Lecture 3: The Polynomial Multiplication Problem

A More General Divide-and-Conquer Approach

Divide: Divide a given problem into subproblems (ideally of approximately equal size).
No longer only TWO subproblems

Conquer: Solve each subproblem (directly or **recursively**), and

Combine: Combine the solutions of the subproblems into a global solution.

The Polynomial Multiplication Problem

another divide-and-conquer algorithm

Problem:
Given two polynomials of degree n

$$\begin{aligned} A(x) &= a_0 + a_1 x + \cdots + a_n x^n \\ B(x) &= b_0 + b_1 x + \cdots + b_n x^n, \end{aligned}$$

compute the product $A(x)B(x)$.

Example:

$$\begin{aligned} A(x) &= 1 + 2x + 3x^2 \\ B(x) &= 3 + 2x + 2x^2 \\ A(x)B(x) &= 3 + 8x + 15x^2 + 10x^3 + 6x^4 \end{aligned}$$

Question: How can we efficiently calculate the coefficients of $A(x)B(x)$?
Assume that the coefficients a_i and b_i are stored in arrays $A[0 \ldots n]$ and $B[0 \ldots n]$.
Cost of any algorithm is number of scalar multiplications and additions performed.

Convolutions

Let $A(x) = \sum_{i=0}^{n} a_i x^i$ and $B(x) = \sum_{i=0}^{m} b_i x^i$.

Set $C(x) = \sum_{k=0}^{n+m} c_i x^i = A(x)B(x)$.

Then

$$c_k = \sum_{i=0}^{k} a_i b_{k-i}$$

for all $0 \le k \le m + n$.

Definition: The vector $(c_0, c_1, \ldots, c_{m+n})$ is the convolution of the vectors $(a_0, a_1, \ldots, a_n)$ and $(b_0, b_1, \ldots, b_m)$.

Calculating convolutions (and thus polynomial multiplication) is a major problem in digital signal processing.

The Direct (Brute Force) Approach

Let $A(x) = \sum_{i=0}^{n} a_i x^i$ and $B(x) = \sum_{i=0}^{n} b_i x^i$.

Set $C(x) = \sum_{k=0}^{2n} c_i x^i = A(x)B(x)$ with

$$c_k = \sum_{i=0}^{k} a_i b_{k-i}$$

for all $0 \leq k \leq 2n$.

The direct approach is to compute all c_k using the formula above. The total number of multiplications and additions needed are $\Theta(n^2)$ and $\Theta(n^2)$ respectively. Hence the complexity is $\Theta(n^2)$.

Questions: Can we do better?
Can we apply the divide-and-conquer approach to develop an algorithm?

The Divide-and-Conquer Approach

The Divide Step: Define

$$\begin{aligned} A_0(x) &= a_0 + a_1 x + \cdots + a_{\lfloor \frac{n}{2} \rfloor - 1} x^{\lfloor \frac{n}{2} \rfloor - 1}, \\ A_1(x) &= a_{\lfloor \frac{n}{2} \rfloor} + a_{\lfloor \frac{n}{2} \rfloor + 1} x + \cdots + a_n x^{n - \lfloor \frac{n}{2} \rfloor}. \end{aligned}$$

Then $A(x) = A_0(x) + A_1(x) x^{\lfloor \frac{n}{2} \rfloor}$.

Similarly we define $B_0(x)$ and $B_1(x)$ such that

$$B(x) = B_0(x) + B_1(x) x^{\lfloor \frac{n}{2} \rfloor}.$$

Then

$$\begin{aligned} A(x)B(x) = \; & A_0(x)B_0(x) + A_0(x)B_1(x)x^{\lfloor \frac{n}{2} \rfloor} + \\ & A_1(x)B_0(x)x^{\lfloor \frac{n}{2} \rfloor} + A_1(x)B_1(x)x^{2\lfloor \frac{n}{2} \rfloor}. \end{aligned}$$

Remark: The original problem of size n is divided into 4 problems of input size $\frac{n}{2}$.

Example:

$$\begin{aligned} A(x) &= 2+5x+3x^2+x^3-x^4 \\ B(x) &= 1+2x+2x^2+3x^3+6x^4 \\ A(x)B(x) &= 2+9x+17x^2+23x^3+34x^4+39x^5 \\ &\quad +19x^6+3x^7-6x^8 \end{aligned}$$

$$\begin{array}{ll} A_0(x)=2+5x, & A_1(x)=3+x-x^2, \\ & A(x)=A_0(x)+A_1(x)x^2 \\ B_0(x)=1+2x, & B_1(x)=2+3x+6x^2, \\ & B(x)=B_0(x)+B_1(x)x^2 \end{array}$$

$$\begin{aligned} A_0(x)B_0(x) &= 2+9x+10x^2 \\ A_1(x)B_1(x) &= 6+11x+19x^2+3x^3-6x^4 \\ A_0(x)B_1(x) &= 4+16x+27x^2+30x^3 \\ A_1(x)B_0(x) &= 3+7x+x^2-2x^3 \\ A_0(x)B_1(x)+A_1(x)B_0(x) &= 7+23x+28x^2+28x^3 \end{aligned}$$

$$\begin{aligned} &A_0(x)B_0(x)+(A_0(x)B_1(x)+A_1(x)B_0(x))x^2+A_1(x)B_1(x)x^4 \\ &=2+9x+17x^2+23x^3+34x^4+39x^5+19x^6+3x^7-6x^8 \end{aligned}$$

The Divide-and-Conquer Approach

The Conquer Step: Solve the four subproblems, i.e., computing

$$A_0(x)B_0(x), \quad A_0(x)B_1(x),$$
$$A_1(x)B_0(x), \quad A_1(x)B_1(x)$$

by recursively calling the algorithm 4 times.

The Divide-and-Conquer Approach

The Combining Step: Adding the following four polynomials

$$
\begin{aligned}
& A_0(x)B_0(x) \\
& \quad + A_0(x)B_1(x)x^{\lfloor \frac{n}{2} \rfloor} \\
& \quad + A_1(x)B_0(x)x^{\lfloor \frac{n}{2} \rfloor} \\
& \quad + A_1(x)B_1(x)x^{2\lfloor \frac{n}{2} \rfloor}.
\end{aligned}
$$

takes $\Theta(n)$ operations. Why?

The First Divide-and-Conquer Algorithm

PolyMulti1$(A(x), B(x))$

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$$A_0(x) = a_0 + a_1 x + \cdots + a_{\lfloor \frac{n}{2} \rfloor - 1} x^{\lfloor \frac{n}{2} \rfloor - 1};$$

$$A_1(x) = a_{\lfloor \frac{n}{2} \rfloor} + a_{\lfloor \frac{n}{2} \rfloor + 1} x + \cdots + a_n x^{n - \lfloor \frac{n}{2} \rfloor};$$

$$B_0(x) = b_0 + b_1 x + \cdots + b_{\lfloor \frac{n}{2} \rfloor - 1} x^{\lfloor \frac{n}{2} \rfloor - 1};$$

$$B_1(x) = b_{\lfloor \frac{n}{2} \rfloor} + b_{\lfloor \frac{n}{2} \rfloor + 1} x + \cdots + b_n x^{n - \lfloor \frac{n}{2} \rfloor};$$

$$U(x) = PolyMulti1(A_0(x), B_0(x));$$
$$V(x) = PolyMulti1(A_0(x), B_1(x));$$
$$W(x) = PolyMulti1(A_1(x), B_0(x));$$
$$Z(x) = PolyMulti1(A_1(x), B_1(x));$$

$$\textbf{return} \left(U(x) + [V(x) + W(x)] x^{\lfloor \frac{n}{2} \rfloor} + Z(x) x^{2 \lfloor \frac{n}{2} \rfloor} \right);$$

$\}$

Running Time of the Algorithm

Assume n is a power of 2, $n = 2^h$. By substitution (expansion),

$$
\begin{aligned}
T(n) &= 4\,T\left(\frac{n}{2}\right) + c\,n \\
&= 4\,\left[4\,T\left(\frac{n}{2^2}\right) + c\,\frac{n}{2}\right] + c\,n \\
&= 4^2\,T\left(\frac{n}{2^2}\right) + (1+2)c\,n \\
&= 4^2\,\left[4\,T\left(\frac{n}{2^3}\right) + c\,\frac{n}{2^2}\right] + (1+2)c\,n \\
&= 4^3 T\left(\frac{n}{2^3}\right) + (1+2+2^2)c\,n \\
&\;\;\vdots \\
&= 4^i T\left(\frac{n}{2^i}\right) + \sum_{j=0}^{i-1} 2^j c\,n \quad \text{(induction)} \\
&\;\;\vdots \\
&= 4^h T\left(\frac{n}{2^h}\right) + \sum_{j=0}^{h-1} 2^j c\,n \\
&= n^2\,T(1) + c\,n(n-1) \\
&\quad \left(\text{since } n = 2^h \text{ and } \sum_{j=0}^{h-1} 2^j = 2^h - 1 = n - 1\right) \\
&= \Theta(n^2).
\end{aligned}
$$

The same order as the brute force approach!

Comments on the Divide-and-Conquer Algorithm

Comments: The divide-and-conquer approach makes no essential improvement over the brute force approach!

Question: Why does this happen.

Question: Can you improve this divide-and-conquer algorithm?

Problem: Given 4 numbers

$$A_0,\ A_1,\ B_0,\ B_1$$

how many multiplications are needed to calculate the three values

$$A_0B_0,\ A_0B_1 + A_1B_0,\ A_1B_1?$$

This can obviously be done using 4 multiplications but there is a way of doing this using only the following 3:

$$\begin{aligned} Y &= (A_0 + A_1)(B_0 + B_1) \\ U &= A_0B_0 \\ Z &= A_1B_1 \end{aligned}$$

U and Z are what we originally wanted and

$$A_0B_1 + A_1B_0 = Y - U - Z.$$

Improving the Divide-and-Conquer Algorithm

Define

$$\begin{aligned} Y(x) &= (A_0(x) + A_1(x)) \times (B_0(x) + B_1(x)) \\ U(x) &= A_0(x)B_0(x) \\ Z(x) &= A_1(x)B_1(x) \end{aligned}$$

Then

$$Y(x) - U(x) - Z(x) = A_0(x)B_1(x) + A_1(x)B_0(x).$$

Hence $A(x)B(x)$ is equal to

$$U(x) + [Y(x) - U(x) - Z(x)]x^{\lfloor \frac{n}{2} \rfloor} + Z(x) \times x^{2\lfloor \frac{n}{2} \rfloor}$$

Conclusion: You need to call the multiplication procedure 3, rather than 4 times.

The Second Divide-and-Conquer Algorithm

PolyMulti2$(A(x), B(x))$
{

$A_0(x) = a_0 + a_1 x + \cdots + a_{\lfloor \frac{n}{2} \rfloor - 1} x^{\lfloor \frac{n}{2} \rfloor - 1}$;
$A_1(x) = a_{\lfloor \frac{n}{2} \rfloor} + a_{\lfloor \frac{n}{2} \rfloor + 1} x + \cdots + a_n x^{n - \lfloor \frac{n}{2} \rfloor}$;

$B_0(x) = b_0 + b_1 x + \cdots + b_{\lfloor \frac{n}{2} \rfloor - 1} x^{\lfloor \frac{n}{2} \rfloor - 1}$;
$B_1(x) = b_{\lfloor \frac{n}{2} \rfloor} + b_{\lfloor \frac{n}{2} \rfloor + 1} x + \cdots + b_n x^{n - \lfloor \frac{n}{2} \rfloor}$;

$Y(x) = PolyMulti2(A_0(x) + A_1(x), B_0(x) + B_1(x))$
$U(x) = PolyMulti2(A_0(x), B_0(x))$;
$Z(x) = PolyMulti2(A_1(x), B_1(x))$;

return $\left(U(x) + [Y(x) - U(x) - Z(x)] x^{\lfloor \frac{n}{2} \rfloor} + Z(x) x^{2 \lfloor \frac{n}{2} \rfloor} \right)$;
}

Running Time of the Modified Algorithm

Assume $n = 2^h$. Let $\lg x$ denote $\log_2 x$.
By the substitution method,

$$
\begin{aligned}
T(n) &= 3\,T\left(\frac{n}{2}\right) + c\,n \\
&= 3\left[3\,T\left(\frac{n}{2^2}\right) + c\,\frac{n}{2}\right] + c\,n \\
&= 3^2\,T\left(\frac{n}{2^2}\right) + \left(1 + \frac{3}{2}\right)c\,n \\
&= 3^2\left[3\,T\left(\frac{n}{2^3}\right) + c\,\frac{n}{2^2}\right] + \left(1 + \frac{3}{2}\right)c\,n \\
&= 3^3 T\left(\frac{n}{2^3}\right) + \left(1 + \frac{3}{2} + \left[\frac{3}{2}\right]^2\right)c\,n \\
&\;\;\vdots \\
&= 3^h T\left(\frac{n}{2^h}\right) + \sum_{j=0}^{h-1}\left[\frac{3}{2}\right]^j c\,n.
\end{aligned}
$$

We have

$$3^h = (2^{\lg 3})^h = 2^{h \lg 3} = (2^h)^{\lg 3} = n^{\lg 3} \approx n^{1.585},$$

and

$$\sum_{j=0}^{h-1}\left[\frac{3}{2}\right]^j = \frac{(3/2)^h - 1}{3/2 - 1} = 2 \cdot \frac{3^h}{2^h} - 2 = 2\,n^{\lg 3 - 1} - 2.$$

Hence

$$T(n) = \Theta(n^{\lg 3} T(1) + 2\,c\,n^{\lg 3}) = \Theta(n^{\lg 3}).$$

Comments

- The divide-and-conquer approach doesn't always give you the best solution.
  Our original D-A-C algorithm was just as bad as brute force.

- There is actually an $O(n \log n)$ solution to the polynomial multiplication problem.
  It involves using the *Fast Fourier Transform* algorithm as a subroutine.
  The FFT is another classic D-A-C algorithm (Chapt 30 in CLRS gives details).

- The idea of using 3 multiplications instead of 4 is used in large-integer multiplications.
  A similar idea is the basis of the classic Strassen matrix multiplication algorithm (CLRS, Chapter 28).