Primal-Dual Approximation Algorithms

We just saw how the primal-dual schema permits sometimes *designing* efficient combinatorial algorithms for solving certain problems. We will now see an example of how a related technique can sometimes be used to design efficient *approximation* algorithms

The major tool that we will use will be the
RELAXED Complementary Slackness conditions

The problem we examine will again be
weighted set-cover.

Recall that given canonical primal

$$\begin{array}{lll} \text{minimize} & \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & a_i' x \geq b_i, & i = 1, \ldots, m \\ & x_j \geq 0, & j = 1, \ldots, n \end{array}$$

the dual is

$$\begin{array}{lll} \text{maximize} & \sum_{i=1}^{m} b_i \pi_i & \\ \text{subject to} & \pi A_j \leq c_j, & j = 1, \ldots, n \\ & \pi_i \geq 0, & i = 1, \ldots, m \end{array}$$

Theorem (Complementary Slackness):

Let x and π respectively be primal and dual feasible solutions. Then x and π are *both* optimal if and only if all of the following conditions are satisfied.

Primal Complementary Slackness conditions

$$\forall 1 \leq j \leq n: \text{ either } x_j = 0 \text{ or } \pi' A_j = c_j$$

Dual Complementary Slackness conditions

$$\forall 1 \leq i \leq m: \text{ either } \pi_i = 0 \text{ or } a_i' x = b_i$$

Theorem (RELAXED Complementary Slackness): **Let x and y respectively be primal and dual feasible solutions. Suppose further that for some $\alpha > 1$, x and y satisfy all of**

Primal Complementary Slackness conditions

$$\forall 1 \leq j \leq n: \text{ either } x_j = 0 \text{ or } \pi A_j = c_j$$

RELAXED Dual C.S. conditions

$$\forall 1 \leq i \leq m: \text{ either } \pi_i = 0 \text{ or } \mathbf{a_i'x} \leq \alpha \mathbf{b_i}$$

Then

$$\sum_{j=1}^{n} c_j x_j \leq \alpha \cdot \sum_{i=1}^{m} b_i \pi_i$$

Proof:

$$\sum_{j=1}^{n} c_j x_j = \sum_{j=1}^{n} (\pi A_j)\, x_j = \pi A x = \sum_{i=1}^{m} (a_i' x)\, \pi_i \leq \alpha \sum_{i=1}^{m} b_i \pi_i.$$

Given such an x, π we immediately know that x is within α of OPT, the minimum cost optimum solution.

Recall *Weighted Set Cover problem* where each set F has a *weight* $Cost(F) = C(F)$, and the problem is to find a *Set Cover* of $\mathcal{C}$ of *Minimum Weight*, $Cost(\mathcal{C}) = \sum_{F \in \mathcal{C}} C(F)$.

For example $X = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{F}$ contains the subsets

$$
\begin{array}{lcll}
F_1 & = & \{1,3,5\}; & C(F_1) = 1 \\
F_2 & = & \{2,3,6\}; & C(F_2) = 1 \\
F_3 & = & \{2,5,6\}; & C(F_1) = 3 \\
F_4 & = & \{2,3,4,6\}; & C(F_1) = 5 \\
F_5 & = & \{1,4\}; & C(F_5) = 1
\end{array}
$$

For example $\mathcal{C} = \{F_1, F_4\}$ is a minimal cardinality solution but not a minimum weight one. $\mathcal{C} = \{F_1, F_2, F_5\}$ is a minimum weight solution.

We previously saw that weighted-set-cover is NP-Hard but developed an H_n approximation algorithm where $n = |X|$ and $H_n = \sum_{i=1}^{n} \frac{1}{i} \sim \ln n$.

This means that, for every input, our algorithm generated a cover $\mathcal{C}$ such that

$$Cost(\mathcal{C}) \leq H_n \cdot OPT$$

where OPT is the cost of the real optimal solution. Duality theory was used in our proof to lower bound OPT.

Now let the *frequency* of element e be

$$\text{freq}(e) = \{F \in \mathcal{F} \,:\, e \in F\}.$$

Let $f = \max_{e \in U} \text{freq}(e)$ be the max number of sets an element can appear in. We will now use a primal-dual schema based on the relaxed complementary-slackness conditions to design a f-approximation algorithm for set-cover.

While an f-approximation algorithm for set-cover might not appear interesting, consider following application.

Let $G = (V, E)$ be a graph.
A *vertex cover* of G is a subset $V' \subseteq V$ such that *every edge in E has at least one endpoint in V'*. Finding a mimimum-cardinality vertex cover is an interesting and NP-hard problem. A straightforward generaliztion of the problem is to assign a cost $C(v)$ to every $v \in V$ and set $cost(V') = \sum_{v \in V'} c(v)$. Finding a min-weight vertex cover is also NP-hard.

Now create a weighed set cover problem with universe $X = E$ and one set corresponding to each $v \in V$. For $e \in E$ write $e = (e_x, e_y)$ and set

$$F_v = \{e \in E \,:\, e_x = v \text{ or } e_y = v\}, \quad C(F_v) = c(v)$$

Then V' is a vertex cover of G iff $\cup_{v \in V'} \{F_v\}$ is a set cover of X. Furthermore, $c(V')$ is equal to the cost of the associated set-cover.

Finally, note that, *since edge e appears in exactly two sets F_v*, $f = 2$.

So, an f-approximation algorithm for set-cover yields a 2-approximation algorithm for vertex cover.

- Our general approach will be to start with some **primal-infeasible** and **dual-feasible** solution and to iterate.

- During each iteration we will improve the feasibility of the primal and the optimality of the dual (always keeping the dual solution feasible).

- At the end we will produce both a **feasible-primal** and **feasible-dual** solution that satisfy the relaxed complimentary slackness conditions.

- The cost of the dual solution will be a lower bound on the cost of OPT.

- This will then give an α-approximation algorithm for the primal problem.

Recall the set-cover LP formulations:
The integer LP will be

Minimize $\sum_{F \in \mathcal{F}} C(F) x_F$
subject to conditions
$\forall e \in U, \qquad \sum_{e \in F} x_F \geq 1$
$\forall F \in \mathcal{F} \qquad x_F \in \{0, 1\}$

The relaxation of the LP is

Minimize $\sum_{F \in \mathcal{F}} C(F) x_F$
subject to conditions
$\forall e \in U, \qquad \sum_{e \in F} x_F \geq 1$
$\forall F \in \mathcal{F}, \qquad x_F \geq 0$

The *dual* of the relaxed LP is then

Maximize $\sum_{e \in U} y_e$
subject to conditions
$\forall F \in \mathcal{F}, \qquad \sum_{e \in F} y_e \leq C(F)$
$\forall e \in U, \qquad y_e \geq 0$

Primal:

Minimize $\sum_{F\in\mathcal{F}} C(F)x_F$

subject to conditions

$\forall e \in U, \qquad \sum_{e\in F} x_F \geq 1$

$\forall F \in \mathcal{F}, \qquad x_F \geq 0$

Dual:

Maximize $\sum_{e\in U} y_e$

subject to conditions

$\forall F \in \mathcal{F}, \qquad \sum_{e\in F} y_e \leq C(F)$

$\forall e \in U, \qquad y_e \geq 0$

Our schema will be to start with all of the $x_F = 0, y_e = 0$, and then iteratively change some of the x_F to 1 while also changing the y_e (but keeping y feasible). Setting $x_F = 1$ means that we put F in the cover.

At the end we will have constructed feasible solutions for both the primal and dual that satisfy the relaxed complementary slackness conditions with $\alpha = f$.

Primal C.S:

$$\forall F \in \mathcal{F} : x_F \neq 0 \Rightarrow \sum_{e: e \in F} y_e = C(F)$$

Relaxed Dual C.S:

$$\forall e : y_e \neq 0 \Rightarrow \sum_{F: e \in F} x_F \leq f \cdot 1 = f$$

We will say that F is *tight* if $\sum_{e: e \in F} y_e = C(F)$.
Our rule will be that we

Pick only tight sets for the cover

Note that, by definition, every x is covered at most f times.

Primal-Dual Set-Cover

1. Set $\forall F$, $x_F = 0$, $\forall e$, $y_e = 0$.
2. Until all elements are covered do
   Pick an uncovered element e, and increase y_e until some set becomes tight.
   Add all newly tight sets to the cover. by setting $x_F = 1$ for those sets.
3. Output the cover

In the algorithm an element e is *covered* at a given step if, at that time, there is an F in the current cover s.t. $e \in F$.

Theorem: The algorithm generates a feasible pair x, y that satisfies the relaxed complementary slackness conditions. The algorithm is therefore a f-approximation algorithm.

Proof Sketch: Algorithm starts with feasible y and x that satisfies the primal complementary slackness conditions with $\alpha = f$. At every step, changing y_e keeps y feasible and setting the new $x_F = 1$ keeps the primal c.s. conditions satisfied. At the end, every e is covered so the primal setting has become a feasible solution.