Sampling and Meshing a Surface with Guaranteed Topology and Geometry*

Siu-Wing Cheng[†] Tamal K. Dey[‡] Edgar A. Ramos[§] Tathagata Ray[‡]

September 10, 2004

Abstract

This paper presents an algorithm for sampling and triangulating a smooth surface $\Sigma \subset \mathbb{R}^3$ where the triangulation is homeomorphic to Σ. The only assumption we make is that the input surface representation is amenable to certain types of computations, namely computations of the intersection points of a line with the surface, computations of the *critical points* of some height functions defined on the surface and its restriction to a plane, and computations of some *silhouette points*. The algorithm ensures bounded aspect ratio, size optimality, and smoothness of the output triangulation. Unlike previous algorithms, this algorithm does not need to compute the local feature size for generating the sample points which was a major bottleneck. Experiments show the usefulness of the algorithm in remeshing and meshing CAD surfaces that are piecewise smooth.

1 Introduction

The need for triangulating a surface is ubiquitous in science and engineering. A set of points from the input surface needs to be generated and be connected with triangles for such a triangulation. The underlying space of the resulting triangulation should have the exact topology and approximate geometry of the input. Variety in input specifications of the surface leads to different problems in surface triangulations.

When the surface is given only through a set of point samples, the problem requires to approximate the surface with guaranteed topology and geometry from these samples. This problem, called *surface reconstruction*, has been recently addressed [1, 2, 4, 14]. When the surface is polyhedral, i.e. made out of planar patches, the Delaunay refinement techniques [7, 9, 10, 12, 20] solve the problem elegantly.

The case where the surface is smooth and is input with an implicit or parametric equation occurs in a variety of applications such as in geometric modeling and computer graphics. Maintaining topology and geometry of the input surface in the output approximation is an important issue in these applications [3, 23, 25]. Furthermore, in applications that involve finite

*Research partly supported by NSF CARGO grants DMS-0138456, DMS-0310642, ARO grant DAAD19-02-1-0347 and by RGC, Hong Kong, China, under grants HKUST6088/99E, HKUST6190/02E.

†Department of Computer Science Science, HKUST, Hong Kong. Email: `scheng@cs.ust.hk`

‡Department of Computer Science and Engineering, Ohio State University, Ohio, USA. Email: `tamaldey@cis.ohio-state.edu` and `rayt@cis.ohio-state.edu`

§Department of Computer Science, University of Illinois at Urbana-Champaign, Urbana-Champaign, USA. Email: `eramosn@cs.uiuc.edu`

element methods, it is important to generate triangles that are well shaped [21]. In this paper we present an algorithm that can triangulate a smooth implicit or parametric surface without boundary with the following guarantees: (i) the output surface has the same topology as the input, (ii) the triangulation is "smooth" in certain sense that we explain later, (iii) all triangles have bounded aspect ratio, (iv) the number of generated sample points is asymptotically optimal.

Related work. Chew [11] introduced a "furthest-point" sampling strategy that inserts points into a sample where the Voronoi edges intersect the surface. In effect, this algorithm attempts to compute the *restricted Delaunay triangulation* of the surface. Edelsbrunner and Shah [16] showed that a *topological ball property* is sufficient for the restricted Delaunay triangulation to be homeomorphic to the input surface. The algorithm of Chew does not guarantee this property and thus does not have any kind of topological guarantee. Following the "furthest-point" strategy Cheng, Dey, Edelsbrunner and Sullivan [8] proposed an algorithm for meshing a special type of surface called skin surface where they guarantee both topology and geometry. This algorithm maintains the surface triangulation under deformation and exploits the local feature sizes which are easily computable for skin surfaces.

Boissonnat and Oudot [5] carried forward the "furthest-point" strategy for general curved surfaces. They show how an initial seed triangle for each surface component can unfurl into a full triangulation of that surface component with topological and geometric guarantees. It requires to compute the local feature sizes for points on the surface which are their distances to the medial axis. The medial axis computation for surfaces is hard and hence exact local feature size computation is difficult, if not impossible, for surfaces in general. Of course, one can approximate the medial axis with existing algorithms [1, 4, 13]. However, these algorithms require a dense sample with respect to the local feature size in the first place. In this paper we improve the "furthest-point" strategy by eliminating the need for local feature size computation. In its place we use some critical and silhouette point computations that are less demanding than the local feature size computations. In contrast to the algorithm of Boissonnat and Oudot which computes the local feature size for each sample point generation, our algorithm needs to compute the critical and silhouette points only sparsely. The algorithm of Boissonnat and Oudot can be made to work with an one-time computation of the minimum local feature size, but then, the output triangulation becomes uniform and unnecessarily dense at places where feature sizes are not small at all.

A related work by Boissonnat, Cohen-Steiner and Vegter [6] considers meshing isosurfaces from a function $E : \mathbb{R}^3 \to \mathbb{R}$ evaluated at grid points. The method constructs a triangulation of a box in $\mathbb{R}^3$ which provides a piecewise linear interpolant $\hat{E}$ of the function E. The isosurface $E = 0$ is approximated with the isosurface $\hat{E} = 0$. The authors provide conditions on sampling and an associated algorithm to guarantee that the computed surface $\hat{E} = 0$ has the same topology as that of the surface given by $E = 0$. This method samples E rather than the surface $E = 0$. It requires the computation of the critical points of E as well as their indices (a more involved computation). In contrast, our method only samples the surface $E = 0$ and computes the critical points of a height function defined on the surface $E = 0$ and its restriction to planes. Furthermore, our approach is based on Delaunay/Voronoi geometry as opposed to octree subdivisions. An advantage of Delaunay/Voronoi based approach is that it fits well with the successful paradigm of Delaunay refinement [7, 20] for mesh generation.

Our approach. We maintain the restricted Delaunay triangulation of a set of points sampled on the surface and generate more sample points dictated by certain conditions while updating the triangulation. The algorithm kicks off with a set of initial points, called the *seeds*. These points are critical points of a height function on the surface which maps a point of the surface to one of its co-ordinates. We show that, given such seeds, one can use a simple "topological-disk" test and certain other critical and silhouette point computations to guarantee the topology. The critical and silhouette point computations could be costly depending upon the complexity of the surface, but they are less demanding than the local feature size computations.

The initial seed set is used to capture the topology. Once the topology is recovered, we delete these seeds. We show that we can maintain the topology of the surface despite these deletions. This enables us to prove a size optimality result for the output. In the geometry recovery phase we enforce the aspect ratio and smoothness of the restricted Delaunay triangulation of the generated sample.

2 Preliminaries

2.1 Necessary concepts

Voronoi diagram. Let P be a finite set of points in $\mathbb{R}^3$. The *Voronoi cell* of $p \in P$ is given as

$$V_p = \{x \in \mathbb{R}^d : \forall q \in P - \{p\}, \|x - p\| \leq \|x - q\|\}.$$

The sets V_p are convex polyhedra. Closed faces shared by j Voronoi cells for $2 \leq j \leq 4$ are called $(4 - j)$-dimensional *Voronoi faces*. The 0-, 1-, 2-dimensional Voronoi faces are called Voronoi *vertices, edges and facets* respectively. The *Voronoi diagram* $\operatorname{Vor} P$ of P is the collection of all Voronoi faces.

Delaunay triangulation. The *Delaunay triangulation* of a set of points P is dual to the Voronoi diagram of P. Assuming general position, the convex hull of $j \leq 4$ points defines a $(j - 1)$-dimensional *Delaunay simplex* if the intersection of their corresponding Voronoi cells is not empty. The 1-, 2-, 3-dimensional Delaunay simplices are called *Delaunay edges, triangles and tetrahedra* respectively. They define a decomposition of the convex hull of all points in P called the *Delaunay triangulation* $\operatorname{Del} P$.

Input Surface. The input is a smooth, compact surface $\Sigma \subset \mathbb{R}^3$ without boundary.

The *medial axis* of Σ is the closure of the set of points that are the centers of maximal balls called *medial balls* whose interiors are empty of any points from Σ. The *local feature size* $f(x)$ at a point $x \in \Sigma$ is the Euclidean distance of x from the medial axis. A Lipschitz property holds for $f()$, that is, $f(x) \leq f(y) + \|x - y\|$ for any two points x, y in Σ, see [1]. A point set $P \subset \Sigma$ is an ε-sample if each $x \in \Sigma$ has a point $p \in P$ within $\varepsilon f(x)$ distance.

We will need some specific numerical computations on the surface Σ. We assume that the input representation of Σ is amenable to these computations. We describe these computations for the case where Σ is given with the implicit equation $E(\mathbf{x}) = 0$ where $\mathbf{x} = (x, y, z)$ is the position vector.

Critical points. Let $d \in \mathbb{S}^2$ be an arbitrary direction with d_x, d_y and d_z being its components in the $x-$, $y-$ and $z-$directions respectively. Define a height function $h\colon \Sigma \to \mathbb{R}$ as $h(\mathbf{x}) = \mathbf{x} \cdot d$.

The *critical points* of h defined through the local co-ordinate patches on Σ are the points on Σ which have normals along d or $-d$; see, for example, Wallace [24]. The vector $\mathbf{n}_\mathbf{x} = (E_x, E_y, E_z)$ with the partial derivatives of E as components is normal to Σ at $\mathbf{x}$. Therefore, this vector is parallel to d or $-d$ when $E_x/d_x = E_y/d_y = E_z/d_z$. Hence the system of equations $E = 0, E_x d_y - E_y d_x = 0$ and $E_x d_z - E_z d_x = 0$ solves to provide the critical points of h. In what follows we refer to these points as the *critical points* of Σ in the direction d and denote the set as Z_d.

We will also use critical points of a height function defined on some curves.

Genericity condition. We assume that Σ is generic in the sense that the critical points in the set Z_d are non-degenerate (Hessians are non-singular) for all $d \in \mathbb{S}^2$.

Silhouette. One of our goals will be to ensure that Σ intersects a Voronoi cell in a topological disk. The concept of silhouette becomes helpful for this purpose. The *silhouette* of Σ with respect to a direction d is

$$J_d = \{\mathbf{x} \in \Sigma \,|\, \mathbf{n}_\mathbf{x} \cdot d = 0\}.$$

It is known that, J_d is a 'Jacobi set' which under the genericity condition are a set of smooth, pairwise disjoint closed curves [15, 18].

The next lemma will be useful in our analysis. A similar result with some different conditions can be found in Snyder [22].

Lemma 1 *Let $M \subset \Sigma$ be a connected, compact 2-manifold with boundary where the boundary is a single cycle. The manifold M is a topological disk if there exists a $d \in \mathbb{S}^2$ so that M does not intersect J_d.*

Proof. Let d be a direction satisfying the condition of the lemma. Consider the map $\Pi\colon M \to A$ where A is a plane with the normal d and Π projects each point of M orthogonally to A. Since M is connected, compact and has a single boundary cycle, it is sufficient to prove that Π is one-to-one. Suppose not. Then, there is a line with direction d which intersects M in two or more points. Let p and p' be two such consecutive points on this line. The two normals $\mathbf{n}_p$ and $\mathbf{n}_{p'}$ are oppositely oriented in the sense that one makes an angle smaller than $\pi/2$ with d and the other makes greater than $\pi/2$ angle with d. None of them can make exactly $\pi/2$ angle with d since, in that case, the point in question would be in J_d. Consider a curve joining p and p'. Since M is connected, such a curve always exists. Along this curve the normal to the surface changes from $\mathbf{n}_p$ to $\mathbf{n}_{p'}$. By mean-value theorem there is a point on the curve where the normal is orthogonal to d which is impossible since M does not contain any point of J_d. ◻

2.2 Surface computations

The algorithm uses the following numerical computations on the input surface.

CritSurf(Σ,d) : This subroutine solves the system of equations as mentioned before to compute the critical points Z_d. Either numerical or symbolic computations can be used for this purpose [19]. The complexity of this computation certainly depends on the nature of E.

CritCurve(Σ,F): This subroutine computes the critical points of a height function defined on the curve of intersection between Σ and the plane of a Voronoi facet F. Let $\mathbf{x}' = M\mathbf{x}$ be a linear transformation of the co-ordinate axes where the x'-y' plane is identified with the plane of F and the y'-axis is identified with the projection of z-axis on the plane of F. Writing $G(\mathbf{x}') = E(M^{-1}\mathbf{x}') = 0$ gives Σ with the new co-ordinate axis frame. The equation $H(x', y') = G(x', y', 0) = 0$ gives the implicit equation of the curve in which Σ intersects the plane of F. The system of equations $H = 0, H_{x'} = 0$ gives the critical points of the height function h defined on the curve where $h(x', y') = y'$. We call the critical points of h as the critical points of the curve $H(x', y') = 0$.

SilhFacet(Σ,F,d): This subroutine determines the intersection points of the silhouette with the plane of the Voronoi facet F. Let $a \cdot \mathbf{x} = 1$ be the equation of the plane containing F. The required point(s) are the solutions of the system of equations $E(\mathbf{x}) = 0$, $a \cdot \mathbf{x} = 1$ and $\mathbf{n_x} \cdot d = 0$.

CritSilh(Σ,d, d'): This subroutine computes the critical points on the silhouette J_d for the height function $h\colon J_d \to \mathbb{R}$ where $h(\mathbf{x}) = \mathbf{x} \cdot d'$ for a direction d' orthogonal to d. The silhouette is given by two implicit equations $E(\mathbf{x}) = 0$ and $G(\mathbf{x}) = \mathbf{n_x} \cdot d = 0$. The tangent to the silhouette at $\mathbf{x}$ is given by $\mathbf{n_x} \times G_\mathbf{x}$ where $G_\mathbf{x} = (G_x, G_y, G_z)$. Thus, the critical points on J_d along d' are the solutions of the system of equations $E(\mathbf{x}) = 0$, $\mathbf{n_x} \cdot d = 0$ and $(\mathbf{n_x} \times G_\mathbf{x}) \cdot d' = 0$.

EdgeSurface(Σ, e): This subroutine determines the intersection points of a Voronoi edge e with the surface Σ. One way to do this is to align the x-axis with the line of e, say with the transformation $\mathbf{x}' = M\mathbf{x}$ and then setting y' and z' to zero in the equation $G(\mathbf{x}') = E(M^{-1}\mathbf{x}') = 0$. The equation $H(\mathbf{x}') = G(x', 0, 0) = 0$ is an equation in a single variable x' whose solution gives the intersection points of the line of e with Σ. Among those intersection points, one can compute the points delimited by the endpoints of e.

2.3 Background results

A set of points P on a surface Σ defines a *restricted Voronoi diagram* $\operatorname{Vor} P|_\Sigma$ as the collection of *restricted Voronoi cells* $\{V_p|_\Sigma = V_p \cap \Sigma\}$. Dual to the restricted Voronoi diagram is the *restricted Delaunay triangulation* $\operatorname{Del} P|_\Sigma$. It is a simplicial complex where $\sigma \in \operatorname{Del} P|_\Sigma$ if and only if it is the convex hull of a set of vertices $R \subseteq P$ and $\bigcap_{q \in R} V_q|_\Sigma \neq \emptyset$.

A result of Edelsbrunner and Shah [16] relates the topology of the restricted Delaunay triangulation to the sampled surface as stated in Theorem 1. We say that $\operatorname{Vor} P|_\Sigma$ satisfies the *topological ball property* if each Voronoi face of dimension d intersects Σ in a closed topological $(d-1)$-ball or in an empty set.

Theorem 1 *The underlying space of* $\operatorname{Del} P|_\Sigma$ *is homeomorphic to* Σ *if* $\operatorname{Vor} P|_\Sigma$ *satisfies the topological ball property.*

In what follows we use the following functions

$$\begin{aligned}\alpha(c) &= \frac{c}{1-4c}\\ \beta(c) &= \arcsin c + \arcsin(\frac{2}{\sqrt{3}}c)\end{aligned}$$

and a constant $k > 0$ for which the following inequality holds:

$$\frac{k}{1-k} < \cos(\alpha(k) + 3\beta(k)). \tag{1}$$

We choose $k = 0.06$ for which the above inequality holds and some of the earlier results become applicable.

We use the notation $\angle\mathbf{a},\mathbf{b}$ to denote the acute angle between the lines supporting the vectors $\mathbf{a}$ and $\mathbf{b}$. In what follows $\mathbf{n}_{pqr}$ denotes the normal to a triangle pqr.

An immediate corollary of Lemma 2 of Amenta and Bern [1] is the following.

Lemma 2 *Let x, y be any two points in Σ so that $\|x - y\| \leq cf(x)$ where $c < 1/4$. Then $\angle\mathbf{n}_x, \mathbf{n}_y \leq \alpha(c)$.*

Next result follows from Theorem 5 of Amenta, Choi, Dey and Leekha[2].

Lemma 3 *Let pqr be a triangle where p, q, r are three points on Σ and the circumradius of pqr is smaller than $cf(p)$. Then, for $c < 0.06$, $\angle\mathbf{n}_{pqr}, \mathbf{n}_p \leq \beta(c)$.*

Next two results are taken from Cheng, Dey, Edelsbrunner and Sullivan [8].

Lemma 4 *Let a line L intersect Σ in two points x, y where the angle $\angle L\mathbf{n}_x$ between L and the normal $\mathbf{n}_x$ at x is ξ. Then, $\|x - y\| \geq 2f(x)\cos\xi$.*

Lemma 5 *Let x, y be two points on Σ. The angle $\angle(y - x), \mathbf{n}_x$ between the vector $y - x$ and $\mathbf{n}_x$ is at least*
$\pi/2 - \arcsin(\frac{\|x-y\|}{2f(x)})$.

3 Topological ball property

The output mesh produced by our algorithm is the restricted Delaunay triangulation $\mathrm{Del}\, P|_\Sigma$ where P is the generated sample. So, our goal is to ensure that $\mathrm{Del}\, P|_\Sigma$ is homeomorphic to Σ. We ensure this by sampling Σ so that the *topological ball property* is satisfied. We establish that whenever this topological ball property is violated, we can find a point on Σ which is far away from all other points and thus maintain a lower bound on inter-point distances. This lower bound ensures the termination of the algorithm.

3.1 Voronoi edges

The topological ball property requires that a Voronoi edge intersects Σ only at a single point. Next lemma shows that if this property is violated, there is a point on Σ far away from all existing sample points.

Lemma 6 *Let $e \in V_p$ be a Voronoi edge that intersects Σ either (i) in two or more points, or (ii) tangentially in a single point. Then, the intersection point of e and Σ which is furthest from p is at least $kf(p)$ away from p.*

Proof. Case(i): Let x and y be any two intersection points of e and Σ (Figure 1). Assume that $\|p-x\| \leq \|p-y\|$. Suppose that contrary to the lemma $\|p-x\| \leq \|p-y\| < kf(p)$. Let e make an angle ξ with the normal $\mathbf{n}_x$. If pqr is the dual Delaunay triangle of e, we have

$$\begin{aligned} \xi &\leq \angle \mathbf{n}_{pqr}, \mathbf{n}_p + \angle \mathbf{n}_p, \mathbf{n}_x \\ &\leq \beta(k) + \alpha(k) \end{aligned}$$

by Lemma 3 and Lemma 2.

By Lemma 4, $\|x-y\| \geq 2f(x)\cos\xi$. On the other hand $\|x-y\| < 2kf(p) \leq \frac{2k}{1-k} f(x)$. We reach a contradiction when the following inequality holds:

$$\frac{k}{1-k} < \cos(\alpha(k) + \beta(k)).$$

This inequality holds for our choice of k.

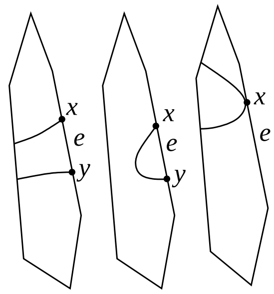

Figure 1: Voronoi edge intersecting Σ at two points or tangentially.

Case (ii): Let x be the point of tangency between e and Σ (Figure 1). As argued above, if $\|x-p\| < kf(p)$, we have ξ, the angle between e and $\mathbf{n}_x$ no more than $\alpha(k) + \beta(k)$ which is much smaller than $\pi/2$ for $k = 0.06$. This contradicts that e meets Σ tangentially at x. $\square$

3.2 Voronoi facets

Lemma 7 *Let F be a Voronoi facet in V_p where $F \cap \Sigma$ contains a cycle C (Figure 2). Let L be any line in the plane of F and normal to C at a point $x \in C$. The furthest point in $L \cap C$ from p is at least $kf(p)$ away from p.*

Proof. Suppose that, on the contrary, all points of $L \cap C$ are within $kf(p)$ distance from p. In particular, $\|x-p\| \leq kf(p)$. Let pq be the dual Delaunay edge of F. We have $\|p-q\| \leq 2\|p-x\| \leq 2kf(p)$. We know that

$$
\begin{aligned}
\angle L\mathbf{n}_x &= \frac{\pi}{2} - \angle(q-p), \mathbf{n}_x \\
&\le \frac{\pi}{2} - \angle(q-p), \mathbf{n}_p + \angle \mathbf{n}_p, \mathbf{n}_x \\
&\le \arcsin(\frac{\|p-q\|}{2f(p)}) + \frac{k}{1-4k} \\
&\quad (\mathit{Lemma}\ \ 5)\ \ (\mathit{Lemma}\ \ 2) \\
&\le \arcsin k + \frac{k}{1-4k} \\
&\le \alpha(k) + \beta(k).
\end{aligned}
$$

Now we can proceed as in the proof of Lemma 6 to reach a contradiction as L must intersect C in at least one other point. □

Lemma 8 *Let F be a Voronoi facet in V_p where $F \cap \Sigma$ contains at least two closed topological intervals. Further, assume that each Voronoi edge intersects Σ in at most one point. The furthest point from p which lies on a Voronoi edge and in $F \cap \Sigma$ is at least $kf(p)$ away from p.*

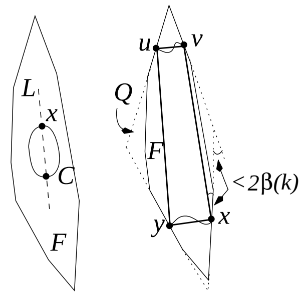

Figure 2: Voronoi facet intersecting Σ in a cycle (left) and topological intervals (right).

Proof. Suppose that, on the contrary, all intersection points of Σ with the Voronoi edges of V_p lie within $kf(p)$ distance of p. Then, by Lemma 3 all these Voronoi edges make angle smaller than $\beta(k)$ with $\mathbf{n}_p$. Let u, v and x, y be the two end points of any two topological intervals in $F \cap \Sigma$. Consider the quadrilateral Q formed by the supporting lines of the four Voronoi edges on which these four end points lie. These Voronoi edges are almost parallel since they make small angle with the common direction of $\mathbf{n}_p$. The quadrilateral $uvxy$ must reside in V_p and hence in Q. It is not hard to verify that at least one edge of $uvxy$ makes a small angle less than $2\beta(k)$ with a Voronoi edge, see Figure 2. Let vx be such an edge. It follows that vx makes an angle less than $\alpha(k)+3\beta(k) = \xi$ with $\mathbf{n}_x$. Also $\|v-x\| \le 2kf(p) \le \frac{2k}{1-k}f(x)$. Now we can reach a contradiction if

$$\frac{k}{1-k} < \cos(\alpha(k) + 3\beta(k))$$

which holds with the assumed value of k. □

3.3 Voronoi cells

The following lemma is used in lower bounding the distances.

Lemma 9 *Let a Voronoi cell V_p contain a point $q \in J_d$ in the silhouette where $d = \mathbf{n}_p$. We have $\|p - q\| \geq kf(p)$.*

Proof. By definition $\mathbf{n}_q \cdot \mathbf{n}_p = 0$. On the other hand, if $\|p - q\| < kf(p)$ we must have $\angle \mathbf{n}_p, \mathbf{n}_q \leq \alpha(k)$ which implies $\mathbf{n}_p \cdot \mathbf{n}_q \neq 0$, a contradiction. $\square$

We will use the following lemma in proving Lemma 11.

Lemma 10 *Let F be a Voronoi facet in V_p where $F \cap \Sigma$ is a topological interval. Further, let the two points where the Voronoi edges of F intersect Σ lie within a distance less than $kf(p)$ from p. Then, all points of $F \cap \Sigma$ lie within a distance less than $kf(p)$ from p.*

Proof. Consider the ball B centered at p with radius $kf(p)$. This ball intersects the plane Π of F in a circle C. We claim that C and hence B contains $F \cap \Sigma$ completely inside. If not, C intersects the interval $F \cap \Sigma$ and hence $\Pi \cap \Sigma$ in at least two disconnected components. One can shrink C into a smaller circle C' which intersects the interval $\Pi \cap \Sigma$ tangentially at two points. First shrink C radially till the closed disk bounded by C intersects $\Pi \cap \Sigma$ in exactly two components, one of them being a single point, say a. Now shrink C further by moving the center towards a till it intersects $\Pi \cap \Sigma$ tangentially at two points, one of them being a, see Figure 3.

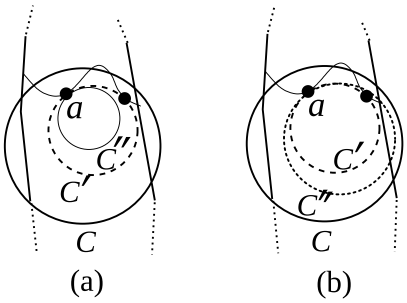

Figure 3: Illustration for the proof of Lemma 10.

The plane Π intersects the two medial balls at a in two circles. Let C'' be the circle among these two on the same side of the tangent plane at a as the circle C' is. Let B'' be the medial ball so that $C'' = B'' \cap \Pi$. We claim that C'' is almost as big as a diametric circle of B''. The distance between p and Π is at most $kf(p)$ as the two Voronoi edges of F are within $kf(p)$ distance from p. The angle between $\mathbf{n}_p$ and Π is no more than $\arcsin(k/2)$ by Lemma 5. Also the angle between $\mathbf{n}_p$ and $\mathbf{n}_a$ is no more than $\alpha(k)$ (Lemma 2). This means that the diametric segment of the medial ball B'' which passes through a makes at most an angle ξ with Π where

$$\xi \leq \arcsin \frac{k}{2} + \alpha(k).$$

Let d_1 and d_2 be the diameters of C'' and B''. Then,

$$\begin{aligned} d_1 &\geq d_2 \cos\xi \\ &\geq 2f(a)\cos\xi \\ &\geq (2(1-k)f(p))\cos\xi. \end{aligned}$$

The radius $d_1/2$ of C'' is more than $kf(p)$ if $\cos\xi > k/(1-k)$, a condition satisfied by our choice of k.

The circles C' and C'' meet only tangentially at a. Also the interior of C'' is empty of any point of Σ other than a. If C' contains or coincides with C'', its radius is bigger than $kf(p)$ contradicting the fact that C' is inside B, see Figure 3(a). If C' is contained inside C'', we reach a contradiction since C' intersects Σ in at least two points one of which must reside inside C'' (Figure 3(b)). It follows that $F \cap \Sigma$ lies within $kf(p)$ distance as claimed. $\square$

Lemma 11 *Let $V_p \cap \Sigma$ be a manifold with boundary where the boundary has at least two cycles none of which resides on a single facet of V_p. Then, there is a point of Σ on a Voronoi edge of V_p which is at least $kf(p)$ away from p.*

Proof.

Let F be any facet of V_p intersecting Σ. We can assume that $F \cap \Sigma$ is a single topological interval. Otherwise, $F \cap \Sigma$ has more than one topological interval since it cannot contain any cycle by the condition of the lemma. In that case we can apply Lemma 8 to have a point on a Voronoi edge of F satisfying the claim of the lemma.

For the sake of contradiction assume that all points of intersection between the Voronoi edges of V_p and Σ lie within a distance less than $kf(p)$ from p.

Now consider a ball B of radius $kf(p)$ centering p. We claim that $B \cap \Sigma$ is a topological disk D. For otherwise B would contain a medial axis point by a result of Boissonnat and Cazals [4] which would imply that B has a radius at least $f(p)$ reaching a contradiction for $k < 1$.

The boundary cycles of the manifold $M = V_p \cap \Sigma$ must lie in B as each point on them has a distance less than $kf(p)$ from p by Lemma 10. Let C be any of these boundary cycles. Consider the Voronoi facets that contain points of C. Let R denote the convex polytope containing p inside and bounded by the planes of these Voronoi facets. The cycle C bounds a smaller disk, disk(C), in the topological disk D. This smaller disk lies inside V_p and hence in R. Consider another boundary cycle C' of M which must exist according to the condition of the lemma. This cycle lies inside R as R contains V_p. If disk(C) and disk(C') are not disjoint, one must lie completely inside the other as C and C' cannot cross each other. Assume, in that case, disk(C) is contained inside disk(C'). Consider a curve joining two points on C and C' and lying outside disk(C). The interior of this curve must intersect R. This is because this curve goes outside R as it goes outside disk(C) and then must reenter R to meet C'. Let G be a facet of R at which this curve reenters R and x be the point of reentry. Let L be the line joining x to its closest point y on the interval of C on G. If y is an interior point of the interval, L intersects C at right angle and we reach a contradiction with similar arguments as in the proof of Lemma 7. If y is an end point of the interval, it must lie on a Voronoi edge of V_p. In this case the angle between L and $\mathbf{n}_y$ is less than the angle between the Voronoi edge and $\mathbf{n}_y$. Again, we can use arguments of the proof of Lemma 7 to reach a contradiction. $\square$

4 Topology sampling

The sampling algorithm first inserts points of Σ into a sample P as long as the restricted Delaunay triangulation $\mathrm{Del}\, P|_\Sigma$ is not homeomorphic to Σ. It ensures the topological ball property by four tests, namely VorEdge(), TopoDisk(), CritCurve() and Silhouette(). The first two tests involve computing the intersections of Voronoi edges with the surface and some simple combinatorial checks. Only the third and fourth tests use more complicated computations such as computing the points by CritSilh and SilhFacet . Formulation of our lemmas allows us to apply these two tests only when the other two fail. Thus, they are used only sparsely. We argue later that these tests are sufficient to achieve the homeomorphism. All four tests also insert a point into the sample P if they fail.

VorEdge($e \in V_p$)

> If EdgeSurface(Σ,e) computes a single point x, check if e meets Σ at x tangentially. If so, return x. Otherwise, if EdgeSurface(Σ, e) computes two or more points, return the point furthest from p among them.

Let T_p be the set of triangles in $\mathrm{Del}\, P|_\Sigma$ incident to p. TopoDisk checks if the triangles in T_p make a topological disk. This is done by first checking if each edge incident to p in T_p has exactly two incident triangles from T_p. If this condition is satisfied, it checks if there is exactly one cycle of triangles in T_p around p. The test fails if either of the conditions is not satisfied.

TopoDisk(p)

> If T_p is not a topological disk, return the intersection point of a Voronoi edge of V_p and Σ which is furthest from p.

The next test checks if a facet F intersects Σ in a cycle though it does not compute the cycle explicitly. It uses CritCurve(Σ,F) instead.

FacetCycle($F \in V_p$)

> Compute X :=CritCurve(Σ,F).
> If any point $x \in X$ lies inside V_p, compute a line L going through x as follows. The line L is the projection of a line L' onto the plane of F where L' goes through x and is parallel to the z-axis. (Notice that L is normal to the intersection curve of Σ and F at x by the definition of CritCurve)
> If L intersects Σ at any point other than x in V_p, return the point among such intersection points which is furthest from p.

Silhouette() checks if a Voronoi cell has any point of the silhouette. For this check it uses CritSilh and SilhFacet computations. If the check succeeds, it introduces a point from the silhouette.

Silhouette(V_p)

> Compute the normal direction $\mathbf{n}_p = (E_x, E_y, E_z)$. Choose a direction d orthogonal to $\mathbf{n}_p$.
> Compute X := CritSilh(Σ, $\mathbf{n}_p$,d).
> If $X \cap V_p \neq \phi$, return one point of X lying inside V_p. Otherwise, for each facet $F \in V_p$ compute Y := SilhFacet(F,d). If $Y \cap F \neq \phi$, return a point from Y lying inside F.

Initially P contains all critical points of Σ along the $z = (0, 0, 1)$ direction computed by CritSurf. It will be clear in Lemma 12 how this initial set helps in satisfying the topological ball property. Topology uses the four tests VorEdge, TopoDisk, FacetCycle and Silhouette, necessarily in this order to insert points into the sample P. This ordering enables the algorithm to postpone the complicated computations necessary in the FacetCycle and Shilouette tests only after the simpler computations in VorEdge and TopoDisk.

Topology(P)

1. Check if any of VorEdge, TopoDisk, FacetCycle, or Silhouette necessarily in this order, can insert a new point. If so, insert it in P and update the Voronoi diagram. Continue the process till no new point is inserted.
2. Return P

Lemma 12 *If the input sample to* Topology *includes all critical points of* Σ *for a direction and* Topology *terminates, the topological ball property holds for* $\mathrm{Vor}\, P|_\Sigma$ *at the end of its execution.*

Proof. No Voronoi edge can intersect Σ in more than one point at the end of Topology since otherwise VorEdge would succeed to insert a new point.

Next, consider the possibility of a Voronoi facet F intersecting Σ in more than one topological intervals, in cycles, or in combination of both. If there is more than one topological interval, Σ must intersect more than two Voronoi edges of F since no Voronoi edge intersects Σ in more than one point. This means the dual Delaunay edge of F is incident to more that two triangles in $\mathrm{Del}\, P|_\Sigma$. But, this violates the topological disk condition imposed by TopoDisk for some point $p \in P$. Next, consider when F intersects Σ in a single topological interval and also in some cycle C. A local minimum $x \in C$ is detected to reside in a Voronoi cell. Also the line L as computed by FacetCycle is normal to C and thus intersects C in another point. Thus, the test in FacetCycle succeeds which would trigger another insertion, a contradiction to the termination of Topology.

The intersection of a Voronoi cell V_p with Σ is a manifold possibly with boundary. This manifold cannot have a component M which is a manifold without boundary since in that case the maxima and minima of M computed by CritSurf as seed points are in P and also in V_p, an impossibility.

Now consider the case where M is a connected, comapct 2-manifold with a single boundary cycle but is not a topological disk. Thanks to Lemma 1, M must intersect the silhouette J_d where $d = \mathbf{n}_p$. Let d' be the direction orthogonal to d chosen by Silhouette for its computation of the set X. If M contains a closed curve from J_d, then there exists a point q in J_d which is critical along the direction d'. This point is computed by Silhouette in the set X which lies inside V_p. It triggers an insertion which violates the termination of Topology. If M does not contain any closed curve from J_d, a curve from it must intersect a Voronoi facet of V_p. In that case Silhouette computes a point in the set Y by SilhFacet which lies inside F. Again, a point from Y is inserted contradicting the termination of Topology.

The only remaining case is that $V_p \cap \Sigma$ is a manifold with more than one boundary cycle where each of these cycles intersect a cycle of Voronoi edges. The dual to this cycle of Voronoi edges is a cycle of triangles in $\mathrm{Del}\, P|_\Sigma$ around p. Since none of the Voronoi edges intersects Σ in more than one point, the two boundary cycles induce two distinct cycles of triangles around p. Therefore, TopoDisk(p) fails and should insert a new point contradicting the termination of TOPOLOGY. ◻

Lemma 13 *Any point inserted by* TOPOLOGY *is more than* $kf(p)$ *away from its closest sample point in* P.

Proof. It follows from Lemma 6 that VOREDGE inserts a point $kf(p)$ away from its closest point p.

TOPODISK inserts a point when it fails. The restricted Delaunay triangles incident to p do not form a topological disk if (i) an edge does not have two restricted Delaunay triangles, (ii) two or more cycles of restricted Delaunay triangles are incident to p.

Case (i): Assume that a restricted Delaunay edge has exactly one triangle in T_p. This means that Σ has intersected the dual facet F of the edge in a single Voronoi edge e. Since Σ has no boundary, $F \cap \Sigma$ cannot have any end point other than on e. This means Σ intersects e in more than one point or tangentially. This is not possible since VOREDGE has taken care of all such Voronoi edges before TOPODISK is called.

Next, consider the case when a restricted Delaunay edge has two or more triangles incident to it from T_p. This means the dual facet F has intersected Σ in two or more topological intervals. Apply Lemma 8 to claim that the point inserted by TOPODISK is at least $kf(p)$ away from its closest sample point p.

Case (ii): In this case $V_p \cap \Sigma$ has at least two boundary cycles none of which lie completely inside a facet. Apply Lemma 11 to claim that the inserted point is at least $kf(p)$ away from p.

When FACETCYCLE inserts a point, the furthest point on $L \cap \Sigma$ is at least $kf(p)$ away from p by Lemma 7.

In the remaining case when SILHOUETTE inserts a point $q \in J_d$ with $d = \mathbf{n}_p$, we have q at least $kf(p)$ away from p by Lemma 9. ◻

4.1 Deleting seeds

All new points other than the seeds are added at least some constant times the local feature size distance away from their closest sample point. This property can be used to show that the output sample size is optimal except that the seeds can be arbitrarily dense. Although the case where the set of critical points is dense is unlikely in practice, the theoretical guarantee of optimality cannot be achieved unless we delete the seeds. We do so after the topology of the input surface is captured.

It is true that if we delete a sample point, the topological ball property may not remain valid. Thanks to the next lemma we can restore it by inserting more points by TOPOLOGY.

Lemma 14 *Let* $p \in P$ *be any point and* $\mathrm{Vor}\, P|_\Sigma$ *satisfy the topological ball property. No Voronoi cell in* $\mathrm{Vor}\,(P \setminus p)$ *can have a connected component of* Σ *inside.*

Proof. Suppose that $V_q \in \mathrm{Vor}\,(P \setminus p)$ has a connected component Q of Σ inside it. Now consider introducing the point p into $\mathrm{Vor}\,(P \setminus p)$. The bisecting plane H of p and q must intersect Q in V_q since otherwise there is a Voronoi cell not satisfying the topological ball property in

Vor $P|_\Sigma$. The plane H intersects Q in a cycle which resides in a Voronoi facet in Vor P. Thus, Vor $P|_\Sigma$ does not satisfy the topological ball property, a contradiction. $\square$

After deleting a seed point, the only thing we need to ensure is that a Voronoi cell V_p does not have any connected component of Σ residing completely inside V_p. Precisely this is the case avoided by seeds in the proof of Lemma 12. Lemma 14 enables us to avoid this case after deleting a seed point. So, we have the following corollary.

Corollary 1 TOPOLOGY *can restore the topological ball property given* $P \setminus p$ *as input where* p *is a seed point and* Vor P *satisfies the topological ball property.*

Putting together we have the topology sampling step as follows.

SAMPLETOPOLOGY(P)

1. TOPOLOGY(P)
2. while there is a seed point $p \in P$
delete p from P and call TOPOLOGY(P)

5 Geometry sampling

Topological guarantee of the output triangulation alone is not sufficient for many applications. In finite element methods, it is important that the surface triangles have bounded aspect ratio. Also, the output approximation should be smooth enough as it approximates a smooth surface. We introduce two sampling steps that take care of these two issues.

5.1 Quality

In order to restore the quality of the triangles, we take the approach of Chew [11]. Define $\rho(t) = \frac{r}{\ell}$ as the radius-edge ratio of a triangle t where r is the circumradius of t and ℓ is its shortest edge length. It is well known that a triangle is well shaped if its radius-edge ratio is bounded from above. Following Chew [11], if there is a triangle t in Del $P|_\Sigma$ with $\rho(t) > \rho_0 > 1$, we introduce the point where the dual Voronoi edge of t intersects Σ. Certainly, if this procedure terminates, all triangles in the output has radius-edge ratio no more than ρ_0. We will show later that this process terminates. We choose $\rho_0 = (1 + k)^2$ which guarantees termination as well as a size optimality result.

QUALITY(P)

While there is a triangle t with $\rho(t) > (1 + k)^2$ in Del $P|_\Sigma$, insert any point of intersection between the dual Voronoi edge of t and Σ into P and update Vor P.

5.2 Smoothness

The output surface is a piecewise linear approximation of Σ. One could take different measures of smoothness such as discrete curvatures for the output. We choose a simple measure for ease in implementation. Let e be any edge on a triangulated surface N with two triangles incident on it. We define the *roughness* $g(e)$ of e as $g(e) = \pi - \theta$ where θ is the internal dihedral angle at e. We sample more points until all edges have roughness below a threshold.

SMOOTH(P)

while there is an edge $pq \in \operatorname{Del} P|_\Sigma$ with $g(pq) > 2\alpha(k)$ insert the point furthest from p among all intersections of the Voronoi edges of V_p and Σ and update $\operatorname{Vor} P$.

Lemma 15 *Any point inserted by* SMOOTH *is more than $kf(p)$ away from its closest neighbor in P.*

Proof. Let pq be the edge which triggers the insertion of a point x by SMOOTH. So x is the intersection point of a Voronoi edge in V_p and Σ furthest from p. Suppose x is within $kf(p)$ distance of p. Then all intersection points of the Voronoi edges of V_p and Σ are within $kf(p)$ distance from p. This means all restricted triangles incident to p have circumradii no more than $kf(p)$. By Lemma 3 the two triangles pqr and pqs incident to p have normals satisfying $\angle \mathbf{n}_{pqr}, \mathbf{n}_p \leq \alpha(k)$ and $\angle \mathbf{n}_{pqs}, \mathbf{n}_p \leq \alpha(k)$. Therefore, $\angle \mathbf{n}_{pqr}, \mathbf{n}_{pqs} \leq 2\alpha(k)$ which means $g(pq) \leq 2\alpha(k)$ contradicting the insertion of x. ◻

6 Meshing and Guarantees

We maintain the restricted Delaunay triangulation $\operatorname{Del} P|_\Sigma$ all the time during sampling. This triangulation is produced as the output mesh of the DELMESH algorithm that combines all steps.

DELMESH(Σ)

1. Compute $P :=$CRITSURF(Σ,$(0,0,1)$).
2. $P_0 := P$
3. SAMPLETOPOLOGY(P)
4. QUALITY(P)
5. SMOOTH(P)
6. Go back to 2 if $P_0 \neq P$
7. Output $\operatorname{Del} P|_\Sigma$.

Notice that after geometry sampling we go back to check the topology since more sample points may disturb the topology of the restricted Delaunay triangulation. In implementation we do not need to search the entire triangulation for possible topology violation. Instead, since each insertion changes the Delaunay triangulation locally, only a local search is sufficient.

6.1 Termination and quality

Theorem 2 DELMESH *terminates.*

Proof. We claim that other than seed points, any two points p and q in the sample P at any stage of DELMESH have distance $\|p - q\| \geq kf(p)/(1 + k)$. We assume the above claim inductively before the insertion of a point p. We show that p is inserted more than $kf(p)$ away from all other existing points. Then, by Lipschitz property of $f()$, it can be shown that any other existing point $q \neq p$ is more than $kf(q)/(k + 1)$ distance away from p establishing the inductive hypothesis.

If p is inserted by SAMPLETOPOLOGY it is more than $kf(p)$ away from all other points by Lemma 13. If p is inserted by QUALITY, it is more than $(k+1)^2\ell$ distance away from its closest sample points where ℓ is the length of an edge in $\mathrm{Del}\, P|_\Sigma$ incident to one of the closest samples q of p. By inductive hypothesis $\ell > kf(q)/(k+1)$ since all seed points are deleted before QUALITY is called. Thus, $\|p-q\| > k(k+1)f(q)$. By Lipschitz property of $f()$, we have $\|p-q\| > kf(p)$. If p is inserted by SMOOTH, Lemma 15 confirms the claim.

Let $f_m = \mathrm{Inf}_{p\in\Sigma} f(p)$. Since Σ is smooth we have $f_m > 0$. It follows that $\ell_m = kf_m/(k+1) > 0$. Since any two inserted points in P are at distance $\ell_m > 0$ or more by the above argument, we can have only fintely many points allowed to be inserted by the standard packing argument. Termination of DELMESH follows. ◻

Theorem 3 *The output surface N of* DELMESH *satisfies the following properties:*
(i) N is homeomorphic to Σ.
(ii) Each triangle in N has radius-edge ratio less than $(1+k)^2$.
(iii) Roughness of each edge of N is less than $2\alpha(k)$.

Proof. Since DELMESH terminates, the conclusion of Lemma 12 holds. Therefore, (i) follows immediately because of Theorem 1. The properties (ii) and (iii) are immediate from the termination of DELMESH. ◻

6.2 Optimality

Theorem 4 *The number of points in P is within a constant factor of any ε-sample of Σ for any $\varepsilon < 1/5$.*

Proof. The number of points in any ε-sample of Σ is $\Omega(\int_\Sigma \frac{1}{f(x)^2}\, dx)$ for any $\varepsilon < 1/5$ [17]. We show that P has at most $c \cdot \int_\Sigma \frac{1}{f(x)^2}\, dx$ points for some suitable constant $c > 0$.

Consider any two points p and q in the final P. By the arguments in the proof of Theorem 2 we have $\|p-q\| \geq kf(p)/(k+1)$. We can center disjoint balls B_p of radii $\frac{k}{2+2k} f(p)$ at the points $p \in P$. For each point $x \in B_p \cap \Sigma$, the Lipschitz condition implies that

$$f(x) \leq f(p) + \|p-x\| \leq \frac{2+3k}{2+2k} f(p). \tag{2}$$

Also, by Lemma 5, $\angle(x-p), \mathbf{n}_p \geq \pi/2 - \arcsin(\frac{k}{4+4k})$. Let x' be the orthogonal projection of x onto the tangent plane at p. It follows that $\|p-x'\| \geq \|p-x\| \cdot \cos(\arcsin(\frac{k}{4+4k})) \geq \|p-x\|/2$. We conclude that

$$\mathrm{area}(B_p \cap \Sigma) \geq \frac{\pi k^2}{16(1+k)^2} f(p)^2. \tag{3}$$

Putting everything together, we obtain

$$
\begin{aligned}
\int_{\Sigma} \frac{1}{f(x)^2} dx &\geq \sum_{p \in P} \int_{B_p \cap \Sigma} \frac{1}{f(x)^2} dx \\
&\overset{(2)\&(3)}{\geq} \sum_{p \in P} \frac{\pi k^2}{4(2+3k)^2} \\
&\geq \frac{\pi k^2}{4(2+3k)^2} \cdot |P|.
\end{aligned}
$$

This proves that the size of our mesh is asymptotically optimal. $\square$

7 Discussions

We presented a provable algorithm for sampling and meshing a smooth surface without boundary. Implicit and parametric surfaces can be meshed with this algorithm with guaranteed topology and geometry and also with guarantees on the mesh size and quality. Although the theory applies to smooth surfaces, we experimented with non-smooth surfaces and the results are encouraging.

We implemented a simplified version of DelMesh using CGAL [27]. We did not implement the FacetCycle and Silhouette tests. Figure 4 shows some of the results of this implementation. For our experiment we took some of the triangulated surfaces obtained by a surface reconstruction software called Tight Cocone as input [26]. Although these surfaces already have sample points, we disregarded all these sample points for our experiments and considered the piecewise linear surface as input. DelMesh generated a new sample and mesh.

We took the local maxima, minima and saddle points in the input surfaces as seed points. Figure 4 shows the progression of the output at different stages. We set the roughness angle at 10°.

We experimented with some CAD surfaces as well though they are not smooth. They are represented with standard STL files where the curved surface patches are triangulated. Third column in Figure 4 shows one such surface. In this case we took the sample points on the edges and vertices where the original surface patches meet as the seeds. The output of DelMesh is a surface mesh with guaranteed quality.

These examples show that DelMesh can be used for remeshing triangulated surfaces and meshing CAD surfaces while guaranteeing bounded aspect ratio. An open question remains if the method or its variant can be proved to mesh non-smooth surfaces with guarantees. The critical point computations are the most costly computations in the algorithm. Can we avoid them and under what circumstances?

Acknowledgements. We thank Günter Rote immensly for detecting an error in an earlier draft which led us to introduce the concept of silhouette. We also thank anonymous referees whose comments were helpful improving the paper.

References

[1] N. Amenta and M. Bern. Surface reconstruction by Voronoi filtering. *Discr. Comput. Geom.* **22** (1999), 481–504.

[2] N. Amenta, S. Choi, T. K. Dey and N. Leekha. A simple algorithm for homeomorphic surface reconstruction. *Internat. J. Comput. Geom. Applications* **12** (2002), 125–141.

[3] J. Bloomenthal. Polygonization of implicit surfaces. *Proc. Comput. Aided Geom. Design* **5** (1988), 341–355.

[4] J.-D. Boissonnat and F. Cazals. Natural neighbor coordinates of points on a surface. *Comput. Geom. Theory Appl.* **19** (2001), 87–120.

[5] J.-D. Boissonnat and S. Oudot. Provably good surface sampling and approximation. *Eurographics Sympos. Geom. Process.* (2003), 9–18.

[6] J.-D. Boissonnat, D. Cohen-Steiner and G. Vegter. Isotopic implicit surface meshing. *36th ACM Sympos. Theory Comput.* (2004).

[7] S.-W. Cheng and T. K. Dey. Quality meshing with weighted Delaunay refinement. *SIAM J. Computing* **33** (2003), 69–93.

[8] H.-L. Cheng, T. K. Dey, H. Edelsbrunner and J. Sullivan. Dynamic skin triangulation. *Discrete Comput. Geom.* **25** (2001), 525–568.

[9] S.-W. Cheng, T. K. Dey, E. A. Ramos and T. Ray. Quality meshing for polyhedra with small angles. *Proc. 20th Annu. Sympos. Comput. Geom.* (2004), 290–299.

[10] S.-W. Cheng and S.-H. Poon. Graded conforming Delaunay tetrahedralization with bounded radius-edge ratio. *Proc. 14th Annu. ACM-SIAM Sympos. Discr. Algorithms* (2003), 295–304.

[11] L. P. Chew. Guaranteed-quality mesh generation for curved surfaces. *Proc. 9th Annu. ACM Sympos. Comput. Geom.*, (1993), 274–280.

[12] D. Cohen-Steiner, E. C. de Verdière and M. Yvinec. Conforming Delaunay triangulations in 3D. *Proc. Annu. Sympos. Comput. Geom.* (2002), 199–208.

[13] T. K. Dey and W. Zhao. Approximating the medial axis from the Voronoi diagram with a convergence guarantee. *Algorithmica* **38** (2004), 179–200.

[14] T. K. Dey. Curve and surface reconstruction. Chapter in *Handbook on Discrete and Computational Geometry*, 2nd Edition (2004), eds. J. Goodman and J. O'Rourke, CRC press, Boca Raton, Florida.

[15] H. Edelsbrunner and J. Harer. Jacobi sets of multiple functions. *Foundations of Comput. Mathematics*, ed. F. Cucker, Cambridge Univ. Press, to appear.

[16] H. Edelsbrunner and N. Shah. Triangulating topological spaces. *Internat. J. Comput. Geom. Appl.* **7** (1997), 365–378.

[17] J. Erickson. Nice point sets can have nasty Delaunay triangulations. *Discr. Comput. Geom.* **30** (2003), 109–132.

[18] M. Golubitsky and V. Guillemin. *Stable Mappings and Their Singularities.* Springer-Verlag, New York, 1973.

[19] T. G. Kolda, R. M. Lewis and V. Torczon. Optimization by direct search: New perspectives on some classical and modern methods. *SIAM Review* **45** (2003), 385–482.

[20] J. R. Shewchuk. Tetrahedral mesh generation by Delaunay refinement. *Proc. 14th Annu. ACM Sympos. Comput. Geom.*, (1998), 86–95.

[21] J. R. Shewchuk. What is a good linear element? interpolation, conditioning and quality measures. *Proc. 11th Internat. Meshing Roundtable* (2002), 115–126.

[22] J. Snyder. *Generative Modeling for Computer Graphics and CAD.* Academic press, 1992.

[23] B. T. Stander and J. C. Hart. Guaranteeing the topology of an implicit surface polygonalization for interactive modeling. *Proc. SIGGRAPH 97* (1997), 279–286.

[24] A. H. Wallace. *Differential topology : first steps.* W. A. Benjamin, Inc., New York, 1968.

[25] A. P. Witkin and P. S. Heckbert. Using particles to sample and control implicit surfaces. *Proc. SIGGRAPH 94* (1994), 269–278.

[26] http://www.cis.ohio-state.edu/∼tamaldey/cocone.html

[27] http://www.cgal.org

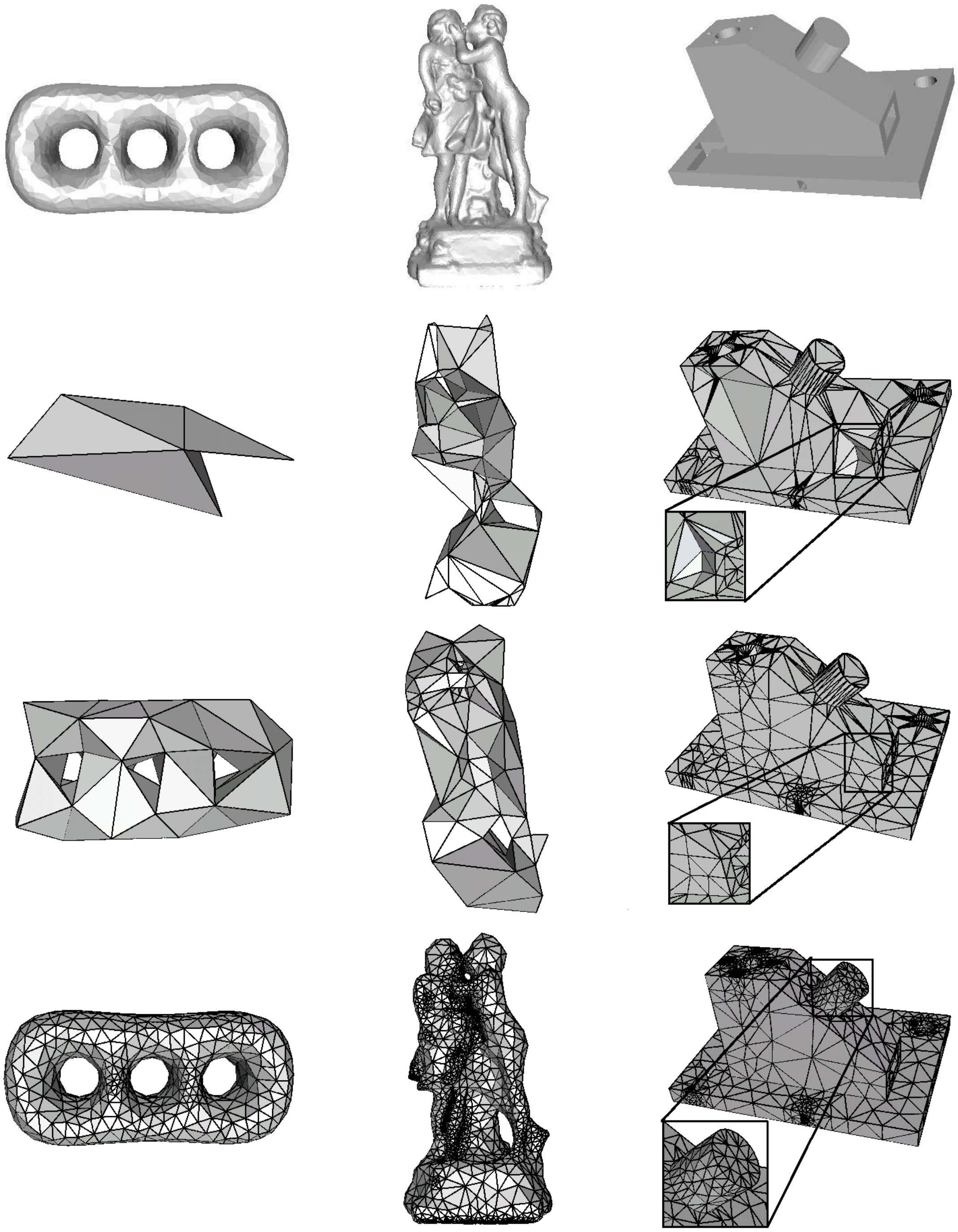

Figure 4: First row shows the surfaces to be sampled. Second row is the restricted Delaunay of the seeds. Third row shows the triangulation after the topology is captured with seeds deleted. Fourth row shows the results after smoothing. For the CAD data, the seeds contain vertices on sharp edges and corners which are not deleted and smoothing are not applied to sharp edges and vertices.