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Unhooking Circulant Graphs:
A Combinatorial Method for Counting Spanning Trees, Hamiltonian Cycles and other Parameters*

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Abstract

It has long been known that the number of spanning trees in n node circulant graphs with fixed jumps satisfies a bounded order, constant coefficient, recurrence relation in n. The proof of this fact was algebraic (evaluating products of eigenvalues of the graphs' adjacency matrices) and not combinatorial. In this paper we derive a straightforward combinatorial proof.

Instead of trying to decompose a large circulant graph into smaller ones, our technique is to instead work on *step-graphs*, the *unhooked* version of circulant graphs, construct a system of linear recurrence relations on the number of different types of forests in the step graph, and then express the number of spanning trees of the circulant graph in terms of the number of different types of forests of the step-graph.

This technique is very general and, unlike the algebraic methods previously employed, can also be used to enumerate other parameters of circulant graphs. We illustrate this by using it to prove that the number of Hamiltonian Cycles in a fixed-jump circulant graph also satisfies a bounded order, constant coefficient, recurrence relation in n.

1 Introduction

This paper was motivated by the desire to provide a *combinatorial* derivation of the recurrence relations on the number of spanning trees of circulant graphs.

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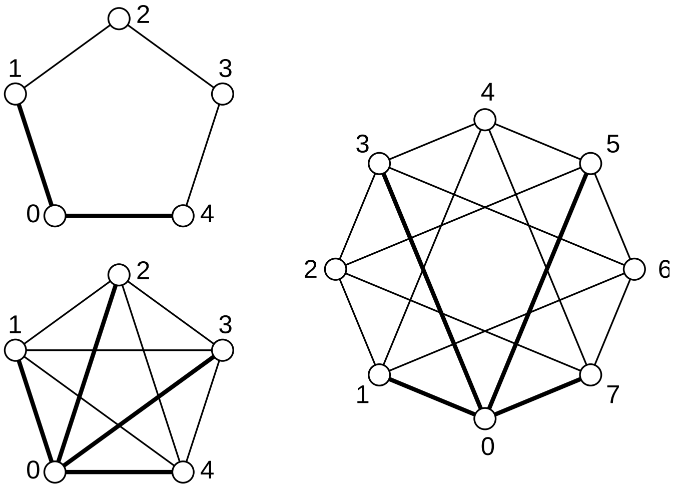

Figure 1: Three examples of circulant graphs; the cycle C_5^1, $C_5^{1,2}$ and $C_8^{1,3}$.

After developing such a derivation we are then able to extend the technique used in order to derive recurrence relations on other parameters of circulant graphs such as the number of Hamiltonian cycles.

We start with some definitions and background.

Definition 1 *The n-node* undirected circulant graph *with jumps $s_1, s_2, \ldots s_k$, is denoted by $C_n^{s_1,s_2,\cdots,s_k}$. This is the $2k$ regular graph*[1] *with n vertices labelled $\{0,\ 1,\ 2,\ \cdots, n-1\}$, such that each vertex i $(0 \le i \le n-1)$ is adjacent to $2k$ vertices $i \pm s_1,\ i \pm s_2,\ \cdots,\ i \pm s_k \bmod n$. Formally,*

$$C_n^{s_1,s_2,\ldots,s_k} = (V(n), E_C(n))$$

where

$$V(n) = \{0, 1, \ldots, n-1\} \quad \textit{and} \quad E_C(n) = \Big\{(i,j) : i-j \bmod n \in \{s_1, s_2, \ldots, s_k\}\Big\}.$$

(See Figure 1.) The simplest circulant graph is the n vertex cycle C_n^1. The next simplest is the *square of the cycle* $C_n^{1,2}$ in which every vertex is connected to its two neighbors and neighbor's neighbors. Circulant Graphs (sometimes known as "loop networks") are very well studied structures, in part because they model practical data connection networks [11, 3].

A *spanning tree* of G is a connected acyclic subgraph of G. For connected graph G, $T(G)$ denotes the number of spanning trees, in G. Counting $T(G)$ is a well

[1] If $\gcd(n, s_1, s_2, \cdots, s_k) > 1$ then the graph is disconnected and contains no spanning trees. Therefore, for the purposes of this paper, we assume that $\gcd(s_1, s_2, \cdots, s_k) = 1$, forcing the graph to be connected. Also note that if $n \le 2s_k$ it is possible that the graph is a *multigraph* with some repeated edges.

studied problem, both for its own sake and because it has practical implications for network reliability, e.g., [7, 8]. For any *fixed* graph G, Kirchhoff's *Matrix-Tree Theorem* [12] efficiently permits calculating $T(G)$ by evaluating a co-factor of the *Kirchoff matrix* of G (this essentially calculates the determinant of a matrix related to the adjacency matrix of G.)

The interesting problem is in calculating the number of spanning trees in graphs chosen from defined *classes* as a function of a parameter. When G is a circulant graph the behavior of $T(G)$ as a function of n has been well studied. The canonical result is that $T(C_n^{1,2}) = nF_n^2$, F_n the *Fibonacci* numbers, i.e., $F_n = F_{n-1}+F_{n-2}$ with $F_1 = F_2 = 1$. This was originally conjectured by Bedrosian [2] and subsequently proven by Kleitman and Golden [13]. The same formula was also conjectured by Boesch and Wang [5] (without the knowledge of [13]). Different proofs can been found in [1, 18, 6, 21, 16]. A formula for $T(C_n^{1,3})$ is given in [16]; formulas for $T(C_n^{1,3})$ and $T(C_n^{1,4})$ are provided in [20]. These formulas were extended by [18] and later [22] to prove the following general theorem: *For any fixed* $1 \le s_1 < s_2 < \cdots < s_k$,

$$T(C_n^{s_1,s_2,\cdots,s_k}) = na_n^2,$$

where a_n *satisfies a recurrence relation of order* 2^{s_k-1} *with constant coefficients.* Knowing the *existence* and *order* of the recurrence relation permits explicitly constructing it by using Kirchoff's theorem to evaluate $T(C_n^{s_1,s_2,\cdots,s_k})$ for $n = 1, 2, \ldots, 2^{s_k-1}$ and then solving for the coefficients of the recurrence relation.

With the exception of the original analysis of $T\left(C_n^{1,2}\right)$ in [13] all of the proofs above work by manipulating the evaluation of the determinant in Kirchoff's theorem. The formula for $T(C_n^{1,3})$ given in [16] manipulates the determinants directly to yield a system of recurrence relations. All of the other results use the following basic schema:

- Let $s_1, s_2, \ldots s_k$ be fixed.
- Find the eigenvalues of the adjacency matrix of $C_n^{s_1,s_2,\ldots s_k}$. This can be done because the adjacency matrix is a *circulant matrix* and eigenvalues of circulant matrices are well understood [4].
- Using Kirchoff's Matrix-Tree theorem and the relationship between determinants and eigenvalues, express $T\left(C_n^{s_1,s_2,\cdots,s_k}\right)$ as a function of these eigenvalues.
- Simplify this function to show that $\sqrt{T\left(C_n^{s_1,s_2,\cdots,s_k}\right)/n}$, as a function of n, satisfies a recurrence relation of the given order.
- To actually *find* the recurrence relation, evaluate the first 2^{s_k} values of $T\left(C_n^{s_1,s_2,\cdots,s_k}\right)$ using the matrix-tree theorem and then use these initial values to solve for the a_n.

The major difficulty with this technique is that, even though it proves the *existence* of the proper order recurrence relation, it does not provide any combinatorial interpretation, e.g., some type of inclusion-exclusion/counting argument, as to what the coefficients mean and why the relation is correct.

As mentioned above, Kleitman and Golden's derivation of $T(C_n^{1,2}) = nF_n^2$ in [13] is an exception to this general technique; their proof is a very clever, fully combinatorial one. Unfortunately, it is also very specific to the special case $C_n^{1,2}$ and can not be extended to cover any other circulant graphs. The major impediment to deriving a general combinatorial proof is that, at first glance, it is difficult to see how to decompose $T\left(C_n^{s_1,s_2,\cdots,s_k}\right)$ in terms of $T\left(C_m^{s_1,s_2,\cdots,s_k}\right)$ where $m < n$; to put it another way, large cycles just do not seem to be easily decomposable into smaller ones.

The main motivation of this paper was to develop a *combinatorial* derivation of the fact that $T\left(C_n^{s_1,s_2,\cdots,s_k}\right)$, as a function of n, satisfies a recurrence relation. Our general technique is *unhooking*, i.e., removing all edges

$$\Big\{ (i,j) \,:\, n - s_k < i < n \text{ and } 0 \le j \Big\}$$

from the graph, creating a new *step graph* $L_n^{s_1,s_2,\cdots,s_k}$. We continue by defining a *fixed* number of classes of forests of $L_n^{s_1,s_2,\cdots,s_k}$ and *combinatorially* derive a linear system of one-step recurrences with constant coefficients counting the number of forests in each class. We then relate this to the original problem by writing $T\left(C_n^{s_1,s_2,\cdots,s_k}\right)$ as a linear combination of the number of forests in each class. Technically, we define a $(m \times 1)$-vector (m, the number of forest classes, will be defined later) $\vec{T}(L_n^{s_1,s_2,\cdots,s_k})$ denoting the number of forests in each class; a $m \times m$ matrix A denoting the system of recurrence relations; and a $(1 \times m)$ row vector $\vec{\beta}$. The process of calculating the entries of β and A will be a simple mechanical combinatorial process. We will then be able to show that

$$\begin{aligned} T\left(C_n^{s_1,s_2,\cdots,s_k}\right) &= \vec{\beta} \cdot \vec{T}(L_n^{s_1,s_2,\cdots,s_k}) \\ &\text{and} \\ \vec{T}(L_n^{s_1,s_2,\cdots,s_k}) &= A \cdot \vec{T}\big(L_{n-1}^{s_1,s_2,\cdots,s_k}\big). \end{aligned}$$

Given these matrix equations, standard techniques, e.g., solving for the generating functions, immediately permit us to derive an order m constant coefficient recurrence relation

This technique of unhooking circulant graphs, i.e., developing a system of recurrences on the resultant line-graphs and then writing the final result as a function of the line-graph values, is actually quite general and can be used to enumerate many other parameters of circulant graphs. In this paper, we further describe how it can be used to derive recurrence relations for the number of Hamiltonian cycles. To the best of our knowledge, this is the first time that general techniques for calculating the value Hamiltonian Cycles of circulant graphs have been developed. We do note, though, that [19] presents a very nice problem-specific combinatorial technique for a related problem, calculating the number of Hamiltonian Cycles in "two-stripe circulant digraphs". These are *directed* circulant graphs with vertex set $\{0,\ 1,\ 2,\ \cdots, n-1\}$, in which each node i has exactly two edges $(i, i+s_1 \bmod n)$ and $(i, i+s_2 \bmod n)$ *leaving* i and two edges $(i-s_1 \bmod n, i)$ and $(i-s_2 \bmod n, i)$, *entering* i, where s_1, s_2 are fixed constants. We also note that, unlike the problem of enumerating spanning trees, the general problem of enumerating Hamiltonian Cycles is #P Hard [17].

The remainder of the paper is structured as follows. In Section 2 we introduce definitions and give a sketch of our general technique. In section 3 we describe how to use the technique to count spanning trees. We first show how to use the general technique to derive recurrence relations for all $T\left(C_n^{s_1,s_2,\cdots,s_k}\right)$ as a function of n. We then then use this to re-derive the formula $T(C_n^{1,2}) = nF_n^2$. In section 4 we discuss Hamiltonian cycles. As in section 3, we start by deriving the general technique for deriving recurrence relations that count the given structures and then apply it to the case $C_n^{1,2}$. Finally, in Section 5, we conclude with some comments about how to extend the developed technique to count other parameters of circulant graphs, such as Eulerian cycles, Eulerian orientations and matchings and discuss some open questions.

Notation: Let $W = \{w_1, \ldots, w_k\}$ be a set. We use $Par(W)$ to denote the set of (set-)partitions of W and $Par_2(W)$ to denote the set of (set-)partitions of W such that each element in the partition is restricted to be of size 1 or 2. As an example, for $W = \{1,2,3\}$:

$$\begin{aligned}
Par(W) &= \{\ \{\{1,2\}\{3\}\},\ \{\{1,3\}\{2\}\},\ \{\{2,3\}\{1\}\},\ \{\{1\}\{2\}\{3\}\},\ \{\{1,2,3\}\}\ \} \\
Par_2(W) &= \{\ \{\{1,2\}\{3\}\},\ \{\{1,3\}\{2\}\},\ \{\{2,3\}\{1\}\},\ \{\{1\}\{2\}\{3\}\}\ \}
\end{aligned}$$

2 The Main Result

In what follows we assume that the step sizes $s_1, s_2, \cdots, s_k$ are fixed and known. To simplify our notation we will often drop the $s_1, s_2, \cdots, s_k$ and just write C_n instead of $C_n^{s_1,s_2,\cdots,s_k}$.

Let $\mathcal{A}(G)$ denote the set of type $\mathcal{A}$ objects in graph G where $\mathcal{A}$ could be Spanning Trees or Hamiltonian Cycles. Let

$$T^{\mathcal{A}}(n) = |\mathcal{A}\left(C_n\right)|$$

be the *number* of $\mathcal{A}$ objects in C_n expressed as a function of n. Our goal is to analyze $T^{\mathcal{A}}(n)$ as a function of n for fixed step sizes $s_1, s_2, \cdots, s_k$.

The major "difficulty" in deriving recurrence relations in n for $T^{\mathcal{A}}(n)$ is that it is difficult to see how larger circulant graphs can be built out of smaller ones. To sidestep this issue we introduce the following.

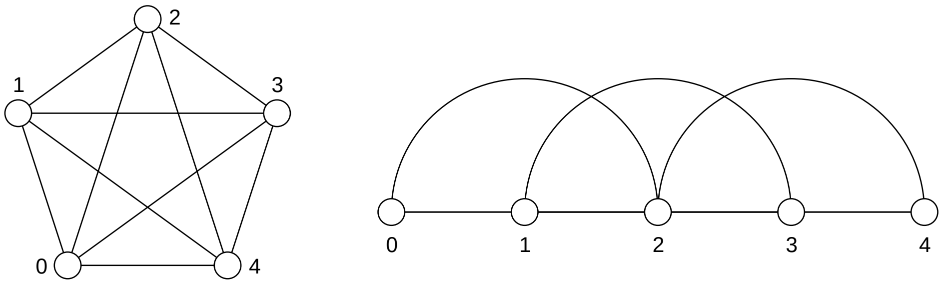

Figure 2: $C_5^{1,2}$ and $L_5^{1,2}$. Hook(5) $= E_C(5) - E_L(5) = \Big\{ (4,0),\ (4,1),\ (3,0) \Big\}$.

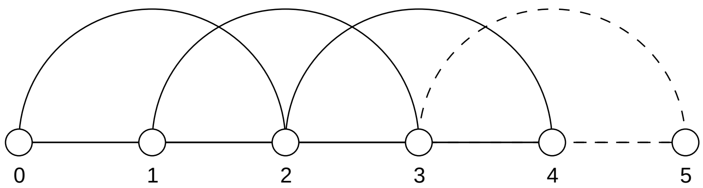

Figure 3: $E_L(6) - E_L(5) = \Big\{ (5,4),\ (5,3),\ \Big\}$.

Definition 2 *The n-node* step graph *with jumps $s_1, s_2, \ldots s_k$, is denoted by*

$$L_n^{s_1,s_2,\ldots,s_k} = (V(n), E_L(n))$$

where

$$V(n) = \{0, 1, \ldots, n-1\} \quad \textit{and} \quad E_L(n) = \Big\{ (i,j) \ : \ i - j \in \{s_1, s_2, \ldots, s_k\} \Big\}.$$

Intuitively, $L_n^{s_1,s_2,\ldots,s_k}$ is what is left when we "unhook" $C_n^{s_1,s_2,\cdots,s_k}$ by removing all edges that *cross over* the interval $(n-1, 0)$ in the circulant graph. See Figure 2 for an example. For notational simplicity, when $s_1, s_2, \ldots, s_k$ are fixed, we will usually write L_n in place of $L_n^{s_1,s_2,\ldots,s_k}$.

We assume that $n > s_k$. Then, by definition, $E_L(n) \subsetneq E_C(N)$ and we set

Definition 3

$$\texttt{Hook}(n) = E_C(n) - E_L(n) = \bigcup_{i=1}^{k} \Big\{ (n-j,\ s_i - j) \ : \ 1 \le j \le s_i \Big\}. \tag{1}$$

See Figure 3 for an example.

It will be useful to have the following definition:

Definition 4

$$W(n) = \{0, 1, \ldots, s_k - 1\} \cup \{n - s_k, n - s_k + 1, \ldots, n-1\}. \tag{2}$$

From (1) an edge $(u,v) \in \texttt{Hook}(n)$ if and only if *both* u and v are in $W(n)$. This fact will enable us to reconstruct properties of the circulant graph from properties of the associated step-graph.

The general schema followed is to assume that we are trying to count the number of $\mathcal{A}$ objects in C_n. Suppose that we were given an $\mathcal{A}$ object, e.g., a spanning tree, and removed all of the edges in $\texttt{Hook}(n)$ from C_n. What would remain would not be a random item but, instead, an object in L_n with a very specific structure. For example, if we removed all edges in $\texttt{Hook}(n)$ from a spanning tree of C_n what would remain would be a *forest* of L_n in which each component of the forest must contain at least one node from the finite set $W(n)$. We will call the objects that remain after removing $\texttt{Hook}(n)$ *legal objects* in L_n.

Furthermore, in the problems we address, it will be possible to efficiently classify all legal objects in L_n using only a small finite set of classifications $\mathcal{P}$. For example, in the spanning tree case, $\mathcal{P} = Par(W(n))$ will be the set of partitions of $W(n)$; the classification of a legal forest will be the partition $X \in \mathcal{P}$ in which two nodes u, v are in the same set of the partition if and only if they are in the same connected component of the forest. See Figure 4. For $X \in \mathcal{P}$ we will then define

$$T_X^{\mathcal{A}}(n) = \text{ Number of legal objects of class } X \text{ in } L_n. \tag{3}$$

Assuming that we know the legal objects in L_n we can reconstruct the legal objects in C_n by adding back the edges from $\texttt{Hook}(n)$. In the spanning tree case we can rebuild a spanning tree of C_n from a legal forest of L_n by adding a subset of edges of $\texttt{Hook}(n)$. The two important observations will be (i) that each legal object in C_n can only be reconstructed from *exactly one* legal object in L_n and (ii) if two legal objects F_1, F_2 in L_n have the same classification, then exactly the same number of legal objects in C_n can be reconstructed from F_1 as from F_2. This will enable us to define constants $B_X^{\mathcal{A}}$ such that

$$T^{\mathcal{A}}(n) = \sum_{X \in \mathcal{P}} B_X^{\mathcal{A}}\, T_X^{\mathcal{A}}(n) \tag{4}$$

The important observation here is that, in all of the problems we examine, $B_X^{\mathcal{A}}$ will *not* depend upon the value of n; the terms $\{n - s_k, n - s_k + 1, \ldots, n-1\}$ are treated as *labels* of vertices and and not numbers.

The above is the general schema for reconstructing legal objects in circulant graphs C_n from legal objects in associated step graphs L_n. The reason for taking this approach is that, unlike for circulant graphs, it is easy to see how to build a larger step graph from a smaller one. In particular (see Figure 3)

$$E_L(n+1) - E_L(n) = \Big\{ (n - s_i,\, n) \;:\; 1 \leq i \leq k \Big\} \tag{5}$$

We will see that the simple recursive structure implied by (5) will enable us to to construct all legal objects in L_{n+1} from knowledge of the legal objects in

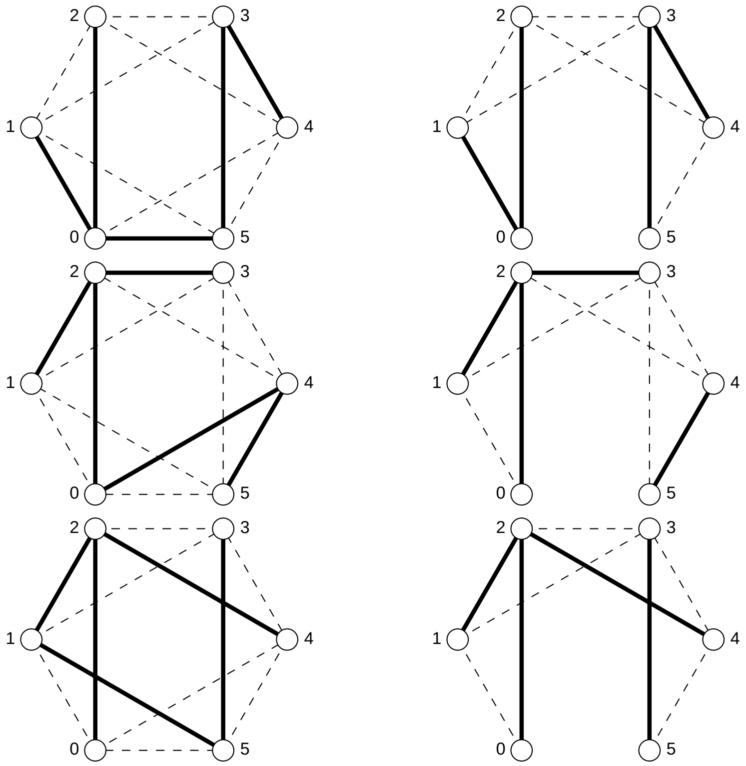

Figure 4: The solid edges in the three graphs on the left form three different spanning trees of $C_6^{1,2}$; Removing the edges $\texttt{Hook}(6) = \{(0,5),(1,5),(4,0)\}$ from the spanning trees leave three different forests in $L_6^{1,2}$. The top two forests both have classification $\{\{0,1\}\,\{n-1,n-2\}\}$. The bottom forest has classification $\{\{0,1,n-2\}\,\{n-1\}\}$.

L_n. Specifically, for $X, X' \in \mathcal{P}$ we will show the existence of constants $\alpha^{\mathcal{A}}_{X,X'}$ such that

$$T^{\mathcal{A}}_{X'}(n+1) = \sum_{X \in \mathcal{P}} \alpha^{\mathcal{A}}_{X',X}\, T^{\mathcal{A}}_X(n). \tag{6}$$

For example, in the spanning tree case, $\alpha^{\mathcal{A}}_{X',X}$ will be the number of subsets of $E_L(n+1) - E_L(n)$ that, added to a legal forest of type X in L_n, yields a legal forest of type X' in L_{n+1}.

(6) says that the $T^{\mathcal{A}}_X(n)$ satisfy a system of recurrence relations with constant coefficients. (4) says that $T^{\mathcal{A}}(n)$, which is what we really want to solve for, is just a linear combination of the $T^{\mathcal{A}}_X(n)$. Combining (4) and (6) and using standard techniques, e.g., generating function tools, we can then derive a recurrence relation in n for $T^{\mathcal{A}}_X(n)$.

In the next two sections we will work through this general schema for both $\mathcal{A}$ being *spanning trees* and $\mathcal{A}$ being *Hamiltonian cycles*.

3 Counting Spanning Trees

In this section $\mathcal{A}$ objects are spanning trees and our problem is to combinatorially rederive the result of [18] and [22] that the number of spanning trees in C_n satisfies a recurrence relation in n. Recall that, given a graph $G = (V, E)$, a spanning tree $T \subseteq E$ is a subset of the edges that forms a connected acyclic graph.

For simplicity we will let $T(n) = T^{\mathcal{A}}(n)$ denote the number of spanning trees in C_n. In the next subsection we will develop the general technique for deriving a recurrence relation for $T(n)$ and then, in Section 3.2 we will specialize to derive the exact solution for $C^{1,2}_n$.

3.1 The General Recurrence

In what follows, assume that $s_1, s_2, \ldots, s_k$ are all fixed.

Let T be a spanning tree of C_n. Removing all edges of $\texttt{Hook}(n) = E_C(n) - E_L(n)$ from T leaves a forest $T \cap E_L(n)$ in L_n. Since all endpoints of edges in $\texttt{Hook}(n)$ are in $W(n)$ we find that every component of the forest $T \cap E_L$ must contain at least one node from $W(n)$. This motivates the following definition (see Figures 4 and 5)

Definition 5 *Let $n \geq 2s_k$.*

1. *A* legal forest *F in L_n is one in which every connected component of F contains at least one node in $W(n)$.*

2. *$\mathcal{P} = Par(W(n))$ is the collection of all* set partitions *of $W(n)$*

3. *Let F be a legal forest of L_n. Then $C(F)$, the* classification of F, *is $X \in \mathcal{P}$ such that $\forall u, v \in W(n)$, u, v are in the same connected component of F if and only if u, v are in the same set in X.*

4. *For $X \in \mathcal{P}$ set*

$$T_X(n) = |\{F : \ F \textit{ is a legal forest of } L_n \textit{ with } C(F) = X\}|$$

Note: The reason for requiring $n \geq 2s_k$ is to guarantee that all of the $2s_k$ elements in $W(n)$ are distinct. Now

Definition 6 *Set*

$$\mathcal{S} = \{S : S \subseteq \texttt{Hook}(n)\}.$$

We make the following straightforward observation (given without proof):

Lemma 1 *Let F, F' be two legal forests of L_n such that $C(F) = C(F')$ and $S \in \mathcal{S}$. Then*
$F \cup S$ is a spanning tree of C_n if and only if $F' \cup S$ is a spanning tree of C_n.

This permits the following definition (see Figures 5 and 6):

Definition 7 *For $X \in \mathcal{P}$ and $S \in \mathcal{S}$ set*

$$\begin{aligned}
\alpha_{S,X} &= \begin{cases} 1 & \textit{if adding } S \textit{ to forest } F \textit{ with } C(F) = X \textit{ yields a spanning tree of } C_n. \\ 0 & \textit{otherwise} \end{cases} \\
\beta_X &= \sum_{S \in \mathcal{S}} \alpha_{S,X} \\
\vec{\beta} &= (\beta_X)_{X \in \mathcal{P}}
\end{aligned}$$

where, in the last equation, $\vec{\beta}$ is a vector ordered using some fixed arbitrary ordering of the elements $\mathcal{P}$.

The crucial observation in the above definitions is that $\alpha_{S,X}$ is *independent* of n and can be easily evaluated just by looking at S and X. For example suppose, $(s_1, s_2) = (1, 2)$ and, for some n, F is a legal forest of $L_n^{1,2}$ with $C(F) = X = \Big\{\{0, n-1\}, \{1, n-2\}\Big\}$, i.e., it has exactly two connected components partitioning the nodes in $V(n)$; one of the components contains $0, n-1$ and the other $1, n-2$. Now, if $S = \{\,(0, n-2)\,\}$ then $\alpha_{S,X} = 1$ since the single edge in S connects the two components to form a spanning tree while if $S = \{\,(0, n-1)\,\}$ then $\alpha_{S,X} = 0$ since the single edge in S creates a cycle in the component containing $0, n-1$. So, the values $\alpha_{S,X}$ are properties of $s_1, s_2, \ldots, s_k$ and independent of n.

We can now prove our first relationship:

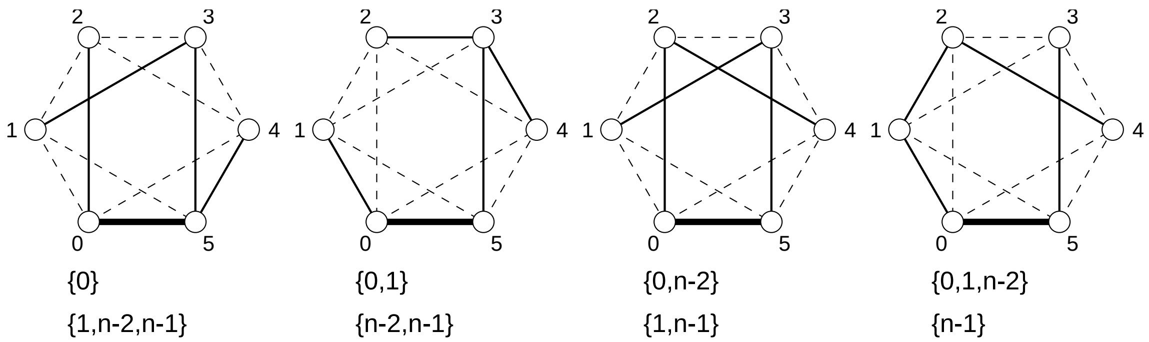

Figure 5: $n = 6$. Each forest shown is from a different class in $L_6^{1,2}$ (the class is shown beneath the forest). Since each class has exactly two components with 0 in the first and $n-1$ in the second, adding $(0,5)$ to forests from these classes of $L_6^{1,2}$ all yield spanning trees of $C_6^{1,2}$. In the notation of Definition 7, if X is one of the 4 classes illustrated and $S = \{(0,5)\}$ then $\alpha_{S,X} = 1$.

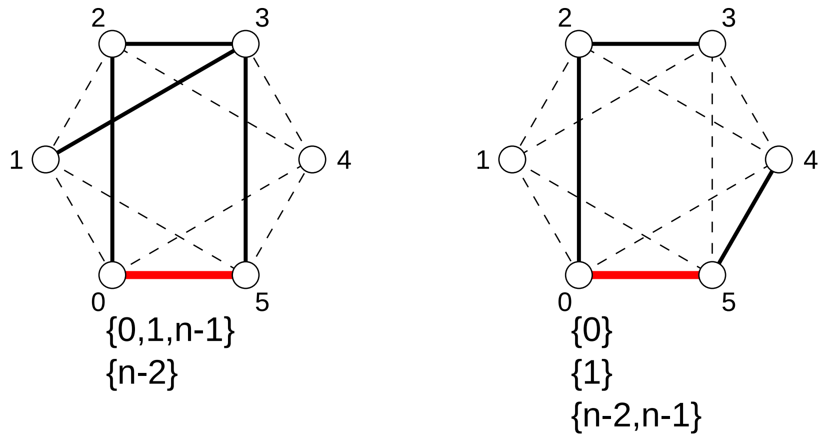

Figure 6: $n = 6$. In the lefthand class 0 and $n-1$ are already in the same component so adding edge $(0,5)$ to forests from these classes induces a cycle and therefore does not yield spanning trees of $C_6^{1,2}$. In the rightmost class, even after adding edge $(0,5)$ the graph is still not connected and therefore a spanning trees of $C_6^{1,2}$ is not formed. In the notation of Definition 7, if X is one of the 2 classes illustrated and $S = \{(0,5)\}$ then $\alpha_{S,X} = 0$.

Lemma 2

$$T\left(C_n\right)=\sum_{X\in\mathcal{P}}\left(\sum_{S\in\mathcal{S}}\alpha_{S,X}\right)T_X(n)=\sum_{X\in\mathcal{P}}\beta_X T_X(n). \tag{7}$$

Letting $\vec{T}\left(L_n\right)$ be the column vector $\left(T_X(n)\right)_{X\in\mathcal{P}}$, this can also be written as

$$T\left(C_n\right)=\vec{\beta}\cdot\vec{T}\left(L_n\right). \tag{8}$$

Proof. Let

$$\text{SPAN}(n) \quad = \quad \{T\ :\ T \text{ is a spanning tree of } C_n\}$$

Since removing the edges in $E_C - E_L$ from a spanning tree always leaves a legal forest,

$$|\text{SPAN}(n)|=\sum_{X\in\mathcal{P}}\sum_{S\in\mathcal{S}}|\{T\in\text{SPAN}(n)\ :\ T\cap(E_C-E_L)=S,\ \text{ and } C(T-S)=X\}| \tag{9}$$

Lemma 1 tells us that

$$|\{T\in\text{SPAN}(n)\ :\ T\cap(E_C-E_L)=S,\ \text{ and } C(T-S)=X\}|=\begin{cases} T_X(n) & \text{if } \alpha_{S,X}=1\\ 0 & \text{otherwise}\end{cases} \tag{10}$$

We can therefore rewrite (9) as

$$T\left(C_n\right)=|\text{SPAN}(n)|=\sum_{X\in\mathcal{P}}\sum_{S\in\mathcal{S}}\alpha_{S,X}T_X(n) \tag{11}$$

and are done.

□

So far we have only shown that the number of spanning trees of C_n is a linear combination of the number of different legal forests of the associated L_n. We will now show the the number of different legal forests can be written as a system of linear recurrences in n. The main observation is the following lemma:

Lemma 3 *Let F be a legal forest in L_{n+1} and*

$$U=F\cap\left(E_L(n+1)-E_L(n)\right).$$

Then $F-U$ is a legal forest in L_n, (where $F-U$ denotes the graph created from F by deleting all edges in U as well as vertex n).

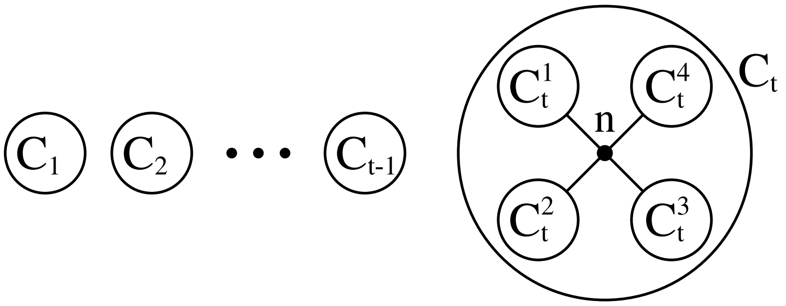

Figure 7: Sketch of the proof of Lemma 3. Forest F is is partitioned into t trees $C_1, C_2, \ldots C_t$; node n is in tree C_t.

Proof. (See Figure 7.) Suppose that F is a legal forest in L_n.

Then F can be partitioned into, say, t trees $C_1, C_2, \ldots C_t$ where each tree contains at least one vertex of

$$W(n+1) = \{0, 1, \ldots, s_k - 1\} \cup \{n - s_k + 1, n - s_k + 2, \ldots, n\}.$$

Without loss of generality let C_t be the tree of F containing node n.

Now remove the vertex n and all edges in

$$E_L(n+1) - E_L(n) = \{(n - s_i,\, n) \,:\, 1 \leq i \leq k\}$$

from F to get a new forest F' in L_n. To show that this is a *legal* forest in L_n we must show that every tree in F' contains at least one vertex in

$$W(n) = \Big(W(n+1) \cup \{n - s_k\}\Big) - \{n\}.$$

The trees of F' are the $C_1, C_2, \ldots C_{t-1}$ and all of the new trees (components) created by removing the edges $\Big\{(n - s_i,\, n) \,:\, 1 \leq i \leq k\Big\}$ from C_t. By the original definition of the C_i we know that for $i < t$, each C_i contains at least one vertex in

$$W(n+1) - \{n\} \subseteq W(n).$$

Furthermore, each new tree created from C_t was created by removing an edge of the form $(n - n_i, n)$ so each tree created must contain a vertex of the form $n - s_i \in W(n)$. The proof is complete. $\square$

Note that this lemma implies that *every* legal forest of L_{n+1} can be constructed by adding edges in $E_L(n+1) - E_L(n)$ to a legal forest of L_n. To continue we will need the following observation:

Lemma 4 *Let F, F' be legal forests in L_n such that $C(F) = C(F')$. Let $U \subseteq E_L(n+1) - E_L(n)$. Then*

- $F \cup U$ *is a legal forest of* L_{n+1} *if and only if* $F' \cup U$ *is a legal forest of* L_n *and*

- *if both* $F \cup U$ *and* $F' \cup U$ *are legal forests of* L_{n+1} *then* $C(F \cup U) = C(F' \cup U)$.

Proof. Straightforward $\square$

Before continuing we should emphasize a subtle point concerning the classification of a legal forest in L_n, which is that *it strongly depends upon* n. For example, suppose, as in Figure 8, we have a legal forest D in $L_n^{1,2}$ with classification $C(D) = \{0\}\{1, n-2, n-1\}$. Since $n = 5$ this means that the forest is partitioned into two components, one of which contains node 0 and the other of which contains nodes 1, $4 (= n-1)$, $3 (= n-2)$. Now suppose that we add a new vertex 6 to D *but no new edges* to get a new forest D'. Then this new forest has three components and the elements of $W(6) = \{0, 1, 4, 5\}$ are partitioned by the forest as $\{0\}$ $\{1, 4\}$ $\{5\}$ which, since $n = 6$, means that $C(D') = \{0\}\{1, n-2\}\{n-1\}$. When calculating how adding vertices and edges to legal forests of L_n change them into different legal forests of L_{n+1} we must take account of this fact.

Lemma 4 permits the next definition (see Figure 8)

Definition 8 *For* $X, X' \in \mathcal{P}$ *and* $U \subseteq E_L(n+1) - E_L(n)$ *set*

$$
\begin{aligned}
\gamma_{X',X,U} &= \begin{cases} 1 & \text{if adding } U \text{ to forest } F \text{ with } C(F) = X' \text{ yields a forest } F' \text{ with } C(F') = X' \\ 0 & \text{otherwise} \end{cases} \\
\alpha_{X',X} &= \sum_{U \subseteq E_L(n+1) - E_L(n)} \gamma_{X',X,U} \\
A &= (a_{X',X})_{X',X \in \mathcal{P}}
\end{aligned}
$$

where, in the last equation, A *is a square matrix whose columns/rows are ordered using the same ordering as in the definition of* $\vec{\beta}$ *in Definition 7. For convenience, we will also define*

$$\mathcal{U} = \{U \,:\, U \subseteq (E_L(n+1) - E_L(n))\}.$$

Note: As in the observation following Lemma 7 we point out that the value of $\gamma_{X',X,U}$ is *independent* of n.

Combining Lemmas 3 and 4 then yield

Lemma 5 $\forall X' \in \mathcal{P}$,

$$T_{X'}(n+1) = \sum_{X \in \mathcal{P}} a_{X',X} T_X(n) \tag{12}$$

or, equivalently,

$$\vec{T}\,(L_{n+1}) = A\vec{T}\,(L_n) \tag{13}$$

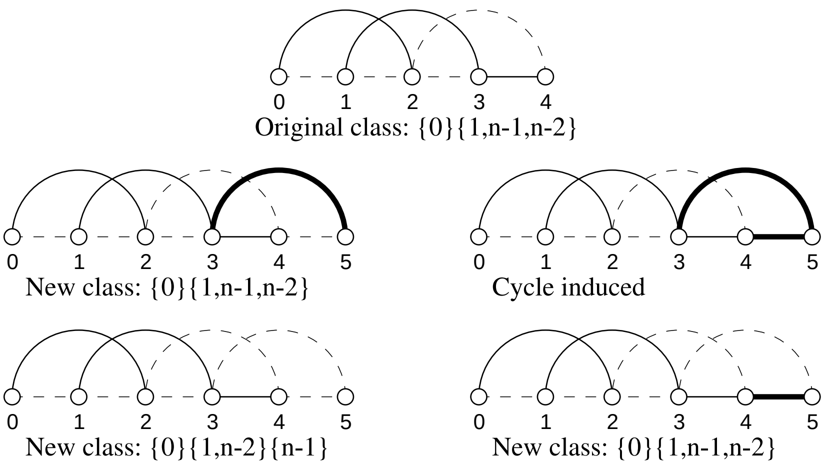

Figure 8: Different ways to add node 5 to a forest F of $L_5^{1,2}$ in class $C(F) = X = \{0\}\{1, n-1, n-2\}$. Bold edges are the ones added with node 5. Adding no edge creates a forest in class $F_{\{0\}\{1,n-2\}\{n-1\}}(6)$; Adding *either* edge $(3, 5)$ *or* edge $(4, 5)$ creates a forest in class $\{0\}\{1, n-1, n-2\}(6)$; adding *both* $(3, 5)$ *and* $(4, 5)$ induces a cycle so no forest is created. In the notation of Definition 8

- Adding no edge creates a forest with classification class $\{0\}\{1, n-2\}\{n-1\}$ so

$$\gamma_{X',X,\emptyset} = \begin{cases} 1 & \text{if } X' = \{0\}\{1, n-2\}\{n-1\} \\ 0 & \text{otherwise} \end{cases}$$

- Adding *either* edge $(3, 5)$ *or* edge $(4, 5)$ creates a forest in class $\{0\}\{1, n-1, n-2\}$ so

$$\gamma_{X',X,\{\{n-2,n\}\}} = \begin{cases} 1 & \text{if } X' = \{0\}\{1, n-1, n-2\} \\ 0 & \text{otherwise} \end{cases}$$

  and

$$\gamma_{X',X,\{\{n-1,n\}\}} = \begin{cases} 1 & \text{if } X' = \{0\}\{1, n-1, n-2\} \\ 0 & \text{otherwise} \end{cases}$$

- Adding *both* $(3, 5)$ *and* $(4, 5)$ induces a cycle so no forest is created and

$$\forall X' \in \mathcal{P},\ \gamma_{X',X,\{\{n-1,n\},\{n-2,n\}\}} = 0.$$

Proof. The proof is very similar to that of Lemma 2. Set

$$\mathrm{Legal}_X(n) = \{F \ : \ \ F \text{ is a legal forest of } L_n \text{ with } C(F) = X\}$$

Lemma 3 says that removing the edges in $E_L(n+1) - E_L(n)$ from a legal forest in $L_{n+1}^{s_1,s_2,\ldots,s_k}$ leaves a legal forest in $L_n^{s_1,s_2,\ldots,s_k}$ so

$$|\mathrm{Legal}_{X'}(n)| = \sum_{U\in\mathcal{U}} \sum_{X\in\mathcal{P}} |\{F \in Legal_{X'}(n+1) \ : \ F \cap (E_L(n+1) - E_L(n)) = U, \ \text{ and } C(F-U) = X\}| \tag{14}$$

Lemma 4 then tells us that

$$|\{F \in Legal_{X'}(n+1) \ : \ F \cap (E_L(n+1) - E_L(n)) = U, \ \text{ and } C(F-U) = X\}| = \begin{cases} T_X(n) & \text{if } \gamma_{X',X,U} = 1 \\ 0 & \text{otherwise} \end{cases}$$

We can therefore rewrite (14) as

$$T_{X'}(n+1) = |\mathrm{Legal}_{X'}(n)| = \sum_{U\in\mathcal{U}} \sum_{X\in\mathcal{P}} \gamma_{X',X,U} T_X(n) \tag{15}$$

and are done.

□

Combining everything in this section proves our main theorem on spanning trees of circulant graphs which is:

Theorem 1 *Let $s_1, s_2, \ldots, s_k$ be given and $T(n)$ denote the number of spanning trees in $C_n^{s_1,s_2,\ldots,s_k}$. Let $\mathcal{P} = Par(W(n))$, $T_X(n)$ denote the number of* legal forests *with classification X and $\vec{T}\,(L_n^{s_1,s_2,\ldots,s_k})$ be the column vector $(T_X(n))_{X\in\mathcal{P}}$ Then, for $n \geq 2s_k$,*

$$\begin{aligned} T\left(C_n^{s_1,s_2,\ldots,s_k}\right) &= \vec{\beta} \cdot \vec{T}\left(L_n^{s_1,s_2,\ldots,s_k}\right) \\ \vec{T}\left(L_{n+1}^{s_1,s_2,\ldots,s_k}\right) &= A\,\vec{T}\left(L_n^{s_1,s_2,\ldots,s_k}\right) \end{aligned}$$

where $\vec{\beta}$ is the constant vector defined in Definition 7 and A is the constant square matrix defined in Definition 8.

This theorem implies that $T\,(C_n^{s_1,s_2,\ldots,s_k})$ satisfies a linear recurrence recurrence relation with constant coefficients of order equal to the rank of the matrix A.

Since the *size* of matrix A is $|\mathcal{P}| = B(2s_k)$ where $B(m)$ is the Bell number[2] of order m the order of the recurrence is at most $B(2s_k)$.

[2] $B(m)$ counts the number of set partitions of m items.

3.2 Evaluating $T(C_n^{1,2})$

In this section we illustrate the unhooking technique by using it to rederive the formula $T(C_n^{1,2}) = nF_n^2$ that was conjectured by Bedrosian [2] and subsequently proven by Kleitman and Golden [13]. In this case $k = 2$ and $(s_1, s_2) = (1, 2)$ so

$$\texttt{Hook}(n) = E_C(n) - E_L(n) = \{\, (n-1, 0),\, (n-1, 1),\, (n-2, 0) \,\}$$

and

$$E_L(n+1) - E_L(n) = \{\, (n, n-1),\, (n, n-2) \,\}$$

Also

$$W(n) = \{0, 1, n-1, n-2\}$$

and $\mathcal{P}$ is the set of all of the partitions of $W(n)$. This gives

$$\vec{T}\,(L_n^{1,2}) = \begin{pmatrix} T_{\{0,1,n-2,n-1\}}(n) \\ T_{\{0\}\{1,n-2,n-1\}}(n) \\ T_{\{1\}\{0,n-2,n-1\}}(n) \\ T_{\{n-2\}\{0,1,n-1\}}(n) \\ T_{\{n-1\}\{0,1,n-2\}}(n) \\ T_{\{0,1\}\{n-2,n-1\}}(n) \\ T_{\{0,n-2\}\{1,n-1\}}(n) \\ T_{\{0,n-1\}\{1,n-2\}}(n) \\ T_{\{0\}\{1\}\{n-2,n-1\}}(n) \\ T_{\{0\}\{n-2\}\{1,n-1\}}(n) \\ T_{\{0\}\{n-1\}\{1,n-2\}}(n) \\ T_{\{1\}\{n-1\}\{0,n-2\}}(n) \\ T_{\{1\}\{n-2\}\{0,n-1\}}(n) \\ T_{\{n-2\}\{n-1\}\{0,1\}}(n) \\ T_{\{0\}\{1\}\{n-2\}\{n-1\}}(n) \end{pmatrix}$$

Mechanically calculating the values $\alpha_{S,X}$ and $\beta_X = \sum_{S \in \mathcal{S}} \alpha_{S,X}$ from Definition 7 and setting $\vec{\beta} = (\beta_X)_{X \in \mathcal{P}}$ gives

$$\vec{\beta} = \begin{pmatrix} 1 & 2 & 1 & 1 & 2 & 3 & 1 & 2 & 2 & 1 & 3 & 1 & 1 & 2 & 1 \end{pmatrix} \tag{16}$$

Mechanically calculating $\gamma_{X',X,U}$ and $\alpha_{X',X} = \sum_{U\subseteq(E_L(n+1)-E_L(n))} \gamma_{X',X,U}$ from Definition 8 and setting $A = (a_{X',X})_{X',X\in\mathcal{P}}$ gives

$$A = \begin{pmatrix}
2 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{pmatrix} \tag{17}$$

Let $G(x) = \sum_{n=1}^{\infty} T\left(C_n^{1,2}\right)$ be the generating function for $T\left(C_n^{1,2}\right)$. We can solve

$$T\left(C_n^{1,2}\right) = \vec{\beta}\cdot\vec{T}\left(L_n^{1,2}\right), \qquad \vec{T}\left(L_{n+1}^{1,2}\right) = A\vec{T}\left(L_n^{1,2}\right)$$

to find that

$$G(x) = \frac{36 - 19x - 116x^2 + 7x^3 + 46x^4 - 12x^5}{1 - 4x + 10x^3 - 4x^5 + x^6}$$

so

$$T(C_n^{1,2}) = 4T(C_{n-1}^{1,2}) - 10T(C_{n-3}^{1,2}) + 4T(C_{n-5}^{1,2}) - T(C_{n-6}^{1,2}) \tag{18}$$

with initial values $36, 125, 384, 1183, 3528, 10404$ for $n = 4, 5, 6, 7, 8, 9$ respectively. This is equivalent to

$$T(C_n^{1,2}) = nF_n^2.$$

In particular we note that since $\vec{T}\left(L_{n+1}^{1,2}\right) = A\vec{T}\left(L_n^{1,2}\right)$ is a one step constant coefficient linear recurrence relation, in order to solve for $G(x)$ it is only necessary to evaluate the one initial vector $\vec{T}\left(L_4^{1,2}\right)$.

We also note that, without much extra work, the technique also permits us to solve for

$$T(L_n^{1,2}) = T_{\{0,1,n-2,n-1\}}(n),$$

the number of spanning trees in $L_n^{1,2}$.

Letting $G(x) = \sum_{n=1}^{\infty} T\left(L_n^{1,2}\right)$ we find that

$$\begin{aligned} G(x) &= \frac{8-3x}{1-3x+x^2} \\ T(L_n^{1,2}) &= 3T(L_{n-1}^{1,2}) - T(L_{n-2}^{1,2}) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{5}}\left(\left(\frac{3+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{3-\sqrt{5}}{2}\right)^{n-1}\right) \\
&= F_n^2 - F_{n-2}^2
\end{aligned}
$$

with initial values $8, 21$ for $n = 4, 5$ respectively.

4 Counting Hamiltonian Cycles

The unhooking technique developed in the previous section is quite general and can be used to count various other parameters of circulant graphs. In this section we describe how to modify it to derive a recurrence relation on the number of Hamiltonian cycles $H\left(C_n^{s_1,s_2,\cdots,s_k}\right)$ in circulant graph $C_n^{s_1,s_2,\cdots,s_k}$. Recall that a Hamiltonian cycle of a graph $G = (V, E)$ is a simple cycle in G containing all of the vertices in V.

In the next subsection we derive our general technique and then, in the subsection following, use it to calculate $H(C_n^{1,2})$. Note that since many of the proofs of our lemmas and theorems are similar to the analogous ones already proven in Section 3 for spanning trees we will not provide all details.

4.1 The General Recurrence

In what follows, we assume that $s_1, s_2, \ldots, s_k$ are all fixed so, as before, we will drop the $s_1, s_2, \ldots, s_k$ and use C_n to denote $C_n^{s_1,s_2,\cdots,s_k}$ and L_n to denote $L_n^{s_1,s_2,\cdots,s_k}$.

Let H be a Hamiltonian Cycle of C_n. Removing all edges of $\texttt{Hook}(n) = E_C(n) - E_L(n)$ from H leaves either (i) a Hamiltonian cycle of L_n or (ii) a collection of disjoint paths (an isolated node is considered to belong to its own path) in L_n that partitions the vertices in $V(n)$. Since all endpoints of edges in $\texttt{Hook}(n)$ are in $W(n)$ we find that all of the endpoints of each path in $H \cap E_L$ must be a node from $W(n)$. This motivates the following definition (See Figure 9):

Definition 9 *Let $n \geq 2s_k$.*

1. *A* Legal Path Decomposition (LPD) *in L_n is either*
   - *(a) a Hamiltonian Cycle of L_n or*
   - *(b) a collection of disjoint paths in L_n in which*
     - *every vertex in $V(n)$ appears on some path in the collection and*
     - *all of the endpoints of the paths are in $W(n)$.*

   Note that a single isolated node is considered to be on a path by itself. In a LPD all isolated nodes must, by definition, be in $W(n)$.

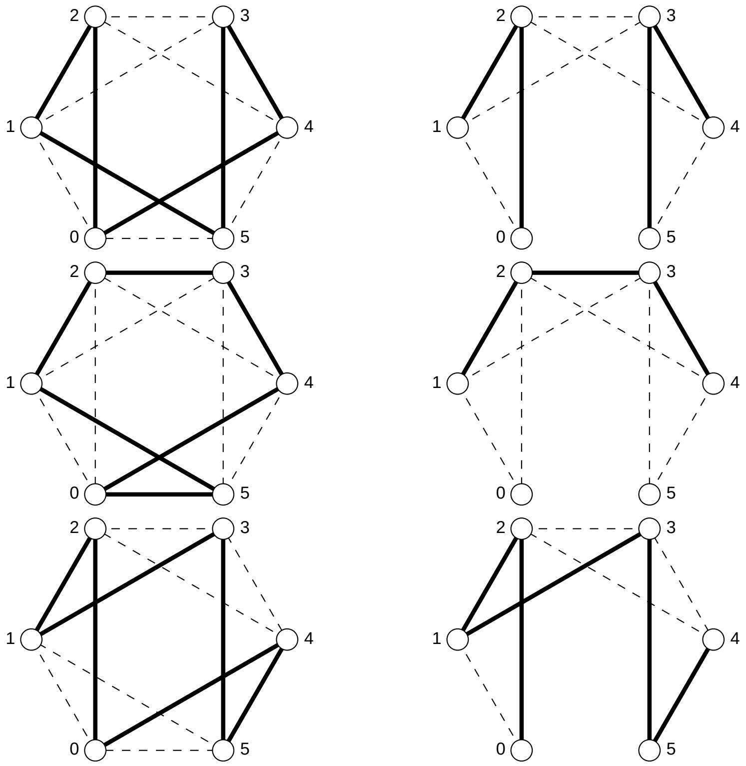

Figure 9: The solid edges in the three graphs on the left form three different Hamiltonian cycles of $C_6^{1,2}$; Removing the edges $\texttt{Hook}(6) = \{(0,5),(1,5),(4,0)\}$ from the cycles leaves three different LPDs in $L_6^{1,2}$. The top LPD has classification $\{0,1\}\{n-1,n-2\}$. The second LPD has classification $\{1,n-2\},\{0\},\{n-1\}$. The bottom LPD has classification $\{0,n-2\}$.

2. $\mathcal{P} = \bigcup_{W \subseteq W(n)} Par_2(W(n))$ *is the collection of all set partitions of subsets of* $W(n)$ *where each element of the set partition is of size 1 or 2.*

3. *Let* D *be a LPD of* $L_n^{s_1,s_2,\ldots,s_k}$. *We define* $C(D)$, *the classification of* D, *as follows:*

   If D *is a Hamiltonian Cycle set* $C(D) = \emptyset$. *Otherwise,*

$$C(D) = \Big\{ \{u_1, v_1\}, \ldots, \{u_t, v_t\}, \{w_1\}, \ldots, \{w_s\} \Big\}$$

   where, for $i = 1, \ldots, t$, u_i, v_i, *are the respective paired endpoints of its component paths and* $w_1, \ldots, w_s$ *are isolated vertices in* D.
   Note, by definition, that if D *is a LPD then* $C(D) \in \mathcal{P}$.

4. *For* $X \in \mathcal{P}$ *set*

$$H_X(n) = \Big| \Big\{ D : \ D \text{ is a LPD of } L_n \text{ with } C(D) = X \Big\} \Big|$$

Recall from Definition 6 that $\mathcal{S} = \{S : S \subseteq \texttt{Hook}(n)\}$. It is now easy to see that

Lemma 6 *Let* D, D' *be two LPDs of* L_n *such that* $C(D) = C(D')$ *and* $S \in \mathcal{S}$. *Then* $D \cup S$ *is a Hamiltonian Cycle of* C_n *if and only if* $D' \cup S$ *is a Hamiltonian Cycle of* C_n.

This permits the following definition

Definition 10 *For* $X \in \mathcal{P}$ *and* $S \in \mathcal{S}$ *set*

$$\begin{aligned}
\alpha_{S,X} &= \begin{cases} 1 & \text{if adding } S \text{ to LPD } D \text{ with } C(D) = X \text{ yields a Hamiltonian cycle of } C_n \\ 0 & \text{otherwise} \end{cases} \\
\beta_X &= \sum_{S \in \mathcal{S}} \alpha_{S,X} \\
\vec{\beta} &= (\beta_X)_{X \in \mathcal{P}}
\end{aligned}$$

where, in the last equation, $\vec{\beta}$ *is a vector ordered using some fixed arbitrary ordering of the elements* $\mathcal{P}$.

As in the spanning tree case the crucial observation in the above definitions is that $\alpha_{S,X}$ is *independent* of n and can be easily evaluated just by looking at S and X.

We can now prove the following relationship. Since the proof is so similar to that of Lemma 2 we do not provide it.

Lemma 7

$$H\left(C_n\right)=\sum_{X\in\mathcal{P}}\left(\sum_{S\in\mathcal{S}}\alpha_{S,X}\right)H_X(n)=\sum_{X\in\mathcal{P}}\beta_X H_X(n). \tag{19}$$

Letting $\vec{H}\left(L_n\right)$ be the column vector $\left(H_X(n)\right)_{X\in\mathcal{P}}$, this can also be written as

$$H\left(C_n\right)=\vec{\beta}\cdot\vec{H}\left(L_n\right). \tag{20}$$

Again, as in the spanning tree case we now show that the number of different LPDs of can be written as a system of linear recurrences in n. We first show that all LPDs of L_{n+1} can be built from LPDs of L_n.

Lemma 8 *Let D be a LPD in L_{n+1} and*

$$U=D\cap\Big(E_L(n+1)-E_L(n)\Big).$$

Then $D-U$ is a LPD in L_n.

Proof. The proof is straightforward but requires a case-by-case analysis.

Recall that $E_L(n+1)-E_L(n)=\cup_{i=1}^{k}\{\,(n,n-s_i)\,\}$. We must show that if D is a LPD of L_{n+1} with all endpoints in

$$W(n+1)=\{0,1,\ldots,s_k-1\}\cup\{n-s_k+1,n-s_k+2,\ldots,n\}$$

then $D-U$ is a LPD of L_n with all endpoints in

$$W(n)=\{0,1,\ldots,s_k-1\}\cup\{n-s_k,n-s_k+1,\ldots,n-1\}.$$

First suppose that $C(D)=\emptyset$, i.e., D is a Hamiltonian cycle of L_{n+1}. Let u,v be the two neighbors of n in D. Then $D-U$ will be a single path with endpoints u,v and is therefore a LDP of L_n.

Now suppose that $C(D)\neq\emptyset$. Then

$$C(D)=\Big\{\,\{u_1,v_1\},\ldots,\{u_t,v_t\},\,\{w_1\},\ldots,\{w_s\}\Big\}$$

where, for $i=1,\ldots,t$, u_i,v_i, are the respective paired endpoints of its component paths and $w_1,\ldots,w_s$ are isolated vertices in D.

Let P_i, $i=1,\ldots,t+s$ be the paths in D (an isolated node is on its own path).

Without loss of generality, let P_{t+s} be the unique path in D which contains node n. Then $D-U$ will contain the same first $t+s-1$ paths as D with the only change being that n and the edge(s) attached to n are removed from the last path P_{t+s}.

Since all of $\{u_1,v_1,u_2,v_2,\ldots,u_t,v_t,w_1,\ldots,w_s\}-\{n\}$ are in $W(n+1)-\{n\}\subset W(n)$, to prove that $D-U$ is a LPD of L_n it suffices to prove that the path decomposition given by $P_{t+s}-U$ has endpoints in $W(n)$.

If $P_{t+s} = n$, i.e., P_{t+s} is the isolated vertex n, then $P_{t+s} - U$ is empty and the the Lemma is obviously correct.

Otherwise we must have that P_{t+s} is a real path and not an isolated vertex. Let u, v be the endpoints of P_{t+s} There are two cases:

1. *One of the two endpoints u, v is n.*
   Without loss of generality let $v = n$ and $u \neq n$. In this case $U = \{\,(n, n-s_i)\,\}$ for some i, and $P_{t+s} - U$ is a single path with endpoints $n - s_i, u$ both of which are in $W(n)$.

2. *Both of the endpoints u, v are not n.*
   In this case n is in the interior of path P_{t+s} and its two neighbors must be vertices of the form $n - s_i, n - s_{i'}$ for some i, i'. Then $U = \{\,(n, n - s_i), (n, n - s_{i'})\,\}$ and $P_{t+s} - U$ is composed of two paths (one or both of which might be isolated vertices) whose endpoints are in $\{u, v, n-s_i, n-s_{i'}\}$ all of which are in $W(n)$ and we are done. (Note that it's possible that u and/or v might be equal to $n - s_i$ or $n - s_{i'}$ in which case one or both of the two paths could be isolated vertices.)

$\square$

This Lemma implies that every LPD of L_{n+1} can be built from a LPD of L_n. In analogy to Lemma 4 we will need the following lemma:

Lemma 9 *Let D, D' be LPDs of L_n such that $C(D) = C(D')$.*
Let $U \subseteq E_L(n+1) - E_L(n)$. Then

- *$D \cup U$ is a LPD of L_{n+1} if and only if $D' \cup U$ is a LPD of L_{n+1}, and*
- *if both $D \cup U$ and $D' \cup U$ are LPDs of L_{n+1} then $C(D \cup U) = C(D' \cup U)$.*

Proof. Straightforward $\square$

This permits the next definition (see Figure 10):

Definition 11 *For $X, X' \in \mathcal{P}$ and $U \subseteq E_L(n+1) - E_L(n)$ set*

$$\begin{aligned}
\gamma_{X',X,U} &= \begin{cases} 1 & \text{if adding } U \text{ to LPD } D \text{ with } C(D) = X' \text{ yields a LDP } D' \text{ with } C(D') = X' \\ 0 & \text{otherwise} \end{cases} \\
\alpha_{X',X} &= \sum_{U \subseteq E_L(n+1) - E_L(n)} \gamma_{X',X,U} \\
A &= (a_{X',X})_{X',X \in \mathcal{P}}
\end{aligned}$$

where, in the last equation, A is a square matrix whose columns/rows are ordered using the same ordering as in the definition of $\vec{\beta}$ in Definition 10. For convenience, we will also define

$$\mathcal{U} = \{U \,:\, U \subseteq E_L(n+1) - E_L(n)\}.$$

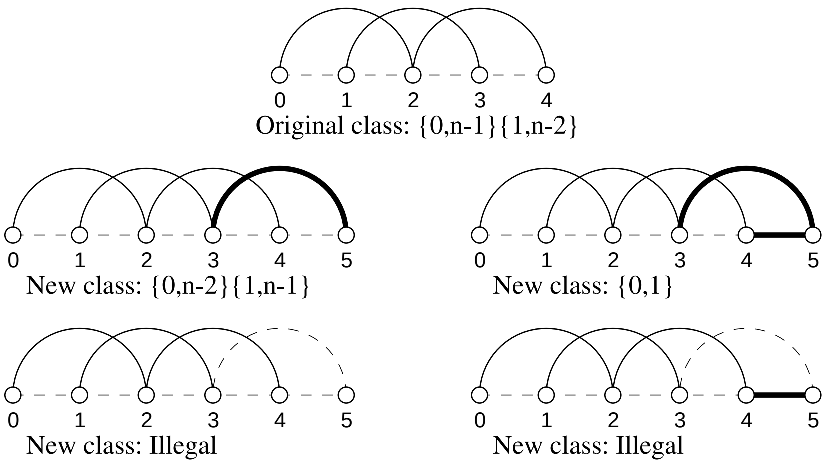

Figure 10: Different ways to add node 5 to a path decomposition of $L_5^{1,2}$ of class $X = \{\,\{0, n-1\},\, \{1, n-2\}\,\}$ to generate different classes of path decomposition of $L_6^{1,2}$. Bold edges are the ones added with node 5. In the notation of Definition 11

- Adding the edge $(3, 5)$ creates a LPD in class $\{0, n-2\}, \{1, n-1\}\}$ so

$$\gamma_{X',X,\{\{n-2,n\}\}} = \begin{cases} 1 & \text{if } X' = \{\,\{0, n-1\},\, \{1, n-2\}\,\} \\ 0 & \text{otherwise} \end{cases}$$

- Adding *both* $(3, 5)$ *and* $(4, 5)$ creates a Hamiltonian Path with endpoints $0, 1$ so

$$\gamma_{X',X,\{\{n-1,n\},\,\{n-2,n\}\}} = \begin{cases} 1 & \text{if } X' = \{\{0, 1\}\} \\ 0 & \text{otherwise} \end{cases}$$

- Adding no edges or the edge $(4, 5)$ creates a path decomposition in which 3 is one of the endpoints of a path. these are *not* legal decompositions since $3 \notin W(6)$. Therefore, $\forall X' \in \mathcal{P}$,

$$\begin{aligned} \gamma_{X',X,\emptyset} &= 0 \\ \gamma_{X',X,\{\{n-1,n\}} &= 0. \end{aligned}$$

Note: As in the observation following Lemma 10 we point out that the value of $\gamma_{X',X,U}$ is *independent* of n.

Combining Lemmas 8 and 9 then yield

Lemma 10 $\forall X' \in \mathcal{P}$,

$$H_{X'}(n+1) = \sum_{X \in \mathcal{P}} a_{X',X} H_X(n) \tag{21}$$

or, equivalently,

$$\vec{H}\left(L_{n+1}\right) = A\vec{H}\left(L_n\right) \tag{22}$$

Proof. The proof is almost exactly the same as that of Lemma 5 and is therefore omitted. □

Finally, combining Lemmas 7 and 10 yields our main theorem of this section which is

Theorem 2 *Let $s_1, s_2, \ldots, s_k$ be given and $H(n)$ denote the number of Hamiltonian Cycles in $C_n^{s_1,s_2,\ldots,s_k}$. Let $\mathcal{P} = \bigcup_{W \subseteq W(n)} Par_2(W)$, $H_X(n)$ denote the number of* Legal Path Decompositions *with classification X and $\vec{H}\left(L_n^{s_1,s_2,\ldots,s_k}\right)$ be the column vector $\left(H_X(n)\right)_{X\in\mathcal{P}}$. Furthermore, let $\vec{\beta} = (\beta_x)_{X\in\mathcal{P}}$ and $A = \left(\alpha_{X',X}\right)_{X,X'\in\mathcal{P}}$ where β_X are the constants defined in Definition 10 and $\alpha_{X',X}$ the constants defined in Definition 11. Then, for $n \geq 2s_k$,*

$$\begin{aligned} H\left(C_n^{s_1,s_2,\ldots,s_k}\right) &= \vec{\beta}\cdot\vec{H}\left(L_n^{s_1,s_2,\ldots,s_k}\right) \\ \vec{T}\left(L_{n+1}^{s_1,s_2,\ldots,s_k}\right) &= A\vec{H}\left(L_n^{s_1,s_2,\ldots,s_k}\right) \end{aligned}$$

This theorem implies that $T\left(C_n^{s_1,s_2,\ldots,s_k}\right)$ satisfies a linear recurrence recurrence relation with constant coefficients of order equal to the rank of the matrix A.

Before continuing on to the next subsection and using Theorem 2 to analyze $H(C_n^{1,2})$ we point out that our definition of $\mathcal{P} = \bigcup_{W\subseteq W(n)} Par_2(W)$ as given in Definition 9 was very loose. It was chosen to simplify the proofs of the lemmas leading up to Theorem 2. Now that we have proven the theorem we quickly describe how to restrict $\mathcal{P}$ to a smaller set which, in practice, can speed up calculations. In particular we can define

Definition 12 *Let $X \in \mathcal{P}$. u* is isolated in *X if u is in its own component in X, i.e., $\{u\}$ is in X. Now define*

$$\mathcal{P}' = \{X \in \mathcal{P} \,:\, s_k - 1 \text{ and } n - s_k \text{ are not isolated in } X\}$$

For example,when discussing $C_n^{1,2}$, $s_k = 2$. Let $X_1 = \{\{0, n-1, n-2\}\{1\}\}$ and $X_2 = \{\{1, n-1, n-2\}\{0\}\}$. Then X_1 and X_2 are both in $\mathcal{P}$ but, since 1 is isolated in X_1, $X_1 \notin \mathcal{P}'$. On the other hand, neither 1 nor $n-2$ are isolated in X_2 so $X_2 \in \mathcal{P}'$

We will now see that $X \notin \mathcal{P}'$ can be ignored. More formally:

Lemma 11 *Let $X \in \mathcal{P} - \mathcal{P}'$, i.e., nodes $(s_k - 1)$ and/or $(n - s_k)$ are isolated in X. Then*

1. $\beta_X = 0$.
2. $\forall X' \in \mathcal{P}'$, $\alpha_{X',X} = 0$.

Proof. To prove (1.) first suppose that node $(s_k - 1)$ is an isolated node in some LPD D with $C(D) = X$. Let $S \subseteq \texttt{Hook}(n)$. In order for $D \cup S$ to be a Hamiltonian cycle of C_n S must contain *two* edges having $(s_k - 1)$ as an endpoint; one edge to "enter" $(s_k - 1)$, the other to "leave" it. But $\texttt{Hook}(n)$ contains only the one edge $(n - 1, s_k - 1)$ with $s_k - 1$ as an endpoint so S can not contain two such edges and $\alpha_{S,X} = 0$.

Now suppose that $(n - s_k)$ is an isolated node in D. Since $\texttt{Hook}(n)$ contains only the one edge $(0, n - s_k)$ having $(n - s_k)$ as an endpoint the proof that $\alpha_{S,X} = 0$ is the same. The proof of (1.) follows for the fact that we just proved $\alpha_{S,X} = 0$ for *every* $S \subseteq \texttt{Hook}(n)$ and $\beta_X = \sum_{S \in \mathcal{S}} \alpha_{S,X} = \sum_{S \in \mathcal{S}} 0 = 0$.

To prove (2.) we will first show that $\forall U \in \mathcal{U}$ and $\forall X \in \mathcal{P}$, $\gamma_{X',X,U} = 0$. (2.) will follow from $\alpha_{X',X} = \sum_{U \in \mathcal{U}} \gamma_{X',X,U}$.

So suppose that node $(s_k - 1)$ is an isolated node in LPD D in L_n with $C(D) = X$. Let $U \in \mathcal{U}$. From (5) and our initial assumption that $n \geq 2s_k$, U does not contain *any* edge with $s_k - 1$ as an endpoint. Therefore, $D \cup U$ continues to have $(s_k - 1)$ as an isolated node. This means that if X has $(s_k - 1)$ as an isolated node and, for some $X' \in \mathcal{P}$, we have $\gamma_{X',X,U} = 1$, then $X' \notin \mathcal{P}'$. This in turn implies that $\forall X' \in \mathcal{P}'$, $\alpha_{X',X} = 0$.

Now suppose that node $(n - s_k)$ is an isolated node in LPD D in L_n with $C(D) = X$. Let $U \in \mathcal{U}$. Note that $E_L(n+1) - E_L(n)$ contains only *one* edge with endpoint $(n - s_k)$, the edge $(n - s_k, n)$. Then the path decomposition of $D \cup U$ must contain $(n - s_k)$ as an endpoint of one of its paths (possibly as an isolated point, which can occur if $(n - s_k, n) \notin U$). But $(n - s_k) \notin W(n + 1)$ so $D \cup U$ is *not* a LPD of L_{n+1}. This means that if $X' \in \mathcal{P}$, i.e., X' *is* any legal classification, then $\gamma_{X',X,U} = 0$, so $\forall X' \in \mathcal{P}$, $\alpha_{X',X} = 0$. $\square$

This lemma immediately implies that we can replace $\mathcal{P}$ with $\mathcal{P}'$ in Theorem 2 and still get the same result. We encapsulate this in the next theorem:

Theorem 3 *Let $s_1, s_2, \ldots, s_k$ be given and $H(n)$ denote the number of Hamiltonian Cycles in $C_n^{s_1,s_2,\ldots,s_k}$. Let $\mathcal{P}'$ be as defined in Definition 12, $H_X(n)$ denote the number of* Legal Path Decompositions *with classification X and $\vec{H}\left(L_n^{s_1,s_2,\ldots,s_k}\right)$ be the column vector $(H_X(n))_{X \in \mathcal{P}'}$. Furthermore, let $\vec{\beta} = (\beta_x)_{X \in \mathcal{P}'}$ and $A = (\alpha_{X',X})_{X,X' \in \mathcal{P}'}$ where β_X are the constants defined in Definition 10 and $\alpha_{X',X}$ the constants defined in Definition 11. Then, for $n \geq 2s_k$,*

$$
\begin{aligned}
H\left(C_n^{s_1,s_2,\ldots,s_k}\right) &= \vec{\beta} \cdot \vec{H}\left(L_n^{s_1,s_2,\ldots,s_k}\right) \\
\vec{T}\left(L_{n+1}^{s_1,s_2,\ldots,s_k}\right) &= A\,\vec{H}\left(L_n^{s_1,s_2,\ldots,s_k}\right)
\end{aligned}
$$

4.2 Analyzing $H(C_n^{1,2})$

In this section we use the unhooking technique to derive a formula for $H(C_n^{1,2})$ via Theorem 3. In this case $k = 2$ and $(s_1, s_2) = (1, 2)$ so

$$\texttt{Hook}(n) = E_C(n) - E_L(n) = \Big\{ (n-1,0),\ (n-1,1),\ (n-2,0) \Big\}$$

and

$$E_L(n+1) - E_L(n) = \Big\{ (n, n-1),\ (n, n-2) \Big\}$$

Also

$$W(n) = \{0, 1, n-1, n-2\}.$$

and $\mathcal{P}'$ is all of the partitions of subsets of $W(n)$ in which all of the sets in the partition have size one or two and $(s_2 - 1) = 1$ and $(n - s_2) = (n-2)$ are not allowed to be isolated nodes.

This gives

$$\vec{H}(L_n^{1,2}) = \begin{pmatrix} H_{\{0,1\}}(n) \\ H_{\{0,n-2\}}(n) \\ H_{\{0,n-1\}}(n) \\ H_{\{1,n-2\}}(n) \\ H_{\{1,n-1\}}(n) \\ H_{\{n-2,n-1\}}(n) \\ H_{\{0,1\}\{n-1\}}(n) \\ H_{\{0,n-2\}\{n-1\}}(n) \\ H_{\{1,n-2\}\{0\}}(n) \\ H_{\{1,n-2\}\{n-1\}}(n) \\ H_{\{1,n-2\}\{0\}\{n-1\}}(n) \\ H_{\{1,n-1\}\{0\}}(n) \\ H_{\{n-2,n-1\}\{0\}}(n) \\ H_{\{0,1\}\{n-2,n-1\}}(n) \\ H_{\{0,n-2\}\{1,n-1\}}(n) \\ H_{\{0,n-1\}\{1,n-2\}}(n) \\ H_{\emptyset}(n) \end{pmatrix}$$

Mechanically calculating the values $\alpha_{S,X}$ and $\beta_X = \sum_{S \in \mathcal{S}} \alpha_{S,X}$ from Definition 10 and setting $\vec{\beta} = (\beta_X)_{X \in \mathcal{P}'}$ gives

$$\vec{\beta} = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \tag{23}$$

Mechanically calculating $\gamma_{X',X,U}$ and $\alpha_{X',X} = \sum_{U \subseteq (E_L(n+1) - E_L(n))} \gamma_{X',X,U}$ from

Definition 11 and setting $A = \left(a_{X',X}\right)_{X',X \in \mathcal{P}'}$ gives

$$A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Let $G(x) = \sum_{n=1}^{\infty} H\left(C_n^{1,2}\right)$ be the generating function for $T\left(C_n^{1,2}\right)$. We can solve

$$H\left(C_n^{1,2}\right) = \vec{\beta} \cdot \vec{H}\left(L_n^{1,2}\right), \qquad \vec{H}\left(L_{n+1}^{1,2}\right) = A\vec{H}\left(L_n^{1,2}\right)$$

to find that

$$G(x) = \frac{9 - 6x - 8x^2 - 5x^4 + 8x^5}{1 - 2x + x^3 + x^5 - x^6}$$

so

$$H(C_n^{1,2}) = 2H(C_{n-1}^{1,2}) - H(C_{n-3}^{1,2}) - H(C_{n-5}^{1,2}) + H(C_{n-6}^{1,2})$$

with initial values $9, 12, 16, 23, 29, 41$ for $n = 4, 5, 6, 7, 8, 9$ respectively. This can be solved to show that

$$H(C_n^{1,2}) \sim 1.46557^n.$$

We also note that, without any extra work, the technique also permits us to solve for

$$H(L_n^{1,2}) = H_\emptyset(n),$$

the number of Hamiltonian Cycles in $L_n^{1,2}$. Letting $G(x) = \sum_{n=1}^{\infty} H_\emptyset(n)$ we find that

$$\begin{aligned} G(x) &= \frac{1}{1-x}, \\ H_\emptyset(n) &= 1. \end{aligned}$$

This can also be easily derived from first principles, leading to a simple check on our techniques. (See Figure 11.)

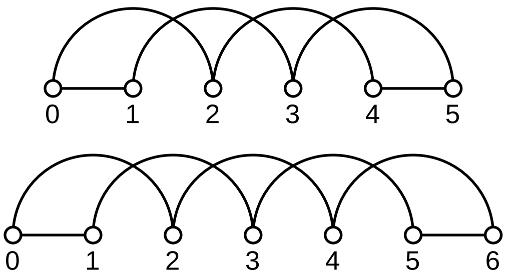

Figure 11: $H(L_n^{1,2}) = 1$. Illustrated are the unique Hamiltonian cycles for $n = 6$ and $n = 7$.

5 Conclusion

In this paper we developed the first general *combinatorial* technique for showing that the number of spanning trees in fixed-jump circulant graphs $C_n^{s_1,s_2,\cdots,s_k}$ satisfy fixed-order constant coefficient recurrence relations in n.

This contrasts to the only previously known general method which used algebraic (spectral/eigenvalue) methods.

Our basic approach, unhooking, permits decomposing a problem on *circulant* graphs into many problems on *step*-graphs. We then used the fact that step-graphs are much more amenable to recursive decompositions than circulant graphs to yield our results.

A nice consequence of our technique is that it can be easily modified to work for other parameters of fixed-jump circulant graph. To illustrate this, in this paper we derived an analogous result for Hamiltonian cycles.

As another example we now quickly sketch how to use the technique to count *perfect matchings* in $C_n^{s_1,s_2,\cdots,s_k}$. Given graph $G = (V, E)$ recall that a *matching* $M \subseteq E$ is a set of edges such that *every $v \in V$ is an endpoint of at most one edge in M*. A matching is *perfect* if *every $v \in V$ is an endpoint of exactly one edge in M*. Note that if M is a perfect matching of $C_n^{s_1,s_2,\cdots,s_k}$ then $M' = M - \texttt{Hook}(n)$ satisfies the following properties:

- $M' \subseteq E_L(n)$.
- Every $v \in V(n) - W(n)$ is an endpoint of exactly one edge in M.
- Every $v \in W(n)$ is an endpoint of at most one edge in M.

We call M' that satisfy these three properties *legal semi-matchings of L_n* and can define their classification as

$$C(M') = \{u \in W(n) : u \text{ is not an endpoint of some edge in } M'\}.$$

Note that the set of classifications of legal semi-matchings is a subset of $\mathcal{P} = \{W : W \subseteq W(n)\}$, the *power-set* of $W(n)$. We can follow almost[3] exactly the same step-by-step approach that was used in for spanning trees in section 3 and Hamiltonian cycles in section 4 to show that the number of prefect matchings in $C_n^{s_1,s_2,\cdots,s_k}$ satisfies a constant coefficient recurrence relation in n of order $|\mathcal{P}| = 2^{s_k}$. A very similar argument lets us show that the number of matchings (not just the perfect ones) also satisfies constant coefficient recurrence relations in n of order $|\mathcal{P}| = 2^{s_k}$.

To the best of our knowledge this is the first time that parameters other than spanning trees, e.g., Hamiltonian cycles and matchings, have been generally analyzed for circulant graphs.

The unhooking technique also permits showing that the Eulerian cycles and Eulerian orientations in $C_n^{s_1,s_2,\cdots,s_k}$ satisfy fixed-order constant coefficient recurrence relations in n. Although the general unhooking approach utilized is the same, counting Eulerian cycles and Eulerian orientations requires introducing an additional set of technical tools and we therefore defer that analysis to a sequel [10].

We also point out that, even though our technique was described only for *undirected* circulant graphs, it is quite easy to extend it to *directed* circulant graphs as well.

We conclude with two open questions. The first deals with the usefulness of our results and the "tightness" of the orders of our recurrence relations. The second, with extending our techniques to *non fixed-jump* circulant graphs.

The order of the recurrence relation derived by our technique grows very quickly. For example, in the spanning tree case, our matrix size (and therefore the order of our recurrence relation) grows as the Bell number $B(2s_k)$. Since $B(n) \sim n^n \left(\frac{n}{\log n}\right)^{1-1\log n} (\log n)^{-n}$, our matrices quickly become very large and unmanageable so our result is only useful as a proof that the recurrence relation exists but not as a practical method for deriving it. In particular, the algebraic derivations in [18, 22] showed that the quantity $T(n)$ that we are calculating grows as

$$T(C_n^{s_1,s_2,\cdots,s_k}) = na_n^2,$$

where a_n satisfies a recurrence relation of order 2^{s_k-1} with constant coefficients. This means that $T(C_n^{s_1,s_2,\cdots,s_k})$ itself satisfies a recurrence relation of order no greater than $2^{4(s_k-1)}$,, something which grows in s_k much slower than $B(2s_k)$. The question then is whether there is a combinatorial proof that a recurrence relation of this order exists. It is unlikely that our technique could be easily modified to answer this question since our technique, being so general, does not use much spanning-tree specific structure. A more special case approach would seem to be called for.

We also point out that our analysis implicitly assumed that $s_1, s_2, \ldots, s_k$, the

[3] Leaving out some technicalities related to the parity of the classifications

jumps in the circulant graph, are *fixed* (this was needed to perform the unhooking). One could also define circulant graphs in which the jump size depends upon the number of vertices in the graph. The canonical example of such a graph is the three-regular *Möbius ladder*[4] $C_{2n}^{1,n}$ for which it is known that the number of spanning trees grows as

$$T\left(C_{2n}^{1,n}\right) = \frac{n}{2}\left[\left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n + 2\right]. \tag{24}$$

There are many proofs of this result e.g., [6, 14, 15], both combinatorial and algebraic, but all of them are very problem specific. Recent work [9] has managed to slightly extended the algebraic approach of [6] to show that for a very small number of *non-fixed-jump* circulant graphs, the number of spanning trees does satisfy a recurrence relation but it is still an open question as to whether there is *any* derivation, combinatorial or algebraic, that shows that the number of spanning trees in general *non-fixed-jump* circulant graphs satisfies a recurrence relation.

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