Enumerating Minimally Rigid Body-Hinge Graphs with Application to Architectural Design

Naoki Katoh, Kyoto Univeristy

It remains a big challenging open problem in the field of combinatorial rigidity to derive combinatorial characterizations for 3-dimensional bar-joint frameworks. However, for a special class of 3-dimensional bar-joint frameworks such as body-bar and body-hinge frameworks, combinatorial characterization was developed by Tay and Whiteley. Recently, for panel-hinge frameworks, Tanigawa and I succeeded in developing combinatorial characterizations.

In this talk, we propose an algorithm for enumerating minimally rigid body-hinge simple graphs. The running time is polynomial per output. We also mention how we can apply the algorithm to enumerate various types of architetctural designs by showing experimental results.

A d-dimensional body-hinge framework is a collection of d-dimensional rigid bodies connected by *hinges*, where a hinge is a $(d-2)$-dimensional affine subspace, e.g. pin joints in 2-space, line-hinges in 3-space, plane-hinges in 4-space. Bodies are allowed to move continuously in $\mathbb{R}^d$ so that the relative motion of any two bodies connected by a hinge is a rotation around it and the framework is called rigid if every motion provides a framework isometric to the original one.

We consider a body-hinge framework as a pair (G, p) of a graph $G = (V, E)$ and a mapping p from $e \in E$ to a $(d-2)$-dimensional affine subspace $p(e)$ in $\mathbb{R}^d$. Namely, $v \in V$ corresponds to a body and $uv \in E$ corresponds to a hinge $p(uv)$ which joints the two bodies corresponding to u and v. Then G is said to be *realized* as a body-hinge framework (G, p) in $\mathbb{R}^d$, and is called a body-hinge graph. Tay and Whiteley independently proved that the generic infinitesimal rigidity of a body- hinge framework is determined by the underlying graph G. Recently, Tanigawa and I proved the counterpart for panel-hinge frameworks. More precisely, for a multigraph $G = (V, E)$ and a positive integer k, the graph obtained by replacing each edge by k parallel edges is denoted by kG. For our special interest in $(D-1)G$, we shall use the simple notation $\widetilde{G}$ to denote $(D-1)G$ where $D = \binom{d}{2}$. The following fact is known.

Proposition 1. *A multigraph G can be realized as an infinitesimally rigid body-hinge and panel-hinge framework in $\mathbb{R}^d$ if and only if $\widetilde{G}$ has D edge-disjoint spanning trees.*

Notice that for $d = 3$, $D = 5$ holds.

In contrast with body-bar frameworks, it is not straightforward to develop an efficient algorithm for enumerating body-hinge graphs since there does not exist a matroidal property for body-hinge graphs that satsify Proposition 1.

The proposed algorithm applies a sequence of the following four operations which produce from a minimally rigid body-hinge simple graph $G = (V, E)$ that of a larger size.

Operation 1 (edge split): For an edge ab, insert a new vertex v on that edge if the resulting graph is rigid.
Operation 2 (edge split plus 1-addition): For an edge ab, insert a new vertex v on that edge and add a new edge e if the resulting graph is minimally rigid.
Operation 3 (vertex 2-addition): Add a new vertex v, choose two existing vertices a and B such that the graph obtained by introducing edges va and vb is minimally rigid.
Operation 4 (triangle addition): Choose an arbitray vertex v, add two new vertices a and b as well as edges va, vb, ab.

This is a joint work with Yuya Higashikawa and Yuki Kobayashi.

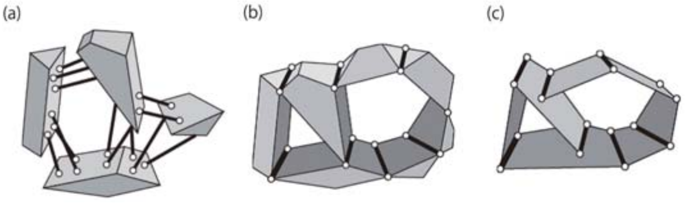

Fig. 1. Illustration of (a) body-bar, (b) body-hinge, and (c) panel-hinge frameworks