\frac{\sin(\delta_1/2)}{1 - \sin \delta_1} \right) |pr| \leq \frac{2\delta_1}{1 - \delta_1}|pr| \leq \frac{4\delta_1}{1 - \delta_1}|pq|,$$

where we have used the fact that $|pr| \leq 2|pq|$. Since the minimum distance between the boundary of the balls B'' and B' is $(1/(1 + \delta_2) - 1/(1 + \epsilon))|pq|$, it is now easy to verify that for the choice of δ_1 and δ_2 specified in the statement of the lemma, $v \in int(B')$ (that is, interior of B'). Thus $v \in int(X)$ and, by convexity of X, $r' \in int(X)$, which contradicts the fact shown earlier that no site of R_p lies in $int(X)$. This completes the proof. $\square$

Let $\delta_1, \delta_2, p, R_p$ be as defined in Lemma 9. In the remainder we will show that $AVN_{\delta_2}(p, R_p)$ having size $O((1/\epsilon)^{(d-1)/2} \log(1/\epsilon))$ can be computed in time $O((1/\epsilon)^{3(d-1)/2} \log(1/\epsilon))$, which will yield the desired total construction time. Henceforth, as a convenience, we write S for R_p and S' for $R_p \setminus \{p\}$. Note that the size of S is $O(1/\epsilon^{d-1})$ as R_p is computed using Lemma 8. Our method for constructing $AVN_{\delta_2}(p, S)$

closely follows the proof of Theorem 3. We recall that the proof involves two tasks for each cone $C \in \mathcal{C}_i$: (i) finding the point $q \in Q'_i \cap C$ that is closest to p and (ii) finding the representative set $R_{i,C}$. By Lemmas 2 and 4, it is is clear that the second task can be performed in time $O((\alpha_i/\sqrt{\delta_2})^{d-1}|S|)$; summed over all the cones the total time is $O(((1/\delta_2)^{(d-1)/2}\log(1/\delta_2))|S|)$.

For the first task we proceed as follows. For each site $r \in S'$ we identify, in constant time, the point y_r that is closest to p among points $x \in C \cap \Sigma'_r$ such that $\lambda_r(x) \in (\lambda_{i-1}, \lambda_i]$. Clearly, q is the point closest to p in the set $\{y_r \ : \ r \in S'\}$ and it can be computed in $O(|S|)$ time. Thus the time for finding the q's for all the cones is $O((1/\delta_2^{(d-1)/2})|S|)$. Taking into account the time for finding the representatives, the total time is therefore $O(((1/\delta_2)^{(d-1)/2}\log(1/\delta_2))|S|)$. Recalling that $|S| = O(1/\epsilon^{d-1})$ and $\delta_2 = \epsilon/4$, the desired bound follows.

We have just described our algorithm to compute approximate Voronoi neighborhoods for all the sites in S. If we are interested in only one site (that we denote by p), the time bound can be substantially improved. Instead of using the result in Lemma 8 to obtain approximate neighborhoods, we use a cone based decomposition. Without loss of generality, suppose that p is at the origin. We consider the hypercube $H = [-1,1]^d$ and we impose a regular grid of size $\epsilon/(16\sqrt{d-1})$ on its boundary; the set of grid points is denoted by G. For any point $q \in \mathbb{R}^d$, we denote by $h(q)$ the intersection of the ray $[pq)$ with the boundary of H. We denote by $g(q)$ the point in G that is closest to $h(q)$ (if there are several such points, we choose the one smallest in lexicographic order). This point $g(q)$ can be easily found in $O(1)$ time using the floor function. For all $g \in G$, if there is a site $s \in S'$ such that $g = g(s)$, we pick one such point that is closest to p as an approximate neighbor. It is easy to see that this set is an $(\epsilon/16)$-approximate neighborhood of p with cardinality $O((1/\epsilon)^{d-1})$ that can be computed in time $O(n + (1/\epsilon)^{d-1})$. Then we can compute an $(\epsilon/4)$-approximate Voronoi neighborhood of this approximate neighborhood in time $O((1/\epsilon)^{3(d-1)/2}\log(1/\epsilon))$ as described above.

5 Lower Bound

To conclude, we give a lower bound for the size of approximate Voronoi neighborhoods. It shows that our construction is within a constant factor from optimal in the plane and within an $O(\log(1/\epsilon))$ factor from optimal in higher dimension. We conjecture that the following lower bound is tight.

Theorem 5 *In the worst case, the size of an ϵ-approximate Voronoi neighborhood in $\mathbb{R}^d$ is $\Omega(1/\epsilon^{(d-1)/2})$.*

Proof Let p be a point in $\mathbb{R}^d$, and let Σ be a sphere centered at p with radius 2. So the Voronoi cell of p with respect to $S \cup \{p\}$ is the unit ball B centered at p. It can be easily seen that any approximate Voronoi cell $\mathcal{V} = P(p, AVN_\epsilon(p,S))$ is contained in the ball B' centered at p and with radius $1+\epsilon$. Hence $\mathcal{V}$ is a polyhedral ϵ-approximation of B in terms of the Hausdorff distance. Such an approximation has $\Omega(1/\epsilon^{(d-1)/2})$ faces (see the remark in Bronshteyn and Ivanov [6]), so the size of $AVN_\epsilon(p,S)$ is also $\Omega(1/\epsilon^{(d-1)/2})$. $\square$

Acknowledgments

We would like to thank David Mount for his valuable comments.

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Appendix: approximation by scaling

In this appendix, all the disks are closed disks, and a polygon includes its boundary and its interior. The distance between two compact sets A and B is the minimum of the distance between any two points $a \in A$ and $b \in B$. The Hausdorff distance between A and B when $A \subset B$ is the maximum of the distance between A and $\{b\}$ for all $b \in B$.

Definition 1 *Given two convex polygons $P \subseteq P'$, a point p lying in the interior of P and a real number $\epsilon > 0$, we say that P' is a (p, ϵ)-approximation of P by scaling if $P' \subseteq p + (1 + \epsilon)(P - p)$.*

We will show that the number of faces required to perform such an approximation is unbounded. To be more specific, we will prove that the following claim is false:

Claim 1 *There is an integer k such that any convex polygon P containing a point p in its interior admits a $(p, 1)$-approximation by scaling that has k vertices.*

Lemma 10 *If Claim 1 is true, then for any disk D and for any point p in its interior, there is a convex polygon Q with k vertices such that $D \subseteq Q \subseteq p + 3(D - p)$*

Proof Let $D' = p + 2(D - p)$ and $D'' = p + 3(D - p)$. Let d denote the distance between the boundary of D and the boundary of D'. Then d is also the distance between the boundary of D' and the boundary of D''. Let P be a convex polygon containing D such that the Hausdorff distance between P and D is less than $d/2$. Let $P' = p + 2(P - p)$ (see Fig. 10). The Hausdorff distance between P' and D' is twice the

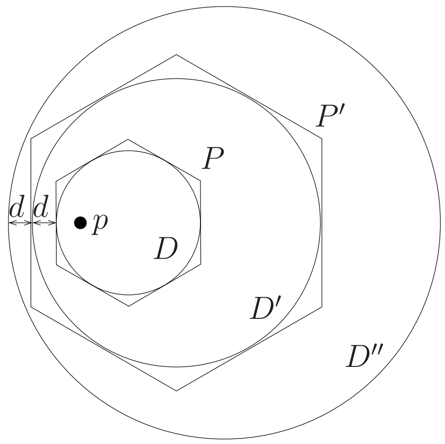

Figure 10: Proof of Lemma 10.

Hausdorff distance between P and D, hence it is less than d. Since the distance between the boundaries of D' and D'' is d and the Hausdorff distance between P' and D' is less than d, we have $P' \subset D''$. So we have $D \subseteq P \subseteq P' \subseteq D''$. By Claim 1, we can fit a polygon Q with k vertices between P and P'. Clearly $D \subseteq Q \subseteq D'' = p + 3(D - p)$. $\square$

Lemma 11 *Assume Claim 1 is true. Let D be a disk and let p be a point on its boundary. There is a convex polygon Q such that $D \subseteq Q \subseteq p + 3(D - p)$.*

Proof We consider a fixed disk D' and a point p on its boundary. We denote $D = p + (1/3)(D' - p)$. Let (p_n) be a sequence of points in the interior of D' that converges to p. Let $D_n = p_n + (1/3)(D' - p_n)$. Note that D and all the disks D_n have radius one third of the radius of D', we denote this common radius by r. By Lemma 10, for all n, there is a polygon P_n with k vertices such that $D_n \subseteq P_n \subseteq D'$. We denote the vertices of P_n by $v_n^1, v_n^2, \ldots, v_n^k$. Since D' is compact, there is a subsequence (P_{φ_n}) of (P_n) such that for all i, the sequence $v_{\varphi_n}^i$ converges to a point v^i. Hence (P_{φ_n}) converges to a polygon P with vertices $v^1, v^2, \ldots, v^k$. By compacity of D', we have $P \subseteq D'$. Now we want to prove that $D \subseteq P$. For all points x and for all polygons Q with k vertices, we denote by $d(x, \partial Q)$ the distance between $\{x\}$ and the boundary of Q. Clearly $d(\cdot, \partial\cdot)$ is continuous in the coordinates of x and the coordinates of the k vertices of Q. For all n, let c_n denote the center of D_n. Clearly c_n converges to the center c of D when n goes to infinity. Besides, for all n we have that $d(c_n, \partial P_n)$ is at least equal to r. By continuity of $d(\cdot, \partial\cdot)$, it follows that $d(c, \partial P) \geq r$. In other words, $D \subseteq P$. Our conclusion is that $D \subseteq P \subseteq D' = p + 3(D - p)$. $\square$

Therefore, if Claim 1 is true, there are two tangent disks D and D' and a polygon P such that $D \subseteq P \subseteq D'$. This situation is clearly impossible, so Claim 1 is false.