Extended Generalization of Fuzzy Rough Sets

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***Abstract*—This paper extends and generalizes the approximations of fuzzy rough sets dealing with fuzzy coverings of the universe induced by a weak fuzzy similarity relation. The weak fuzzy similarity relation is considered as a generalization of fuzzy similarity relation in representing a more realistic relationship between two objects in which it has weaker symmetric and transitive properties. Since the conditional symmetry in the weak fuzzy similarity relation is an asymmetric property, there are two distinct fuzzy similarity classes that provide two different fuzzy coverings. The generalization of fuzzy rough sets approximations is discussed based on two interpretations: object-oriented generalization and class-oriented generalization. More concepts of generalized fuzzy rough set approximations are introduced and defined, representing more alternatives to provide level-2 interval-valued fuzzy sets. Moreover, through combining several pairs of proposed approximations of the generalized fuzzy rough sets, it is possible to provide the level-2 type-2 fuzzy sets as an extension of the level-2 interval valued fuzzy sets. Some properties of the concepts are examined.**

***Index Terms*— fuzzy rough sets, fuzzy covering, weak fuzzy similarity relations, conditional probability relations**

I. Introduction

PAWLAK introduced the theory of rough sets in 1982 dealing with granularity of the universe [11]. The theory of rough sets is mainly implemented as mathematical tools to recognize partial or total dependencies, discover hidden patterns, remove redundancy, and others in relational database [10]. A rough set might be considered as a generalization of crisp set by means that a given crisp set is approximate into two approximated subsets, called lower and upper approximations generated from a crisp partition of a universal set of objects [9]. The crisp partition is built by equivalence classes of objects in the universe. It means that similarity relation among the members of the universe is related based on equivalence relation. Formally, the concept of rough sets is discussed as follows. Given U be a non-empty universal set of objects, and R be an equivalence relation on U. The crisp partition of U, denoted by U/R, is a quotient set where $[x]_R$ is an equivalence class in U/R that contains $x \in U$. A rough set of $A \subseteq U$ is represented by a pair of lower and upper approximations. The lower approximation is given by the following equation.

$$Lo(A) = \{x \in U \mid [x]_R \subseteq A\} = \{[x]_R \in U/R \mid [x]_R \subseteq A\} \quad (1)$$

The lower approximation is a union of all equivalence classes in U/R that are subsets of A. On the other hand, the upper approximation is given by a union of all equivalence classes in U/R that overlap with A as given by the following equation.

$$\begin{aligned} Up(A) &= \{x \in U \mid [x]_R \cap A \neq \emptyset\}, \\ &= \{[x]_R \in U/R \mid [x]_R \cap A \neq \emptyset\} \end{aligned} \quad (2)$$

As discussed in [12], the concept of rough sets built on a crisp partition dealing with equivalence relations may be considered less applicable in representing a real-world problem. It is because the equivalence relation used in constructing the crisp partition having too strong symmetric and transitive properties in representing relationships between two objects. Therefore, a covering of the crisp universe [15] as a generalization of the crisp partition was introduced to provide a more general concept of rough sets. Formally, a crisp covering of a non-empty universal set of objects U, denoted by $C = \{C_1, C_2, \cdots, C_n\}$ is given by a family of subsets of U such that $U = \cup\{C_i \mid i = 1, \cdots, n\}$. Since it is a covering, $C_i \in C$ and $C_j \in C$ as two distinct elements in C may not be necessarily disjoint. In this case, similarity of relationship between two distinct objects in U is an asymmetric and non-transitive relationship. To provide a more realistic and relationship between two objects in the universe, Intan, Mukaidono and Yao [3] introduced *a weak fuzzy similarity relation* as a generalization of fuzzy similarity relation to provide a more realistic and applicable relation representing relationships between two objects. The weak fuzzy similarity relation has weaker symmetric and transitive properties than the fuzzy similarity relation. (Fuzzy) conditional probability relation proposed in [2] was an example of formulation characterized by the weak fuzzy similarity relation. A fuzzy covering [4] is then constructed by relationship between two objects based on the weak fuzzy similarity relation.

Using the concept of fuzzy covering, Intan and Mukaidono [5,6] proposed a concept of generalized fuzzy rough sets in 2002. This paper is an extended work of the concept of fuzzy rough sets. Therefore, this paper firstly recalls and discusses the generalized concept of fuzzy rough sets as proposed by Intan and Mukaidono in [5] and [6]. The concept of generalized rough sets is dealing with two generalized interpretations: *object-oriented generalization* and *class-oriented generalization*. This paper then proposes more concepts of approximations based on the concept of

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generalized fuzzy rough set. The proposed concepts of approximations can be considered to represent more alternatives in providing the interval-valued fuzzy sets. Some properties to show the relations of the proposed concepts are examined. Finally, an illustrative example is given to clarify the proposed concepts.

II. Weak Fuzzy Similarity Relations

The fuzzy similarity relation or fuzzy equivalence relation proposed by Zadeh [16] is a weaker similarity relation than the equivalence relation. The properties of fuzzy similarity relation are given by Definition 1.

Definition 1 A fuzzy similarity relation is a mapping, $R: U \times U \rightarrow [0,1]$, such that for $x, y, z \in U$,
(a) Reflexivity: $R(x,x) = 1$,
(b) Symmetry: $R(x,y) = R(y,x)$,
(c) Max−min Transitivity:
$R(x,z) \geq \max_{y \in U}[\min [R(x,y), R(y,z)]]$,

The weak fuzzy similarity relation as proposed in [3] has weaker symmetric and transitive properties than the fuzzy similarity relation as seen in the following definition.

Definition 2 A *weak fuzzy similarity relation* is a mapping, $R: U \times U \rightarrow [0,1]$, such that for $x, y, z \in U$,
(a) Reflexivity: $R(x,x) = 1$,
(b) Conditional Symmetry: if $R(x,y) > 0$ then $R(y,x) > 0$,
(c) Conditional Transitivity: if $R(x,y) \geq R(y,x) > 0$ and $R(y,z) \geq R(z,y) > 0$ then $R(x,z) \geq R(z,x)$.

The fuzzy conditional probability relation proposed in [2] was an example of formulation characterized by the weak fuzzy similarity relation. Fuzzy conditional probability relation is an extended concept of a conditional probability relation dealing with the fuzzy sets. The concept of conditional probability relation is given by Definition 3.

Definition 3 A *conditional probability relation* is a mapping, $R: U \times U \rightarrow [0,1]$, such that for $x, y \in U$,
$R(x,y) = P(x|y) = P(y \rightarrow x)$, (3)
where $R(x,y)$ means the degree y supports x or the degree y is similar to x.

The probability values, x and y in Definition 3, may be regarded as the semantic relationships between two objects by using the epistemological point of view in terms of probability theory. Calculating degree of relationship between x and y can be illustrated by assuming x and y are two objects in a given binary information table. Simply, degree of similarity, $R(x,y)$, can be calculated by (4),

$$R(x,y) = P(x|y) = \frac{|x \cap y|}{|y|}, \quad (4)$$

where $|\cdot|$ denotes the cardinality of a set.
For illustrative example, given Table I shows a binary information table, where set of objects, $U = \{O_1, O_2, O_3, O_4, O_5, O_6, O_7\}$, is characterized by a set of eight attributes, $At = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$. Therefore, every object in U might be represented by a subset of attributes that belong to the object. From Table I, it can be seen that $O_1 = \{a_1, a_2, a_3, a_8\}$, $O_2 = \{a_2, a_3, a_4, a_6, a_8\}$, $O_3 = \{a_1, a_5, a_7\}$, etc. Using (4), it can be followed that,

$$R(O_1, O_2) = \frac{3}{5}, \quad R(O_2, O_1) = \frac{3}{4},$$

$$R(O_1, O_3) = \frac{1}{3}, \quad R(O_2, O_1) = \frac{1}{4},$$

$$R(O_2, O_3) = 0, \quad R(O_3, O_2) = 0.$$

It can be verified that $R(O_1, O_2) > 0 \Rightarrow R(O_2, O_1) > 0$ means that the conditional probability relation satisfies conditional symmetry of the weak fuzzy similarity relation. The property of conditional symmetry might be asymmetry as shown in the calculation, $R(O_1, O_2) \neq R(O_2, O_1)$.

TABLE I. Binary Information Table

U	At							
	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
O_1	1	1	1	0	0	0	0	1
O_2	0	1	1	1	0	1	0	1
O_3	1	0	0	0	1	0	1	0
O_4	1	0	0	1	0	0	0	1
O_5	1	0	0	1	1	0	0	1
O_6	1	0	0	1	1	0	1	1
O_7	1	0	1	1	0	1	1	0

When objects are fuzzy sets, the degree of similarity two objects can be calculated by a fuzzy conditional probability relation as proposed by Intan and Mukaidono in [2] as defined in Definition 4.

Definition 4 Let $x, y \in U$ be two objects in U. x and y are fuzzily characterized by a set of attributes A that means x and y are represented as fuzzy sets over a set of attribute At as given by: $x, y: At \rightarrow [0,1]$. A *fuzzy conditional probability relation* is defined by:

$$R(x,y) = \frac{\sum_{a \in At} \min [x(a), y(a)]}{\sum_{a \in At} y(a)} \quad (5)$$

For example, three fuzzy sets, *Short* (*S*), *Medium* (*M*) and *Tall* (*T*), are represented as three objects over set of attributes, $\{3', 4', 5', 6', 7', 8', 9'\}$ in feet as given in Table II.

TABLE II. Fuzzy Information Table of Height

U	$3'$	$4'$	$5'$	$6'$	$7'$	$8'$	$9'$
S	1.0	0.8	0.2	0.0	0.0	0.0	0.0
M	0.0	0.2	0.8	0.2	0.0	0.0	0.0
T	0.0	0.0	0.0	0.8	1.0	1.0	1.0

From Table II, $S(3') = 1.0$ means $3'$ is fully a member of *Short* (*S*). $S(6') = 0$ means $6'$ is fully non-member of *Short* (*S*). Asymmetric degrees of similarity relationship between two fuzzy objects, *S* and *M*, are calculated by:

$$R(S,M) = \frac{\min(1,0)+\min(0.8,0.2)+\min(0.2,0.8)+\min(0,0.2)}{0.2+0.8+0.2}$$

$$R(S,M) = 0.33$$

$$R(M,S) = \frac{\min(1,0)+\min(0.8,0.2)+\min(0.2,0.8)+\min(0,0.2)}{1.0+0.8+0.2}$$

$$R(M,S) = 0.2$$

Similarly, it can be verified that $R(S,M) > 0 \Rightarrow R(M,S) > 0$ means that the fuzzy conditional probability relation also satisfies conditional symmetry of the weak fuzzy similarity relation.

It can be clearly proved that fuzzy conditional probability relation R in Definition 4 satisfies all properties of *weak fuzzy similarity relation*. More properties have been discussed clearly in [5] and [6] as summarized by the following properties:

For $x, y, z \in U$,

(r0) $R(x,y) = R(y,x) = 1 \Leftrightarrow x = y$,

(r1) $[R(y,x) = 1, R(x,y) < 1] \Leftrightarrow x \subset y$,

(r2) $R(x,y) = R(y,x) > 0 \Rightarrow |x| = |y|$,

(r3) $R(x,y) < R(y,x) \Rightarrow |x| < |y|$,

(r4) $R(x,y) > 0 \Leftrightarrow R(y,x) > 0$,

(r5) $[R(x,y) \geq R(y,x) > 0, R(y,z) \geq R(z,y) > 0]$ $R(x,z) \geq R(z,x)$.

III. Generalization of Fuzzy Rough Sets

The concept of rough fuzzy sets and fuzzy rough sets as a hybrid concept of fuzzy sets and rough sets was introduced by Dubois and Prade in 1990 [1]. To provide a more general concept of fuzzy rough sets, Intan and Mukaidono [5,6] discussed and proposed a generalization of fuzzy rough sets in 2002. The concept of generalized fuzzy rough sets is derived from a fuzzy covering of the universal set of objects. First, it is necessary to define fuzzy covering before discussing how to generalize the fuzzy rough sets. In the relation to the fuzzy conditional probability relations that have a weaker symmetric property, the property can be used to provide two asymmetric similarity classes of a particular object x as a foundation of generating two asymmetric coverings, as given by the following definition.

Definition 5 Let U be a non-empty universal set of objects, and R be a fuzzy conditional probability relation on U. For any object $x \in U$, R_x^s and R_x^p are defined fuzzy similarity class that supports x and fuzzy similarity class supported by x, respectively as follows.

$$\forall y \in U, R_x^s(y) = R(x,y), \qquad (6)$$

$$\forall y \in U, R_x^p(y) = R(y,x) \qquad (7)$$

where $R(x,y) \in [0,1]$ be degree y that supports x or the degree y is similar to x as calculated by the fuzzy conditional probability.

Here, $R_x^s(y)$ and $R_x^p(y)$ are also interpreted as membership degree of y in R_x^s and R_x^p, respectively. By using both asymmetric fuzzy similarity classes, two asymmetric fuzzy coverings of the universe can be generated as follows.

$$\Psi^s = \{R_x^s | x \in U\}, \qquad (8)$$

$$\Psi^p = \{R_x^p | x \in U\}, \qquad (9)$$

Fuzzy covering might be regarded as a case of *fuzzy granularity* in which its similarity class is a fuzzy set.

Now, it is time to discuss how to generalize fuzzy rough sets. As mentioned before, Intan and Mukaidono extended the concept of fuzzy rough sets induced by asymmetric fuzzy covering [6]. The proposed concept of fuzzy rough sets provided more generalized concept of fuzzy rough sets than the initial concept of fuzzy rough sets. Extended concept of fuzzy rough sets as proposed in [6] is formally defined as follows.

Definition 6 Let U be a non-empty universal set of objects, and A be a given fuzzy set on U. Two oriented generalizations of fuzzy rough sets (namely, *object-oriented generalization* and *class-oriented generalization*) are approximated on fuzzy covering Ψ^s as follows.

(i) *object-oriented generalization*:

$$Lo(A)_e^1(x) = \inf_{\{y \in U | R_x^s(y) > 0\}}\{\min[R_x^s(y), A(y)]\}, x \in U, \qquad (10)$$

$$Up(A)_e^1(x) = \sup_{\{y \in U | R_x^s(y) > 0\}}\{\min[R_x^s(y), A(y)]\}, x \in U, \qquad (11)$$

(ii) *class-oriented generalization, for $y \in U$*:

$$Lo(A)_{cm}^1(y) = \inf_{\{x \in U | R_x^s(y) > 0\}}\{\inf_{\{z \in U | R_x^s(z) > 0\}}\{\min[R_x^s(z), A(z)]\}\}, \qquad (12)$$

$$Up(A)_{cm}^1(y) = \inf_{\{x \in U | R_x^s(y) > 0\}}\{\sup_{\{z \in U | R_x^s(z) > 0\}}\{\min[R_x^s(z), A(z)]\}\}, \qquad (13)$$

$$Lo(A)_{cM}^1(y) = \sup_{\{x \in U | R_x^s(y) > 0\}}\{\inf_{\{z \in U | R_x^s(z) > 0\}}\{\min[R_x^s(z), A(z)]\}\}, \qquad (14)$$

$$Up(A)_{cM}^1(y) = \sup_{\{x \in U | R_x^s(y) > 0\}}\{\sup_{\{z \in U | R_x^s(z) > 0\}}\{\min[R_x^s(z), A(z)]\}\}, \qquad (15)$$

Where $Lo(A)_e^1(x)$ and $Up(A)_e^1(x)$ are grades of membership of x in $Lo(A)_e^1(x)$ and $Up(A)_e^1(x)$, respectively. Similarly, $Lo(A)_{c*}^1(y)$ and $Up(A)_{c*}^1(y)$ are grades of membership of y in $Lo(A)_{c*}^1(y)$ and $Up(A)_{c*}^1(y)$, respectively (note: $* \in \{m, M\}$).

From Definition 5, it is seen that $R_y^p(x) = R_x^s(y)$ by which all equations in Definition 6 may be represented using $R_y^p(x)$ as follows:

(i) *object-oriented generalization:*

$$Lo(A)_e^1(x) = \inf_{\{y \in U | R_y^p(x) > 0\}}\{\min[R_y^p(x), A(y)]\}, x \in U, \qquad (16)$$

$$Up(A)^1_e(x) = \sup_{\{y\in U|R^p_y(x)>0\}}\{\min[R^p_y(x), A(y)]\}, x \in U,$$
(17)

(ii) *class-oriented generalization, for* $y \in U$:

$$Lo(A)^1_{cm}(y) = \inf_{\{x\in U|R^p_y(x)>0\}}\{\inf_{\{z\in U|R^p_z(x)>0\}}\{\min[R^p_z(x), A(z)]\}\},$$
(18)

$$Up(A)^1_{cm}(y) = \inf_{\{x\in U|R^p_y(x)>0\}}\{\sup_{\{z\in U|R^p_z(x)>0\}}\{\min[R^p_z(x), A(z)]\}\},$$
(19)

$$Lo(A)^1_{cM}(y) = \sup_{\{x\in U|R^p_y(x)>0\}}\{\inf_{\{z\in U|R^p_z(x)>0\}}\{\min[R^p_z(x), A(z)]\}\},$$
(20)

$$Up(A)^1_{cM}(y) = \sup_{\{x\in U|R^p_y(x)>0\}}\{\sup_{\{z\in U|R^p_z(x)>0\}}\{\min[R^p_z(x), A(z)]\}\},$$
(21)

Obviously, $Lo(A)^1_e$ and $Up(A)^1_e$ as well as $Lo(A)^1_{c*}$ and $Up(A)^1_{c*}$ are considered as fuzzy sets, where we have some relations and properties such as $\forall y \in U$,

$$Lo(A)^1_e(y) \le A(y) \le Up(A)^1_e(y),$$

$$Lo(A)^1_{cm}(y) \le Lo(A)^1_{cM}(y) \le A(y) \le Up(A)^1_{cm}(y) \le Up(A)^1_{cM}(y).$$

Moreover, some relationships between lower and upper approximations based on object-oriented generalization and class-oriented generalization is represented by $Lo(A)^1_e(y) \le Lo(A)^1_{cM}(y)$ and $Up(A)^1_e(y) \le Up(A)^1_{cM}(y)$, where relation between $Lo(A)^1_e(y)$ and $Lo(A)^1_{cm}(y)$ as well as relation between $Up(A)^1_e(x)$ and $Up(A)^1_{cm}(y)$ cannot be inquired. Similar to Definition 6, when fuzzy set A is approximated on fuzzy covering Ψ^p, it will provide distinct generalized fuzzy rough sets as given by the following equations.

(i) *object-oriented generalization:*

$$Lo(A)^2_e(x) = \inf_{\{y\in U|R^p_x(y)>0\}}\{\min[R^p_x(y), A(y)]\}, x \in U,$$
(22)

$$Up(A)^2_e(x) = \sup_{\{y\in U|R^p_x(y)>0\}}\{\min[R^p_x(y), A(y)]\}, x \in U,$$
(23)

(ii) *class-oriented generalization, for* $y \in U$:

$$Lo(A)^2_{cm}(y) = \inf_{\{x\in U|R^p_x(y)>0\}}\{\inf_{\{z\in U|R^p_x(z)>0\}}\{\min[R^p_x(z), A(z)]\}\},$$
(24)

$$Up(A)^2_{cm}(y) = \inf_{\{x\in U|R^p_x(y)>0\}}\{\sup_{\{z\in U|R^p_x(z)>0\}}\{\min[R^p_x(z), A(z)]\}\},$$
(25)

$$Lo(A)^2_{cM}(y) = \sup_{\{x\in U|R^p_x(y)>0\}}\{\inf_{\{z\in U|R^p_x(z)>0\}}\{\min[R^p_x(z), A(z)]\}\},$$
(26)

$$Up(A)^2_{cM}(y) = \sup_{\{x\in U|R^p_x(y)>0\}}\{\sup_{\{z\in U|R^p_x(z)>0\}}\{\min[R^p_x(z), A(z)]\}\},$$
(27)

Similarly, by considering $R^p_x(y) = R^s_y(x)$, equations (22) to (27) might be reformulated by the following equations.

(i) *object-oriented generalization:*

$$Lo(A)^2_e(x) = \inf_{\{y\in U|R^s_y(x)>0\}}\{\min[R^s_y(x), A(y)]\}, x \in U,$$
(28)

$$Up(A)^2_e(x) = \sup_{\{y\in U|R^s_y(x)>0\}}\{\min[R^s_y(x), A(y)]\}, x \in U,$$
(29)

(ii) *class-oriented generalization, for* $y \in U$:

$$Lo(A)^2_{cm}(y) = \inf_{\{x\in U|R^s_y(x)>0\}}\{\inf_{\{z\in U|R^s_z(x)>0\}}\{\min[R^s_z(x), A(z)]\}\},$$
(30)

$$Up(A)^2_{cm}(y) = \inf_{\{x\in U|R^s_y(x)>0\}}\{\sup_{\{z\in U|R^s_z(x)>0\}}\{\min[R^s_z(x), A(z)]\}\},$$
(31)

$$Lo(A)^2_{cM}(y) = \sup_{\{x\in U|R^s_y(x)>0\}}\{\inf_{\{z\in U|R^s_z(x)>0\}}\{\min[R^s_z(x), A(z)]\}\},$$
(32)

$$Up(A)^2_{cM}(y) = \sup_{\{x\in U|R^s_y(x)>0\}}\{\sup_{\{z\in U|R^s_z(x)>0\}}\{\min[R^s_z(x), A(z)]\}\},$$
(33)

IV. More Generalization of Fuzzy Rough Sets

Besides the above generalizations of fuzzy rough sets as proposed in [5,6], this paper introduces a new concept of generalized fuzzy rough sets by focusing on objects in the given fuzzy set as an extended version of [7]. The reason behind the idea is that all objects in the given fuzzy set should be regarded as centers of approximating its fuzzy rough sets. Formally, the concept of the generalized fuzzy rough sets is defined as follows.

Definition 7 Let U be a non-empty universal set of objects, and A be a given fuzzy set on U. Generalized fuzzy rough sets of A on fuzzy covering Ψ^s is simply calculated by the following equations.
For $x \in U$,

$$Lo(A)^1_f(x) = \begin{cases} \inf_{\{y\in U|R^s_x(y)>0\}}\{\min[R^s_x(y), A(y)]\}, & A(x) > 0 \\ 0, & A(x) = 0 \end{cases}.$$
(34)

For $y \in U$,

$$Up(A)^1_{fm}(y) = \inf_{\{x\in U|(A(x)>0)\wedge(R^s_x(y)>0)\}}\{\sup_{\{z\in U|R^s_x(z)>0\}}\{\min[R^s_x(z), A(z)]\}\} \quad (35)$$

$$Up(A)^1_{fM}(y) = \sup_{\{x\in U|(A(x)>0)\wedge(R^s_x(y)>0)\}}\{\sup_{\{z\in U|R^s_x(z)>0\}}\{\min[R^s_x(z), A(z)]\}\} \quad (36)$$

Definition 7 introduces triple approximations (one lower and two upper approximations) of generalized fuzzy rough sets as shown in (34), (35) and (36). Relation among these three approximations is given by:

$$Lo(A)^1_f(x) \le A(x) \le Up(A)^1_{fm}(x) \le Up(A)^1_{fM}(x).$$

Related to the previous generalized fuzzy rough sets, it can be examined that,

$$Lo(A)^1_f(x) = Lo(A)^1_e(x).$$

However, it can be verified that it is faster to find $Lo(A)^1_f(x)$ than $Lo(A)^1_e(x)$, since the process of constructing $Lo(A)_f^1(x)$ only considers all objects in A. Compared to other class-oriented generalization, another relation is given by:

$$Up(A)_{fM}^1(y) \leq Up(A)_{cM}^1(y),$$

$$Up(A)_{cm}^1(y) \leq Up(A)_{fm}^1(y).$$

Based on the previous explanation of $R_y^p(x) = R_x^s(y)$, equations in (34), (35) and (36) are reformulated to be the following equations.
Similar to Definition 7, when fuzzy set A is approximated on fuzzy covering Ψ^p, the approximation of A will provide other distinct fuzzy rough sets as given by the following equations.

For $x \in U$,

$$Lo(A)_f^1(x) = \begin{cases} \inf_{\{y \in U | R_y^p(x) > 0\}} \{\min[R_y^p(x), A(y)]\}, & A(x) > 0 \\ 0, & A(x) = 0 \end{cases}. \quad (37)$$

For $y \in U$,

$$Up(A)_{fm}^1(y) = \inf_{\{x \in U | (A(x)>0) \wedge (R_y^p(x)>0)\}} \{\sup_{\{z \in U | R_z^p(x)>0\}} \{\min\,[R_z^p(x), A(z)]\}\} \quad (38)$$

$$Up(A)_{fM}^1(y) = \sup_{\{x \in U | (A(x)>0) \wedge (R_y^p(x)>0)\}} \{\sup_{\{z \in U | R_z^p(x)>0\}} \{\min\,[R_z^p(x), A(z)]\}\} \quad (39)$$

Similar to the previous explanation that actually $R_y^s(x) = R_x^p(y)$, equations (40), (41) and (42) are reformulated to be the following equations.
For $x \in U$,

$$Lo(A)_f^2(x) = \begin{cases} \inf_{\{y \in U | R_x^p(y) > 0\}} \{\min[R_x^p(y), A(y)]\}, & A(x) > 0 \\ 0, & A(x) = 0 \end{cases}. \quad (40)$$

For $y \in U$,

$$Up(A)_{fm}^2(y) = \inf_{\{x \in U | (A(x)>0) \wedge (R_x^p(y)>0)\}} \{\sup_{\{z \in U | R_x^p(z)>0\}} \{\min\,[R_x^p(z), A(z)]\}\} \quad (41)$$

$$Up(A)_{fM}^2(y) = \sup_{\{x \in U | (A(x)>0) \wedge (R_x^p(y)>0)\}} \{\sup_{\{z \in U | R_x^p(z)>0\}} \{\min\,[R_x^p(z), A(z)]\}\} \quad (42)$$

Similar to the previous explanation that actually $R_y^s(x) = R_x^p(y)$, equations (40), (41) and (42) are reformulated to be the following equations.
For $x \in U$,

$$Lo(A)_f^2(x) = \begin{cases} \inf_{\{y \in U | R_y^s(x) > 0\}} \{\min[R_y^s(x), A(y)]\}, & A(x) > 0 \\ 0, & A(x) = 0 \end{cases}. \quad (43)$$

For $y \in U$,

$$Up(A)_{fm}^2(y) = \inf_{\{x \in U | (A(x)>0) \wedge (R_y^s(x)>0)\}} \{\sup_{\{z \in U | R_z^s(x)>0\}} \{\min\,[R_z^s(x), A(z)]\}\} \quad (44)$$

$$Up(A)_{fM}^2(y) = \sup_{\{x \in U | (A(x)>0) \wedge (R_y^s(x)>0)\}} \{\sup_{\{z \in U | R_z^s(x)>0\}} \{\min\,[R_z^s(x), A(z)]\}\} \quad (45)$$

V. Illustrative Example

To be more understandable, it is necessary to demonstrate the proposed concept using a simple example. Given a binary information tables as shown in Table I, firstly it is necessary to construct two asymmetric fuzzy coverings, Ψ^s and Ψ^p. Since the objects are simply characterized by crisp attributes as represented by a binary information table, the degree of similarity between two objects is calculated dealing with the conditional probability relations using Equation (4). Furthermore, fuzzy similarity class of a given object is then constructed based on the degree of similarity between the object and others. The number of fuzzy similarity classes is the same as the number of objects. The fuzzy similarity classes are then used to construct the fuzzy covering of objects. There are two kinds of fuzzy similarity classes. They are the fuzzy similarity class that supports O_i and fuzzy similarity class supported by O_i, denoted by $R_{O_i}^s$ and $R_{O_i}^p$, respectively. These two kinds of fuzzy similarity classes are used to construct two different fuzzy coverings, Ψ^s and Ψ^p, as given as follows.

$\Psi^s = \{R_{O_1}^s, R_{O_2}^s, R_{O_3}^s, R_{O_4}^s, R_{O_5}^s, R_{O_6}^s, R_{O_7}^s\}$, where

$$R_{O_1}^s = \left\{\frac{1}{O_1}, \frac{3/5}{O_2}, \frac{1/3}{O_3}, \frac{2/3}{O_4}, \frac{2/4}{O_5}, \frac{2/5}{O_6}, \frac{2/5}{O_7}\right\},$$
$$R_{O_2}^s = \left\{\frac{3/4}{O_1}, \frac{1}{O_2}, \frac{0}{O_3}, \frac{2/3}{O_4}, \frac{2/4}{O_5}, \frac{2/5}{O_6}, \frac{3/5}{O_7}\right\},$$
$$R_{O_3}^s = \left\{\frac{1/4}{O_1}, \frac{0}{O_2}, \frac{1}{O_3}, \frac{1/3}{O_4}, \frac{2/4}{O_5}, \frac{3/5}{O_6}, \frac{2/5}{O_7}\right\},$$
$$R_{O_4}^s = \left\{\frac{2/4}{O_1}, \frac{2/5}{O_2}, \frac{1/3}{O_3}, \frac{1}{O_4}, \frac{3/4}{O_5}, \frac{3/5}{O_6}, \frac{2/5}{O_7}\right\},$$
$$R_{O_5}^s = \left\{\frac{2/4}{O_1}, \frac{2/5}{O_2}, \frac{2/3}{O_3}, \frac{1}{O_4}, \frac{1}{O_5}, \frac{4/5}{O_6}, \frac{2/5}{O_7}\right\},$$
$$R_{O_6}^s = \left\{\frac{2/4}{O_1}, \frac{2/5}{O_2}, \frac{1}{O_3}, \frac{1}{O_4}, \frac{1}{O_5}, \frac{1}{O_6}, \frac{3/5}{O_7}\right\},$$
$$R_{O_7}^s = \left\{\frac{2/4}{O_1}, \frac{3/5}{O_2}, \frac{2/3}{O_3}, \frac{2/3}{O_4}, \frac{2/4}{O_5}, \frac{2/5}{O_6}, \frac{1}{O_7}\right\}.$$

$\Psi^p = \{R_{O_1}^p, R_{O_2}^p, R_{O_3}^p, R_{O_4}^p, R_{O_5}^p, R_{O_6}^p, R_{O_7}^p\}$, where

$$R_{O_1}^p = \left\{\frac{1}{O_1}, \frac{3/4}{O_2}, \frac{1/4}{O_3}, \frac{2/4}{O_4}, \frac{2/4}{O_5}, \frac{2/4}{O_6}, \frac{2/4}{O_7}\right\},$$
$$R_{O_2}^p = \left\{\frac{3/5}{O_1}, \frac{1}{O_2}, \frac{0}{O_3}, \frac{2/5}{O_4}, \frac{2/5}{O_5}, \frac{2/5}{O_6}, \frac{3/5}{O_7}\right\},$$
$$R_{O_3}^p = \left\{\frac{1/3}{O_1}, \frac{0}{O_2}, \frac{1}{O_3}, \frac{1/3}{O_4}, \frac{2/3}{O_5}, \frac{1}{O_6}, \frac{2/3}{O_7}\right\},$$
$$R_{O_4}^p = \left\{\frac{2/3}{O_1}, \frac{2/3}{O_2}, \frac{1/3}{O_3}, \frac{1}{O_4}, \frac{1}{O_5}, \frac{1}{O_6}, \frac{2/3}{O_7}\right\},$$
$$R_{O_5}^p = \left\{\frac{2/4}{O_1}, \frac{2/4}{O_2}, \frac{2/4}{O_3}, \frac{3/4}{O_4}, \frac{1}{O_5}, \frac{1}{O_6}, \frac{2/4}{O_7}\right\},$$
$$R_{O_6}^p = \left\{\frac{2/5}{O_1}, \frac{2/5}{O_2}, \frac{3/5}{O_3}, \frac{3/5}{O_4}, \frac{4/5}{O_5}, \frac{1}{O_6}, \frac{3/5}{O_7}\right\},$$
$$R_{O_7}^p = \left\{\frac{2/5}{O_1}, \frac{3/5}{O_2}, \frac{2/5}{O_3}, \frac{2/5}{O_4}, \frac{2/5}{O_5}, \frac{3/5}{O_6}, \frac{1}{O_7}\right\}.$$

All values of fuzzy coverings, Ψ^s and Ψ^p, might be represented using $|U| \times |U|$ matrices, M_s and M_p, respectively. Related to the above values of Ψ^s and Ψ^p, matrices M_s and M_p are given by:

$$M_s = \begin{bmatrix} 1 & 3/5 & 1/3 & 2/3 & 2/4 & 2/5 & 2/5 \\ 3/4 & 1 & 0 & 2/3 & 2/4 & 2/5 & 3/5 \\ 1/4 & 0 & 1 & 1/3 & 2/4 & 3/5 & 2/5 \\ 2/4 & 2/5 & 1/3 & 1 & 3/4 & 3/5 & 2/5 \\ 2/4 & 2/5 & 2/3 & 1 & 1 & 4/5 & 2/5 \\ 2/4 & 2/5 & 1 & 1 & 1 & 1 & 3/5 \\ 2/4 & 3/5 & 2/3 & 2/3 & 2/4 & 3/5 & 1 \end{bmatrix}$$

$$M_p = \begin{bmatrix} 1 & 3/4 & 1/4 & 2/4 & 2/4 & 2/4 & 2/4 \\ 3/5 & 1 & 0 & 2/5 & 2/5 & 2/5 & 3/5 \\ 1/3 & 0 & 1 & 1/3 & 2/3 & 1 & 2/3 \\ 2/3 & 2/3 & 1/3 & 1 & 1 & 1 & 2/3 \\ 2/4 & 2/4 & 2/4 & 3/4 & 1 & 1 & 2/4 \\ 2/5 & 2/5 & 3/5 & 3/5 & 4/5 & 1 & 3/5 \\ 2/5 & 3/5 & 2/5 & 2/5 & 2/5 & 3/5 & 1 \end{bmatrix}$$

Clearly, it can be seen that $M_p = (M_s)^T$ (transpose matrix). Now, given A be a fuzzy set on set of objects as given by:

$$A = \left\{\frac{2/6}{O_1}, \frac{5/6}{O_2}, \frac{1}{O_4}, \frac{1}{O_5}, \frac{1/6}{O_6}, \frac{2/6}{O_7}\right\}$$

By (10), it can be calculated that the only non-zero result is $x = O_2$ as given by the following calculation.

$$Lo(A)_e^1(x) = \inf_{\{y \in U | R_x^s(y) > 0\}} \{\min\,[R_x^s(y), A(y)]\}, x \in U,$$

$$Lo(A)_e^1(O_1) = \inf\{\min[R_{O_1}^s(O_1), A(O_1)], \min[R_{O_1}^s(O_2), A(O_2)], \min[R_{O_1}^s(O_3), A(O_3)], \min[R_{O_1}^s(O_4), A(O_4)], \min[R_{O_1}^s(O_5), A(O_5)], \min[R_{O_1}^s(O_6), A(O_6)], \min[R_{O_1}^s(O_7), A(O_7)]\},$$

$$Lo(A)_e^1(O_1) = \inf\{\min[1, {}^2/_6], \min[{}^3/_5, {}^5/_6], \min[{}^1/_3, 0], \min[{}^2/_3, 1], \min[{}^2/_4, 1], \min[{}^2/_5, {}^1/_6], \min[{}^2/_5, {}^2/_6]\} = 0,$$

$$Lo(A)_e^1(O_2) = \inf\{\min[R_{O_2}^s(O_1), A(O_1)], \min[R_{O_2}^s(O_2), A(O_2)], \min[R_{O_2}^s(O_4), A(O_4)], \min[R_{O_2}^s(O_5), A(O_5)], \min[R_{O_2}^s(O_6), A(O_6)], \min[R_{O_2}^s(O_7), A(O_7)]\},$$

$$Lo(A)_e^1(O_2) = \inf\{\min[{}^3/_4, {}^2/_6], \min[1, {}^5/_6], \min[{}^2/_3, 1], \min[{}^2/_4, 1], \min[{}^2/_5, {}^1/_6], \min[{}^3/_5, {}^2/_6]\} = {}^1/_6,$$

$$Lo(A)_e^1(O_3) = \inf\{\min[R_{O_3}^s(O_1), A(O_1)], \min[R_{O_3}^s(O_2), A(O_2)], \min[R_{O_3}^s(O_4), A(O_4)], \min[R_{O_3}^s(O_5), A(O_5)], \min[R_{O_3}^s(O_6), A(O_6)], \min[R_{O_3}^s(O_7), A(O_7)]\},$$

$$Lo(A)_e^1(O_3) = \inf\{\min[{}^1/_4, {}^2/_6], \min[1,0], \min[{}^1/_3, 1], \min[{}^2/_3, 1], \min[{}^3/_5, {}^1/_6], \min[{}^2/_5, {}^2/_6]\} = 0,$$

$$Lo(A)_e^1(O_4) = \inf\{\min[R_{O_4}^s(O_1), A(O_1)], \min[R_{O_4}^s(O_2), A(O_2)], \min[R_{O_4}^s(O_3), A(O_3)], \min[R_{O_4}^s(O_4), A(O_4)], \min[R_{O_4}^s(O_5), A(O_5)], \min[R_{O_4}^s(O_6), A(O_6)], \min[R_{O_4}^s(O_7), A(O_7)]\},$$

$$Lo(A)_e^1(O_4) = \inf\{\min[{}^2/_4, {}^2/_6], \min[{}^2/_5, {}^5/_6], \min[{}^1/_3, 0], \min[1,1], \min[{}^3/_4, 1], \min[{}^3/_5, {}^1/_6], \min[{}^2/_5, {}^2/_6]\} = 0,$$

$$Lo(A)_e^1(O_5) = \inf\{\min[R_{O_5}^s(O_1), A(O_1)], \min[R_{O_5}^s(O_2), A(O_2)], \min[R_{O_5}^s(O_3), A(O_3)], \min[R_{O_5}^s(O_4), A(O_4)], \min[R_{O_5}^s(O_5), A(O_5)], \min[R_{O_5}^s(O_6), A(O_6)], \min[R_{O_5}^s(O_7), A(O_7)]\},$$

$$Lo(A)_e^1(O_5) = \inf\{\min[{}^2/_4, {}^2/_6], \min[{}^2/_5, {}^5/_6], \min[{}^2/_3, 0], \min[1,1], \min[1,1], \min[{}^4/_5, {}^1/_6], \min[{}^2/_5, {}^2/_6]\} = 0,$$

$$Lo(A)_e^1(O_6) = \inf\{\min[R_{O_6}^s(O_1), A(O_1)], \min[R_{O_6}^s(O_2), A(O_2)], \min[R_{O_6}^s(O_3), A(O_3)], \min[R_{O_6}^s(O_4), A(O_4)], \min[R_{O_6}^s(O_5), A(O_5)], \min[R_{O_6}^s(O_6), A(O_6)], \min[R_{O_6}^s(O_7), A(O_7)]\},$$

$$Lo(A)_e^1(O_6) = \inf\{\min[{}^2/_4, {}^2/_6], \min[{}^2/_5, {}^5/_6], \min[1,0], \min[1,1], \min[1,1], \min[1, {}^1/_6], \min[{}^3/_5, {}^2/_6]\} = 0,$$

$$Lo(A)_e^1(O_7) = \inf\{\min[R_{O_7}^s(O_1), A(O_1)], \min[R_{O_7}^s(O_2), A(O_2)], \min[R_{O_7}^s(O_3), A(O_3)], \min[R_{O_7}^s(O_4), A(O_4)], \min[R_{O_7}^s(O_5), A(O_5)], \min[R_{O_7}^s(O_6), A(O_6)], \min[R_{O_7}^s(O_7), A(O_7)]\},$$

$$Lo(A)_e^1(O_4) = \inf\{\min[{}^2/_4, {}^2/_6], \min[{}^3/_5, {}^5/_6], \min[{}^2/_3, 0], \min[{}^2/_3, 1], \min[{}^2/_4, 1], \min[{}^2/_5, {}^1/_6], \min[1, {}^2/_6]\} = 0.$$

From the above results of calculation, $Lo(A)_e^1$ is given by the following equation.

$$Lo(A)_e^1 = \left\{\frac{1/6}{O_2}\right\}.$$

Similarly, result of upper approximation based on object-oriented generalization as calculated by (11) is given as follow.

$$Up(A)_e^1 = \left\{\frac{2/3}{O_1}, \frac{5/6}{O_2}, \frac{2/4}{O_3}, \frac{1}{O_4}, \frac{1}{O_5}, \frac{1}{O_6}, \frac{2/3}{O_7}\right\}.$$

Generalization of fuzzy rough sets based on class-oriented generalization are constructed by (12), (13), (14) and (15) as following results.

$$Lo(A)_{cm}^1 = \{\}.$$

$$Up(A)_{cm}^1 = \left\{\frac{1/2}{O_1}, \frac{2/3}{O_2}, \frac{1/2}{O_3}, \frac{1/2}{O_4}, \frac{1/2}{O_5}, \frac{1/2}{O_6}, \frac{1/2}{O_7}\right\}.$$

$$Lo(A)_{cM}^1 = \left\{\frac{1/6}{O_1}, \frac{1/6}{O_2}, \frac{1/6}{O_3}, \frac{1/6}{O_4}, \frac{1/6}{O_5}, \frac{1/6}{O_6}, \frac{1/6}{O_7}\right\}.$$

$$Up(A)_{cM}^1 = \left\{\frac{1}{O_1}, \frac{1}{O_2}, \frac{1}{O_3}, \frac{1}{O_4}, \frac{1}{O_5}, \frac{1}{O_6}, \frac{1}{O_7}\right\}.$$

Finally, Definition 7 provides other approximations of fuzzy rough sets as calculated by (34), (35) and (36) as follows. By (34), it can be found that $Lo(A)_f^1 = Lo(A)_e^1 = \left\{\frac{1/6}{O_2}\right\}$.
By (35 and (36),

$$Lo(A)_{fm}^1 = \left\{\frac{2/3}{O_1}, \frac{2/3}{O_2}, \frac{2/3}{O_3}, \frac{2/3}{O_4}, \frac{2/3}{O_5}, \frac{2/3}{O_6}, \frac{2/3}{O_7}\right\},$$

$$Up(A)_{fM}^1 = \left\{\frac{1}{O_1}, \frac{1}{O_2}, \frac{1}{O_3}, \frac{1}{O_4}, \frac{1}{O_5}, \frac{1}{O_6}, \frac{1}{O_7}\right\}.$$

In the same way, other approximations of generalized fuzzy rough sets based on Ψ^p may provide different interesting results. The usage of fuzzy coverings either Ψ^sor Ψ^p depends on the contextual application.

The above illustrative example used binary information table. In this case, fuzzy information table as shown in Table II can also be used as a generalization of binary information table. In case of using fuzzy information table, degree of similarity between two fuzzy object can be calculated using fuzzy conditional probability relation as defined in Definition 4. Since the objects are fuzzy sets, fuzzy similarity classes such as R_x^s as well as R_x^p are considered as level-2 fuzzy sets. Moreover, generalized fuzzy rough sets provide several approximations of membership degrees started from lower to upper approximations in which they are regarded as interval valued fuzzy sets. Thus, it can be said that generalized fuzzy rough sets are one of the concepts than can be used to develop level-2 interval valued fuzzy sets from data. Furthermore, several pairs of approximations proposed in this paper may also be combined or used together to develop not only interval valued fuzzy sets, but also type-2 fuzzy sets.

VI. Conclusion

This paper discussed the concept of generalized fuzzy rough sets dealing with weak fuzzy similarity relations. The weak fuzzy similarity relation is considered as a generalization of fuzzy similarity relation in representing a more realistic relationship between two objects. Here, the weak fuzzy similarity relation has conditional symmetry and conditional transitivity that are weaker than symmetric and transitive properties in the fuzzy similarity relation. Conditional symmetry in the weak fuzzy similarity relation is actually an asymmetric property. Consequently, the weak fuzzy similarity relations provide two asymmetric fuzzy similarity classes as basic elements in constructing two asymmetric fuzzy coverings. Some concepts of generalized fuzzy rough sets were recalled and reformulated more detail. In addition, a new concept of generalized fuzzy rough sets was introduced also dealing with the weak fuzzy similarity relations. In order to understand the concept well, an illustrative example was given to demonstrate how to calculate and generate the generalized fuzzy rough sets simply from binary information table using conditional probability relation. Finally, the generalized fuzzy rough sets proposed and discussed in this paper may be regarded as a solution how to generate the level-2 interval valued fuzzy sets directly from the data. Moreover, through combining several pairs of proposed approximations of the generalized fuzzy rough sets, it is possible to provide the level-2 type-2 fuzzy sets as an extended of the level-2 interval valued fuzzy sets.

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