Lecture 2: Maximum Contiguous Subarray Problem

Overview

- Reference: Chapter 8 in *Programming Pearls, (2nd ed)* by Jon Bentley.

- Clean way to illustrate basic algorithm design
  - A $\Theta(n^3)$ brute force algorithm
  - A $\Theta(n^2)$ algorithm that reuses data.
  - A $\Theta(n \log n)$ divide-and-conquer algorithm

- *Cost* of algorithm will be number of primitive operations, e.g., comparisons and arithmetic operations, that it uses.

MCS Example

ACME CORP – PROFIT HISTORY

Year	1	2	3	4	5	6	7	8	9
Profit M$	-3	2	1	-4	5	2	-1	3	-1

Betweeen years 5 and 8 ACME earned
$5 + 2 - 1 + 3 = 9$ Million Dollars

This is the **MAXIMUM** amount that ACME earned in *any* contiguous span of years.

Examples:
Between years 1 and 9 ACME earned
$-3 + 2 + 1 - 4 + 5 + 2 - 1 + 3 - 1 = 4$ M$
and between years 2 and 6
$2 + 1 - 4 + 5 + 2 = 6$ M$.

The **Maximum Contiguous Subarray Problem** is to find the span of years in which ACME earned the most, e.g., $(5, 8)$.

Formal Definition

Input: An array of reals $A[1 \ldots N]$.

The *value* of subarray $A[i \ldots j]$ is

$$V(i,j) = \sum_{x=i}^{j} A(x).$$

The **Maximum Contiguous subarray problem** is to find $i \le j$ such that

$$\forall (i', j'),\ V(i', j') \le V(i,j).$$

Output: $V(i,j)$ s.t. $\forall (i', j'), \quad V(i', j') \le V(i,j)$.

Note: Can modify the problem so it returns indices (i,j).

$\boxed{\Theta(n^3) \textbf{ Solution: Brute Force}}$

Idea: Calculate the value of $V(i, j)$ for each pair $i \leq j$ and return the **maximum** value.

```
VMAX=A[1];
for (i=1 to N) {
 for (j=i to N) {
  // calculate V(i, j)
  V=0;
  for (x= i to j)
   V=V+A[x];
  if (V > VMAX)
   VMAX=V;
 }
}
return VMAX;
```

$\boxed{\Theta(n^2)\text{ solution: Reuse data}}$

Idea: We don't need to calculate each $V(i,j)$ from "scratch" but can exploit the fact that

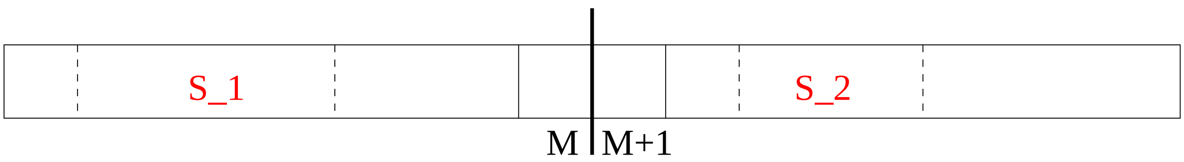
$$V(i,j) = \sum_{x=i}^{j} A[x] = V(i,j-1) + A[j].$$

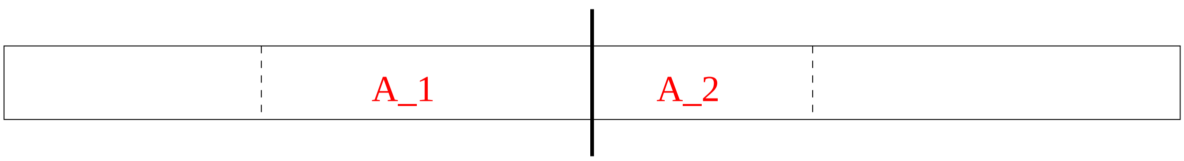
```
VMAX=A[1];
for (i=1 to N) {
 V=0;
 for (j=i to N) {
  // calculate V(i, j)
  V=V+A[j];
  if (V > VMAX)
   VMAX=V;
 }
}
return VMAX;
```

$\Theta(n \log n)$ solution: Divide-and-Conquer

Idea: Set $M = \lfloor (N+1)/2 \rfloor$.
Let A_1 and A_2 be the MCS that MUST contain $A[M]$ and $A[M+1]$ respectively. Note that the MCS must be one of

- S_1 : The MCS in $A[1 \ldots M]$,
- S_2 : The MCS in $A[M+1 \ldots N]$,
- A : Where $A = A_1 \cup A_2$.





A_1 = MCS on left containing A[M]   A_2 = MCS on right containing A[M+1]

A= A_1 U A_2

Example

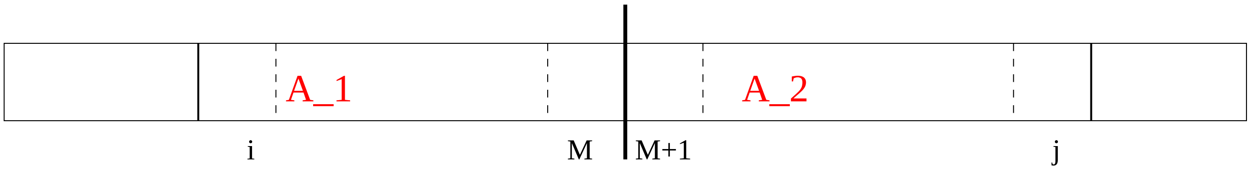
1	-5	4	2	-7	3	6	-1	2	-4	7	-10	2	6	1	-3

1	-5	4	2	-7	3	6	-1	2	-4	7	-10	2	6	1	-3

$S_1 = [3, 6]$ and $S_2 = [2, 6, 1]$.
$A_1 = [3, 6, -1]$ and $A_2 = [2, -4, 7]$;
$A = A_1 \cup A_2 = [3, 6, -1, 2, -4, 7]$

Since $Value(S_1) = 9$, $Value(S_2) = 9$
and $Value(A) = 13$
the solution to the problem is A.

Finding A : The conquer stage



A_1 is in the form $A[i \ldots \mathbf{M}]$:
there are only $M - i + 1$ such sequences, so, A_1 can be found in $O(M - i)$ time.

```
MAX=A[M];
SUM=A[M];
for (k=M-1 down to i)
{
 SUM+=A[k];
 if (SUM > MAX) MAX=SUM;
}
A_1=MAX;
```

Similarly, A_2 is in the form $A[\mathbf{M+1} \ldots j]$:
there are only $j - M$ such sequences, so, A_2 the maximum valued such one, can be found in $O(j - M)$ time.

$A = A_1 \cup A_2$ can therefore be found in $O(j - i)$ time, which is linear to the input size.

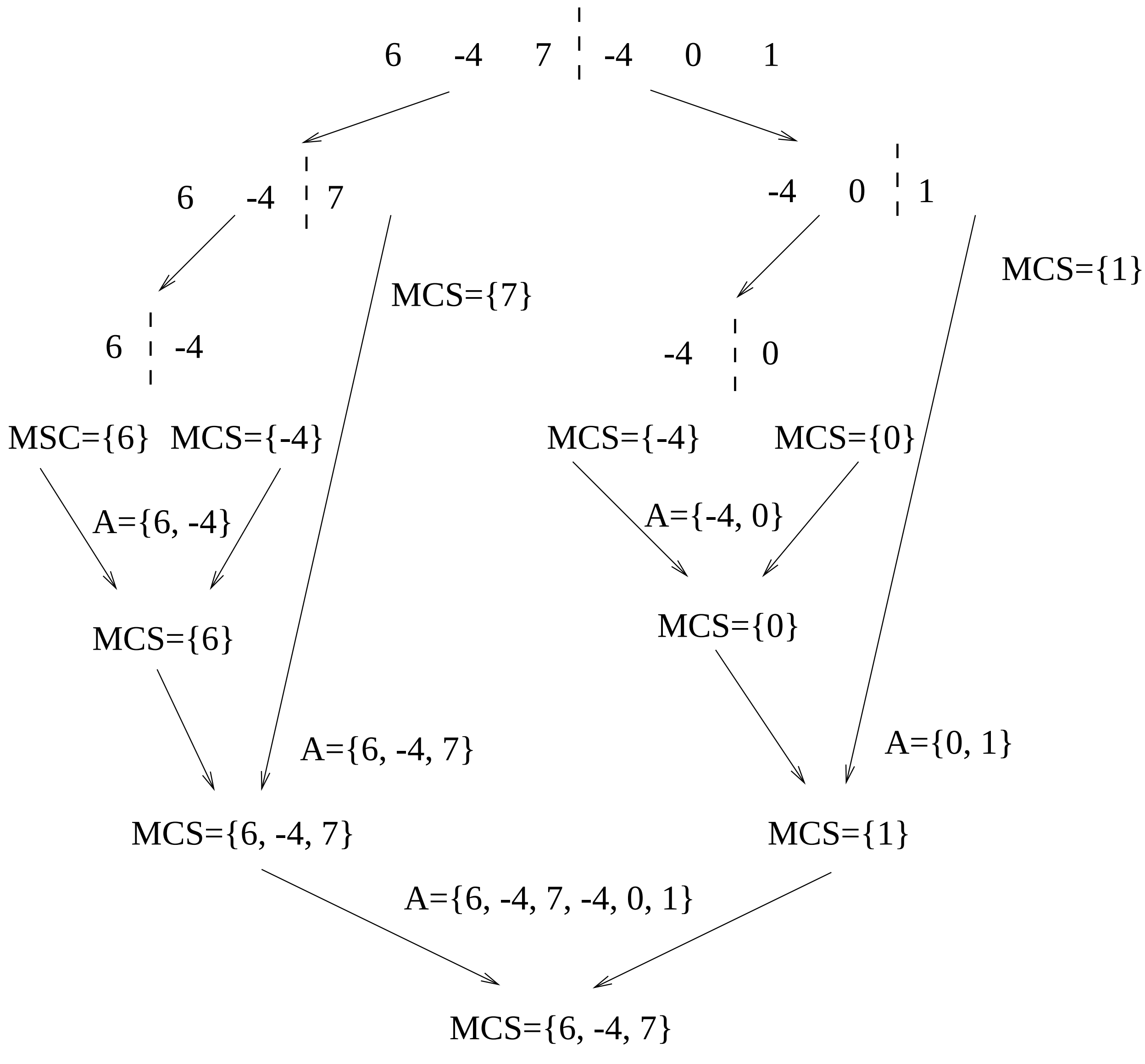
The Full Divide-and-Conquer Algorithm

// Input : $A[i \dots j]$ with $i \le j$
// Output : the MCS of $A[i \dots j]$

$MCS(A, i, j)$

1. If $i == j$ return $A[i]$
2. Else
3. Find $MCS(A, i, \lfloor \frac{i+j}{2} \rfloor)$;
4. Find $MCS(A, \lfloor \frac{i+j}{2} \rfloor + 1, j)$;
5. Find MCS that contains *both* $A\left[\lfloor \frac{i+j}{2} \rfloor\right]$ and $A\left[\lfloor \frac{i+j}{2} \rfloor + 1\right]$;
6. Return Maximum of the three sequences found

A full example



Analysis of the DC Algorithm

Let $T(m)$ (where m is the problem size) be time needed to run

$MSC(A, i, j)$, $(j - i + 1 = m)$

Step (1) requires $O(1)$ time.

Steps (3) and (4) each require $T(m/2)$ time.

Step (5) requires $O(m)$ time.

Step (6) requires $O(1)$ time

Then $T(1) = O(1)$ and

for $n > 1$, $T(n) = 2T(n/2) + O(n)$

Analysis of the DC Algorithm

To simplify the analysis, we assume that n is a power of 2.

$T(n) \leq 2\,T(\frac{n}{2}) + c\,n$. Repeating this recurrence gives

$$
\begin{aligned}
T(n) &\leq 2\,T\left(\frac{n}{2}\right) + c\,n \\
&\leq 2\left[2\,T\left(\frac{n}{2^2}\right) + c\frac{n}{2}\right] + c\,n \\
&= 2^2 T\left(\frac{n}{2^2}\right) + 2c\,n \\
&\leq 2^2\left[2\,T\left(\frac{n}{2^3}\right) + c\frac{n}{2^2}\right] + 2c\,n \\
&= 2^3 T\left(\frac{n}{2^3}\right) + 3c\,n \\
&\leq \ldots \\
&= 2^h T\left(\frac{n}{2^h}\right) + hc\,n
\end{aligned}
$$

Set $h = \log_2 n$, so that $2^h = n$. With this substitution, we have

$$T(n) \leq n\,T(1) + (\log_2 n)c\,n = O(n \log_2 n).$$

Review

In this lecture we saw 3 different algorithms for solving the maximum contiguous subarray problem. They were

- A $\Theta(n^3)$ brute force algorithm
- A $\Theta(n^2)$ algorithm that reuses data.
- A $\Theta(n \log n)$ divide-and-conquer algorithm