THE NUMBER OF SPANNING TREES IN CIRCULANT GRAPHS*

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Abstract

In this paper we develop a method for determining the exact number of spanning trees in (directed or undirected) circulant graphs. Using this method we can, for any class of circulant graph, exhibit a recurrence relation for the number of its spanning trees. We describe the method and give examples of its use.

1 Introduction

The number of spanning trees in a graph (network) is an important, well studied quantity [6]. A classic result known as the *Matrix Tree Theorem* [8] expresses the number of spanning trees $T(G)$ of a graph G as a function of the determinant of a matrix that can be easily constructed from G's

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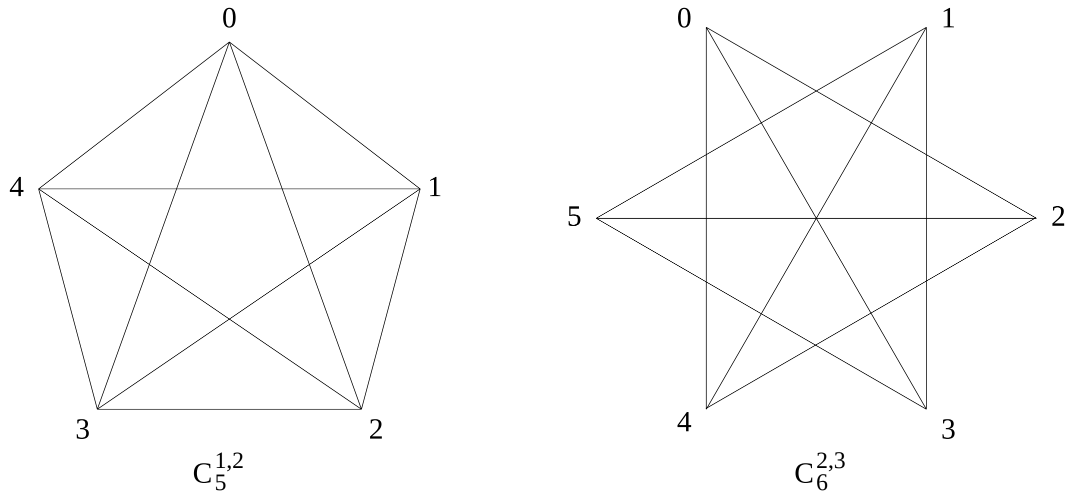

Figure 1: Two examples of circulant graphs.

incidence matrix. In practice, though, this method of counting spanning trees by calculating determinants is infeasible for large graphs.

For some special classes of graphs, it is possible to give explicit, simple formulas for the number of trees. If G is the complete graph K_n, then $Cayley's\ tree\ formula$ [7] states that $T(K_n) = n^{n-2}$. Other special cases can be found in [4, 12, 13]. In this paper we consider the problem restricted to the class of circulant graphs.

Let $1 \le s_1 < s_2 < \cdots < s_k$. The *undirected circulant graph*, $C_n^{s_1,s_2,\cdots,s_k}$, has n vertices labelled $0,\ 1,\ 2,\ \cdots, n-1$, with each vertex i $(0 \le i \le n-1)$ adjacent to $2k$ vertices $i \pm s_1,\ i \pm s_2,\ \cdots,\ i \pm s_k \bmod n$. The *directed circulant graph*, $\vec{C}_n^{s_1,s_2,\cdots,s_k}$, is a digraph on n vertices $0,\ 1,\ 2,\ \cdots, n-1$; for each vertex i $(0 \le i \le n-1)$, there are k arcs from i to vertices $i+s_1,\ i+s_2,\ \cdots,\ i+s_k \bmod n$. Multiple edges and self-loops are permitted. In Figure 1 we give examples of two circulant graphs, $C_5^{1,2}$ and $C_6^{2,3}$.

$T(C_n^{s_1,s_2,\cdots,s_k})$ is the number of spanning trees in the undirected circulant graph $C_n^{s_1,s_2,\cdots,s_k}$; $T(\vec{C}_n^{s_1,s_2,\cdots,s_k})$, the number of spanning trees in the directed circulant graph $\vec{C}_n^{s_1,s_2,\cdots,s_k}$. A spanning tree in a digraph is a rooted tree with directed paths from the root to all nodes. When counting the number of spanning trees in a digraph we count all trees rooted at all possible roots. We note that, in both the undirected and directed cases, if

$\gcd(s_1, s_2, \cdots, s_k, n) \neq 1$ then the graph is not connected so there are no spanning trees.

The formula $T(C_n^{1,2}) = nF_n^2$, F_n the *Fibonacci* numbers, was originally conjectured by Bedrosian [2] and subsequently proven by Kleitman and Golden [9]. The same formula was also conjectured by Boesch and Wang [5] (without the knowledge of [9]). Different proofs can been found in [1, 4, 13]. Formulas for $T(C_n^{1,3})$ and $T(C_n^{1,4})$ are provided in [12].

This paper shows that the above are not special cases. In Section 2 we start from the determinant formula and show that, for fixed $s_1, s_2, \cdots, s_k$, the number of spanning trees in the circulant graph with n vertices always satisfies a recurrence relation and describe how to derive the relation. In Section 3 we apply the method to find the formulas for $T(C_n^{1,5})$, $T(C_n^{2,3})$, $T(C_n^{2,4})$. Other formulas are listed in the tables appended. We should point out that formulas for $T(C_n^{2,4})$ and $T(C_n^{3,4})$ have already been derived in [11]; we list them here for the sake of completeness. We conclude in Section 4 by deriving the asymptotic behavior of these quantities.

2 Basic lemmas

We start from the fact that, in the circulant graph case, the matrix determinant formula for the number of spanning trees is known to reduce to a simpler product formula:

Lemma 1 (Biggs [3], Proposition 3.5 plus Corollary 6.5 and [14])

$$\begin{aligned}
T(C_n^{s_1,s_2,\cdots,s_k}) &= \frac{1}{n}\prod_{j=1}^{n-1}(2k - \varepsilon^{-s_1 j} - \varepsilon^{-s_2 j} - \cdots - \varepsilon^{-s_k j} \\
&\qquad\qquad - \varepsilon^{s_1 j} - \varepsilon^{s_2 j} - \cdots - \varepsilon^{s_k j}), \\
T(\vec{C}_n^{s_1,s_2,\cdots,s_k}) &= \prod_{j=1}^{n-1}(k - \varepsilon^{s_1 j} - \varepsilon^{s_2 j} - \cdots - \varepsilon^{s_k j}),
\end{aligned}$$

where ε^{-j} is the conjugate of ε^j, $\varepsilon = e^{\frac{2\pi i}{n}}$.

The main result of this section will be to show that, for fixed $s_1, s_2, \cdots, s_k$, the above formula actually satisfies a recurrence relation in n. We start by manipulating the formula into a first more convenient form:

Lemma 2 *Let*

$$g_{s_1,s_2,\cdots,s_k}(x) = 2k - x^{-s_1} - x^{-s_2} - \cdots - x^{-s_k} - x^{s_1} - x^{s_2} - \cdots - x^{s_k}. \quad (1)$$

Set

$$f_{s_1,s_2,\cdots,s_k}(x) = -\frac{x^{s_k} g_{s_1,s_2,\cdots,s_k}(x)}{(x-1)^2}. \quad (2)$$

Then

$$f_{s_1,s_2,\cdots,s_k}(x) = \sum_{i=1}^{k} x^{s_k - s_i} \left(\sum_{j=0}^{s_i - 1} x^j \right)^2. \quad (3)$$

Now let $M_{s_1,s_2,\cdots,s_k}$ be the companion matrix of $f_{s_1,s_2,\cdots,s_k}(x)$. This means that if we denote $f_{s_1,s_2,\cdots,s_k}(x)$ as $a_0 + a_1 x + \cdots + a_{2s_k-3}x^{2s_k-3} + x^{2s_k-2}$, then

$$M_{s_1,s_2,\cdots,s_k} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & -a_{2s_k-4} \\ 0 & 0 & \cdots & 1 & -a_{2s_k-3} \end{pmatrix}.$$

If $A_n = nT(C_n^{s_1,s_2,\cdots,s_k})$ then

$$\begin{aligned} A_n &= (-1)^{(s_k-1)(n-1)} \frac{n^2}{f_{s_1,s_2,\cdots,s_k}(1)} \prod_{j=0}^{n-1} f_{s_1,s_2,\cdots,s_k}(\varepsilon^j) \\ &= (-1)^{(s_k-1)(n-1)} \frac{n^2}{f_{s_1,s_2,\cdots,s_k}(1)} |I - M^n_{s_1,s_2,\cdots,s_k}| \end{aligned}$$

where $|\cdot|$ represents the determinant of the matrix.

Proof. Equation (3) can be found by direct calculation.

For the remainder of the proof note that from Lemma 1

$$\begin{aligned} A_n &= \prod_{j=1}^{n-1} (2k - \varepsilon^{-s_1 j} - \varepsilon^{-s_2 j} - \cdots - \varepsilon^{-s_k j} - \varepsilon^{s_1 j} - \varepsilon^{s_2 j} - \cdots - \varepsilon^{s_k j}) \\ &= \prod_{j=1}^{n-1} g_{s_1,s_2,\cdots,s_k}(\varepsilon^j) \\ &= (-1)^{(n-1)} \prod_{j=1}^{n-1} \varepsilon^{-s_k j} \prod_{j=1}^{n-1} (\varepsilon^j - 1)^2 \prod_{j=1}^{n-1} f_{s_1,s_2,\cdots,s_k}(\varepsilon^j). \end{aligned}$$

By calculation

$$\begin{aligned}
&|I - M_{s_1,s_2,\cdots,s_k}| = f_{s_1,s_2,\cdots,s_k}(1),\\
&|\varepsilon^j I - M_{s_1,s_2,\cdots,s_k}| = f_{s_1,s_2,\cdots,s_k}(\varepsilon^j),\\
&\prod_{j=1}^{n-1}(\varepsilon^j - 1) = (-1)^{n-1}n,\\
&\prod_{j=1}^{n-1}\varepsilon^{-s_k j} = (-1)^{s_k(n-1)}.
\end{aligned}$$

Thus

$$\begin{aligned}
A_n &= (-1)^{(s_k-1)(n-1)}\frac{n^2}{f_{s_1,s_2,\cdots,s_k}(1)}\prod_{j=0}^{n-1} f_{s_1,s_2,\cdots,s_k}(\varepsilon^j)\\
&= (-1)^{(s_k-1)(n-1)}\frac{n^2}{f_{s_1,s_2,\cdots,s_k}(1)}|I - M^n_{s_1,s_2,\cdots,s_k}|.
\end{aligned}$$

□

Lemma 3 *Let*

$$\vec{f}_{s_1,s_2,\cdots,s_k}(x) = \sum_{j=1}^{k}\sum_{i=0}^{s_j-1} x^i.$$

Set $\vec{M}_{s_1,s_2,\cdots,s_k}$ to be the companion matrix of $\vec{f}_{s_1,s_2,\cdots,s_k}(x)$. Then

$$T(\vec{C}_n^{s_1,s_2,\cdots,s_k}) = \frac{n}{\vec{f}_{s_1,s_2,\cdots,s_k}(1)}|I - \vec{M}^n_{s_1,s_2,\cdots,s_k}|.$$

Proof. Note that

$$\prod_{j=1}^{n-1}(k - \varepsilon^{s_1 j} - \varepsilon^{s_2 j} - \cdots - \varepsilon^{s_k j}) = \prod_{j=1}^{n-1}(1-\varepsilon^j)\prod_{j=1}^{n-1}\vec{f}_{s_1,s_2,\cdots,s_k}(\varepsilon^j).$$

The remainder of the proof is similar to that of Lemma 2. □

We now show that the A_n must have a special form. This form will permit us to write a recurrence relation for them.

Lemma 4 *For any* $1 \le s_1 < s_2 < \cdots < s_k$, *we have the following formula:*

$$T(C_n^{s_1,s_2,\cdots,s_k}) = \frac{1}{n}A_n = na_n^2,$$

where $a_n = \frac{1}{\sqrt{f_{s_1,s_2,\cdots,s_k}(1)}} \sum_{i=1}^{2^{s_k-1}} r_i^n$ *for some (not necessarily unique) complex numbers* r_i, $1 \le i \le 2^{s_k-1}$.

Proof. We need to show that $f_{s_1,s_2,\cdots,s_k}(x)$ can be expressed in a particular form. We do this by finding restrictions on its roots.

By calculation we find that

$$g_{s_1,s_2,\cdots,s_k}(1) = g'_{s_1,s_2,\cdots,s_k}(1) = 0$$

but $g''_{s_1,s_2,\cdots,s_k}(1) \neq 0$ so 1 is a root of multiplicity 2 of $g_{s_1,s_2,\cdots,s_k}(x)$. By the way $f_{s_1,s_2,\cdots,s_k}(x)$ is defined in (2) this implies that 1 is not a root of $f_{s_1,s_2,\cdots,s_k}(x)$.

Now note that $g_{s_1,s_2,\cdots,s_k}(x) = \sum_{j=1}^{k}(2 - x^{-s_j} - x^{s_j})$ and $\forall j$, $u_{s_j}(x) = 2 - x^{-s_j} - x^{s_j} \ge 0$, when $x = -1$ with $u_{s_j}(-1) = 0$ if and only if S_j is even. Thus $g(-1) = 0$ iff s_1, s_2, $\cdots$, s_k are all even numbers. It is also easy to see that $\forall j$, $u'_{s_j}(x) = s_j x^{-s_j-1} - s_j x^{s_j-1} = 0$, when $x = -1$ so $g'_{s_1,s_2,\cdots,s_k}(-1) = 0$. But, for even s_j, $u''_{s_j}(-1) < 0$ so if $g(-1) = 0$ then $g''(-1) \neq 0$. Plugging back into (2) we find that if -1 is a root of $f_{s_1,s_2,\cdots,s_k}(x)$ it must also be a root of order 2.

Finally, we note that

$$f_{s_1,s_2,\cdots,s_k}\left(\frac{1}{x}\right) = \frac{1}{x^{2s_k-2}} f_{s_1,s_2,\cdots,s_k}(x)$$

so α is a root if and only if $\frac{1}{\alpha}$ is a root. More generally set

$$\overline{f}_{s_1,s_2,\cdots,s_k}(x) = \frac{f_{s_1,s_2,\cdots,s_k}(x)}{(x-\alpha)(x-\frac{1}{\alpha})}.$$

Then

$$\overline{f}_{s_1,s_2,\cdots,s_k}\left(\frac{1}{x}\right) = \frac{1}{x^{2s_k-4}} \overline{f}_{s_1,s_2,\cdots,s_k}(x).$$

Continuing in this fashion shows that α and $\frac{1}{\alpha}$ always have the same multiplicity as roots of $f_{s_1,s_2,\cdots,s_k}(x)$.

Combining all the above observations let us write

$$f_{s_1,s_2,\cdots,s_k}(x) = c\prod_{i=1}^{s_k-1}(x-\alpha_i)(x-\alpha_i^{-1}).$$

for some constant c. But, from (1) and (2) we find that $f(0)=1$ so $c=1$ and

$$f_{s_1,s_2,\cdots,s_k}(x) = \prod_{i=1}^{s_k-1}(x-\alpha_i)(x-\alpha_i^{-1}). \tag{4}$$

By definition

$$f_{s_1,s_2,\cdots,s_k}(1) = |I - M_{s_1,s_2,\cdots,s_k}| = \prod_{i=1}^{s_k-1}(1-\alpha_i)(1-\alpha_i^{-1}).$$

From Lemma 2,

$$A_n = (-1)^{(s_k-1)(n-1)}n^2\prod_{j=1}^{n-1}f_{s_1,s_2,\cdots,s_k}(\varepsilon^j).$$

Thus

$$\begin{aligned} a_n^2 &= (-1)^{(s_k-1)(n-1)}\prod_{j=1}^{n-1}\prod_{i=1}^{s_k-1}(\varepsilon^j-\alpha_i)(\varepsilon^j-\alpha_i^{-1}) \\ &= (-1)^{(s_k-1)(n-1)}\prod_{i=1}^{s_k-1}\frac{(1-\alpha_i^n)(1-\alpha_i^{-n})}{(1-\alpha_i)(1-\alpha_i^{-1})} \\ &= \left(\frac{\prod_{i=1}^{s_k-1}(1-\alpha_i^n)}{\sqrt{f_{s_1,s_2,\cdots,s_k}(1)}\sqrt{(\prod_{i=1}^{s_k-1}(-\alpha_i))^n}}\right)^2. \end{aligned}$$

Consequently $a_n = \frac{1}{\sqrt{f_{s_1,s_2,\cdots,s_k}(1)}}\sum_{i=1}^{2^{s_k-1}} r_i^n$ for some complex numbers $r_i,\ 1\le i\le 2^{s_k-1}$ (where it is possible that that $r_i=r_j$ for some $i\neq j$).

□

Note. It is well known that if $a_n = c\sum_{i=1}^{2^{s_k-1}} r_i^n$ for some constant c then a_n is the coefficient of x^n in the generating function

$$G(x) = \sum_{i=1}^{2^{s_k-1}} \frac{c}{1-r_i x} = c\frac{\sum_{i=1}^{2^{s_k-1}} \prod_{j\neq i}(1-r_i x)}{\prod_{i=1}^{2^{s_k-1}}(1-r_i x)}$$

Thus $G(x) = P(x)/Q(x)$, $P(x)$ and $Q(x)$ polynomials (i.e., $G(x)$ is a rational function) where degree of $Q(x) \leq 2^{s_k-1}$ so there exist $b_i,\ 1 \leq i \leq 2^{s_k-1}$ such that the a_n satisfy a recurrence relation of the form

$$\forall n \geq 2^{s_k-1},\ a_n = \sum_{i=1}^{2^{s_k-1}} b_i a_{n-i}.$$

We can find these b_i by noting that Lemma 2 gives us a direct way of calculating any a_i. We can therefore use Lemma 2 to evaluate all of the a_i for $i \leq 2^{S_k}$ and then solve for the b_i. We will see many examples of this in the next section.

We also note that we can say even more about the structure of $G(x)$. Suppose that $G(x) = \frac{P(x)}{Q(x)}$ where the $P(x)$ and $Q(x)$ are now relatively prime. We claim that every root of $Q(x)$ must be simple. This follows from the fact that if r was not a simple root then the coefficient of x^n in $G(x)$ would include an additive term of the form $h(n)r^n$ where $h(n)$ is a non-constant polynomial, contradicting the lemma.

Lemma 5 *For any $1 \leq s_1 < s_2 < \cdots < s_k$, we have the following formula:*

$$T(\vec{C}_n^{s_1,s_2,\cdots,s_k}) = na_n,$$

where a_n satisfies a linear recurrence relation of order 2^{s_k-1} with initial conditions given by Lemma 3.

Proof. Let $\vec{f}_{s_1,s_2,\cdots,s_k}(x) = \prod_{j=1}^{s_k-1}(x-\alpha_j)$, where $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_{s_k-1}$ are roots of $\vec{f}_{s_1,s_2,\cdots,s_k}(x)$. The proof of the remainder of the theorem is similar to the proof of Lemma 4.

$\square$

We now prove a lemma saying that, in many cases, $f_{s_1,s_2,\cdots,s_k}(\alpha) = 0$ does not have a unit root; this in turn will imply that the minimum modulus root of $Q(x)$ is not only simple (which is true for all roots) but is also the only root of minimum modulus. This fact will later permit us to derive the asymptotic growth of the number of spanning trees as n increases.

Lemma 6 *If* $\gcd(s_1, s_2, \cdots, s_k) = 1$*, and* $f_{s_1,s_2,\cdots,s_k}(\alpha) = 0$*, then* $|\alpha| \neq 1$.

Proof. From equation (3) we know that $\alpha = 1$ is not a root of $f_{s_1,s_2,\cdots,s_k}(x)$.

Recall that

$$f_{s_1,s_2,\cdots,s_k}(x) = -\frac{x^{s_k} g_{s_1,s_2,\cdots,s_k}(x)}{(x-1)^2}.$$

This means that if $\alpha \neq 1$ is a root of $f_{s_1,s_2,\cdots,s_k}(x)$ then α is also a root of $g_{s_1,s_2,\cdots,s_k}(x)$.

Now suppose that some $\alpha \neq 1$ with $|\alpha| = 1$ is a root of $g_{s_1,s_2,\cdots,s_k}(x)$. Then

$$\alpha = e^{i\varphi} = \cos\varphi + i\sin\varphi$$

for some φ. Thus

$$\begin{aligned} g_{s_1,s_2,\cdots,s_k}(\alpha) &= \sum_{j=1}^{k}(2 - \alpha^{-s_j} - \alpha^{s_j}) \\ &= \sum_{j=1}^{k}(2 - 2\cos s_j\varphi) \\ &= 0. \end{aligned}$$

This implies that

$$\forall 1 \leq j \leq k, \quad \cos s_j\varphi = 1.$$

This in turn implies that

$$\forall 1 \leq j \leq k, \quad e^{is_j\varphi} = (\cos\varphi + i\sin\varphi)^{s_j} = 1 \tag{5}$$

which means that α is a unit root of 1. Let m be the minimum positive integer such that α is a unit root of degree m. From (5) $m|s_j$, $j = 1, 2, \cdots, k$; from the fact that $\alpha \neq 1$, we have that $m > 1$. This contradicts the assumption $\gcd(s_1, s_2, \cdots, s_k) = 1$ so $g_{s_1,s_2,\cdots,s_k}(x)$ cannot have a root with $|\alpha| = 1$ and thus neither can $f_{s_1,s_2,\cdots,s_k}(x)$. $\square$

Now let $G(x) = \sum_n a_n x^n$ be the generating function of the a_n. It was previously shown that $G(x)$ is a rational function. The preceding lemma will imply that the minimum modulus pole of $G(x)$ is unique in modulus.

Corollary 1 *Let $P(x)$, $Q(x)$ be relatively prime polynomials such that $G(x) = \frac{P(x)}{Q(x)}$. Then if $\gcd(s_1, s_2, \cdots, s_k) = 1$, there is only one root of $Q(x)$ that has minimum modulus.*

Proof. Referring back to the proof of Lemma 4 we find (equation (4)) that

$$f_{s_1,s_2,\cdots,s_k}(x) = \prod_{i=1}^{s_k-1} (x - \alpha_i)(x - \alpha_i^{-1})$$

for some α_i. Without loss of generality we may assume that $\forall i$, $|\alpha_i| \geq 1$; otherwise we may swap α_i and $\frac{1}{\alpha_i}$.

In the same proof we found that

$$a_n^2 = \left(\frac{\prod_{i=1}^{s_k-1}(1 - \alpha_i^n)}{\sqrt{f_{s_1,s_2,\cdots,s_k}(1)}\ \sqrt{(\prod_{i=1}^{s_k-1}(-\alpha_i))^n}} \right)^2 .$$

We used this to derive that $a_n = \frac{1}{\sqrt{f_{s_1,s_2,\cdots,s_k}(1)}} \sum_{i=1}^{2^{s_k-1}} r_i^n$ for some complex numbers r_i, $1 \leq i \leq 2^{s_k-1}$. By definition, these r_i are the roots of $Q(x)$. Expanding the product we see that each r_j is of the form

$$r_j = \frac{\prod_{i=1}^{s_k-1} \alpha_i^{e_j(i)}}{\sqrt{\prod_{i=1}^{s_k-1}(-\alpha_i)}}$$

where $e_j(i) \in \{0, 1\}$ and, taken over all roots r_i, all 2^{s_k-1} choices of the $e_j(i)$ may occur. Note that because all of the α_j satisfy $|\alpha_j| \geq 1$ the minimum modulus over the r_i is

$$\left| \frac{1}{\sqrt{\prod_{i=1}^{s_k-1}(-\alpha_i)}} \right| ;$$

This modulus is only achieved by roots r_i that satisfy

$$\text{if } e_j(i) = 1 \text{ then } |\alpha_j| = 1.$$

From Lemma 6 we know that $\forall j,\ |\alpha_j| \neq 1$. Thus a minimum modulus root of $Q(x)$ is only realized for the one root r_i such that $\forall j,\ e_j(i) = 0$. This proves the corollary. $\square$

The previous lemmas assumed that $\gcd(s_1, s_2, \cdots, s_k) = 1$. We will now see that this is a reasonable assumption since, otherwise, the problem can easily be transformed to one in which the gcd() really is 1.

Lemma 7 *If* $\gcd(n,\ d) = 1$, *then*

$$\begin{aligned} T(C_n^{s_1,s_2,\cdots,s_k}) &= T(C_n^{ds_1,ds_2,\cdots,ds_k}), \\ T(\vec{C}_n^{s_1,s_2,\cdots,s_k}) &= T(\vec{C}_n^{ds_1,ds_2,\cdots,ds_k}). \end{aligned}$$

Proof. From Lemmas 1 and 2 we have

$$\begin{aligned} T(C_n^{s_1,s_2,\cdots,s_k}) &= \frac{1}{n}\prod_{j=1}^{n-1} g_{s_1,s_2,\cdots,s_k}(\varepsilon^j), \\ T(C_n^{ds_1,ds_2,\cdots,ds_k}) &= \frac{1}{n}\prod_{j=1}^{n-1} g_{ds_1,ds_2,\cdots,ds_k}(\varepsilon^j) = \frac{1}{n}\prod_{j=1}^{n-1} g_{s_1,s_2,\cdots,s_k}(\varepsilon^{dj}). \end{aligned}$$

If $\gcd(n,\ d) = 1$, then $dj \equiv dj' \pmod{n}$ if and only if $j \equiv j' \pmod{n}$ so

$$\{1,\ 2,\ \cdots,\ n-1\} = \{d \bmod n,\ 2d \bmod n,\ \cdots,\ (n-1)d \bmod n\}.$$

Thus

$$T(C_n^{s_1,s_2,\cdots,s_k}) = T(C_n^{ds_1,ds_2,\cdots,ds_k}).$$

Similarly,

$$T(\vec{C}_n^{s_1,s_2,\cdots,s_k}) = T(\vec{C}_n^{ds_1,ds_2,\cdots,ds_k}).$$

$\square$

Note. Lemma 7 can actually be proven directly by noting that, if $\gcd(n,\ d) = 1$, there is an isomorphism between graphs $C_n^{s_1,s_2,\cdots,s_k}$ and $C_n^{ds_1,ds_2,\cdots,ds_k}$. We define the function $f : i \mapsto di \bmod n$ from $\{0,\ 1,\ 2,\ \cdots,\ n-1\}$ to itself. As in the the proof of Lemma 7, we see that f is a bijection. Furthermore, if $\gcd(n,\ d) = 1$, then $i_1 - i_2 = s_j (\bmod n)$ if and only if $di_1 - di_2 = ds_j (\bmod n)$, where $i_1,\ i_2 \in \{0,\ 1,\ 2,\ \cdots,\ n-1\}$ and $s_j \in \{s_1,\ s_2,\ \cdots,\ s_k\}$. Thus $(i, j) \in C_n^{s_1,s_2,\cdots,s_k}$ if and only if $(di, dj) \in C_n^{ds_1,ds_2,\cdots,ds_k}$ and f is an isomorphism between $C_n^{s_1,s_2,\cdots,s_k}$ and $C_n^{ds_1,ds_2,\cdots,ds_k}$. The same method can be applied to the directed case.

Given $1 \le s_1 < s_2 < \cdots < s_k$ the graphs $C_n^{s_1,s_2,\cdots,s_k}$ and $\vec{C}_n^{s_1,s_2,\cdots,s_k}$ are connected if and only if $\gcd(s_1, s_2, \cdots, s_k, n) = 1$. If the graph is unconnected then it does not contain any spanning trees. This proves

Corollary 2 *If* $\gcd(s_1, s_2, \cdots, s_k) = d \neq 1$ *then*

$$T(C_n^{s_1,s_2,\cdots,s_k}) = \begin{cases} 0 & \gcd(n,d) \neq 1 \\ T\left(C_n^{\frac{s_1}{d},\frac{s_2}{d},\cdots,\frac{s_k}{d}}\right) & \gcd(n,d) = 1 \end{cases}$$

and

$$T(\vec{C}_n^{s_1,s_2,\cdots,s_k}) = \begin{cases} 0 & \gcd(n,d) \neq 1 \\ T\left(\vec{C}_n^{\frac{s_1}{d},\frac{s_2}{d},\cdots,\frac{s_k}{d}}\right) & \gcd(n,d) = 1. \end{cases}$$

3 Results

In this section we show how to apply Lemma 4. In the first two proofs we work from basic principles, specializing Lemma 4.

Theorem 8 *Case 1:* $s_1 = 1,\ s_2 = 5$.

$$T(C_n^{1,5}) = \frac{1}{n} A_n = n a_n^2,$$

where a_n *satisfies the recurrence relation:*

$$\begin{aligned} a_n \;=\; & \sqrt{2}a_{n-1} + a_{n-4} + 4\sqrt{2}a_{n-5} - 6a_{n-6} - \sqrt{2}a_{n-7} - \sqrt{2}a_{n-9} \\ & -6a_{n-10} + 4\sqrt{2}a_{n-11} + a_{n-12} + \sqrt{2}a_{n-15} - a_{n-16}, \end{aligned}$$

with the initial conditions $a_1 = 1,\ a_2 = \sqrt{2},\ a_3 = 2,\ a_4 = 2\sqrt{2},\ a_5 = 1,\ a_6 = 4\sqrt{2},\ a_7 = 13,\ a_8 = 16\sqrt{2},\ a_9 = 34,\ a_{10} = 29\sqrt{2},\ a_{11} = 89,\ a_{12} = 128\sqrt{2},\ a_{13} = 325,\ a_{14} = 377\sqrt{2},\ a_{15} = 842,\ a_{16} = 1088\sqrt{2}$.

Proof. We specialize Lemma 4. Let $f_{1,5}(x) = \prod_{i=1}^{4}(x-\alpha_i)(x-\alpha_i^{-1})$. As in the proof of Lemma 2 we find that $f_{1,5}(1) = \prod_{i=1}^{4}(1-\alpha_i)(1-\alpha_i^{-1}) = |I - M_{1,5}| = 26$.

By Lemma 2,

$$A_n = n^2 \prod_{j=1}^{n-1} f_{1,5}(\varepsilon^j).$$

Thus

$$\begin{aligned} a_n^2 &= \prod_{j=1}^{n-1}\prod_{i=1}^{4}(\varepsilon^j - \alpha_i)(\varepsilon^j - \alpha_i^{-1}) \\ &= \prod_{i=1}^{4} \frac{(1-\alpha_i^n)(1-\alpha_i^{-n})}{(1-\alpha_i)(1-\alpha_i^{-1})} \\ &= \left(\frac{(1-\alpha_1^n)(1-\alpha_2^n)(1-\alpha_3^n)(1-\alpha_4^n)}{\sqrt{26}\sqrt{(\alpha_1\alpha_2\alpha_3\alpha_4)^n}}\right)^2. \end{aligned}$$

Consequently $a_n = \frac{1}{\sqrt{26}}\sum_{i=1}^{16} r_i^n$ for some complex numbers $r_i,\ 1 \le i \le 16$.

Thus there exist $b_i,\ 1 \le i \le 16$, such that $a_n = \sum_{i=1}^{16} b_i a_{n-i}$. This gives the following linear equations,

$$\sum_{j=1}^{16} b_j a_{16+i-j} = a_{16+i},\ 1 \le i \le 16.$$

The values of a_i for $i \le 32$ can be calculated using Lemma 2 (the values for $i \le 16$ are shown in the theorem statement). Using these 32 values we can solve for the b_i to derive that $b_1 = \sqrt{2}$, $b_2 = 0$, $b_3 = 0$, $b_4 = 1$, $b_5 = 4\sqrt{2}$, $b_6 = -6$, $b_7 = -\sqrt{2}$, $b_8 = 0$, $b_9 = -\sqrt{2}$, $b_{10} = -6$, $b_{11} = 4\sqrt{2}$, $b_{12} = 1$, $b_{13} = 0$, $b_{14} = 0$, $b_{15} = \sqrt{2}$, $b_{16} = -1$.

□

Theorem 9 *Case 2:* $s_1 = 2,\ s_2 = 3$.

$$T(C_n^{2,3}) = \frac{1}{n}A_n = na_n^2,$$

where a_n *satisfies the recurrence relation:*

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} - a_{n-4},$$

with the initial conditions $a_1 = 1,\ a_2 = 1,\ a_3 = 1,\ a_4 = 3$.

Proof. Repeat the procedure above. Let

$$f_{2,3}(x) = (x-\alpha_1)(x-\alpha_1^{-1})(x-\alpha_2)(x-\alpha_2^{-1}).$$

Then $f_{2,3}(1) = (1-\alpha_1)(1-\alpha_1^{-1})(1-\alpha_2)(1-\alpha_2^{-1}) = |I - M_{2,3}| = 13$.

By Lemma 2,

$$A_n = n^2 \prod_{j=1}^{n-1} f_{2,3}(\varepsilon^j),$$

so

$$\begin{aligned} a_n^2 &= \prod_{j=1}^{n-1} (\varepsilon^j - \alpha_1)(\varepsilon^j - \alpha_1^{-1})(\varepsilon^j - \alpha_2)(\varepsilon^j - \alpha_2^{-1}) \\ &= \left(\frac{(1-\alpha_1^n)(1-\alpha_2^n)}{\sqrt{13}\sqrt{(\alpha_1\alpha_2)^n}} \right)^2, \end{aligned}$$

$a_n = \frac{1}{\sqrt{13}}(r_1^n + r_2^n + r_3^n + r_4^n)$ for some numbers $r_1,\ r_2,\ r_3,\ r_4$, and there exist $b_1,\ b_2,\ b_3,\ b_4$, such that $a_n = b_1 a_{n-1} + b_2 a_{n-2} + b_3 a_{n-3} + b_4 a_{n-4}$. Finally, by solving the linear equations, we have $b_1 = b_2 = b_3 = 1$, $b_4 = -1$.

□

Theorem 10 *Case 3:* $s_1 = 2,\ s_2 = 4$.

$$T(C_n^{2,4}) = \begin{cases} 0 & n \text{ even}, \\ T(C_n^{1,2}) & n \text{ odd}. \end{cases}$$

Proof. Follows directly from Corollary 2.

□

As described in Lemma 4 we can use the methods above for any values of $s_1, s_2, \cdots, s_k$ to find b_i, such that

$$\forall n > 2^{s_k - 1},\ a_n = \sum_{i=1}^{2^{s_k - 1}} b_i a_{n-i}$$

where $T(C_n^{s_1, s_2, \cdots, s_k}) = n a_n^2$. We have done this for all such s_i tuples with $s_k \leq 5$. The results are presented in the two tables. The first table presents the b_i coordinates. The second presents the initial values of a_n for $n \leq 2^{s_k - 1}$.

$\{s_i\}$	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}
1, 2	1	1														
1, 3	$\sqrt{2}$	0	$\sqrt{2}$	-1												
2, 3	1	1	1	-1												
1, 2, 3	$\sqrt{2}$	1	$\sqrt{2}$	-1												
1, 4	1	0	1	3	-1	0	-1	-1								
2, 4																
3, 4	1	0	2	2	-2	0	-1	-1								
1, 2, 4	1	1	1	5	-1	1	-1	-1								
1, 3, 4	$\sqrt{2}$	0	$2\sqrt{2}$	3	$-2\sqrt{2}$	0	$-\sqrt{2}$	-1								
2, 3, 4	1	1	2	4	-2	1	-1	-1								
1, 2, 3, 4	$\sqrt{2}$	1	$2\sqrt{2}$	5	$-2\sqrt{2}$	1	$-\sqrt{2}$	-1								
1, 5	$\sqrt{2}$	0	0	1	$4\sqrt{2}$	-6	$-\sqrt{2}$	0	$-\sqrt{2}$	-6	$4\sqrt{2}$	1	0	0	$\sqrt{2}$	-1
2, 5	1	1	0	0	6	-6	-1	-1	-1	-6	6	0	0	1	1	-1
3, 5	$\sqrt{2}$	0	$\sqrt{2}$	-2	$5\sqrt{2}$	-8	$\sqrt{2}$	-2	$\sqrt{2}$	-8	$5\sqrt{2}$	-2	$\sqrt{2}$	0	$\sqrt{2}$	-1
4, 5	1	0	1	3	4	-5	-3	-4	-3	-5	4	3	1	0	1	-1
1, 2, 5	$\sqrt{2}$	1	0	1	$7\sqrt{2}$	-9	$-2\sqrt{2}$	-1	$-2\sqrt{2}$	-9	$7\sqrt{2}$	1	0	1	$\sqrt{2}$	-1
1, 3, 5	$\sqrt{3}$	0	$\sqrt{3}$	-1	$6\sqrt{3}$	-12	0	0	0	-12	$6\sqrt{3}$	-1	$\sqrt{3}$	0	$\sqrt{3}$	-1
1, 4, 5	$\sqrt{2}$	0	$\sqrt{2}$	5	$5\sqrt{2}$	-10	$-4\sqrt{2}$	-7	$-4\sqrt{2}$	-10	$5\sqrt{2}$	5	$\sqrt{2}$	0	$\sqrt{2}$	-1
2, 3, 5	$\sqrt{2}$	1	$\sqrt{2}$	-2	$8\sqrt{2}$	-13	0	-3	0	-13	$8\sqrt{2}$	-2	$\sqrt{2}$	1	$\sqrt{2}$	-1
2, 4, 5	1	1	1	5	7	-7	-3	-9	-3	-7	7	5	1	1	1	-1
3, 4, 5	$\sqrt{2}$	0	$2\sqrt{2}$	2	$5\sqrt{2}$	-12	$\sqrt{2}$	-5	$\sqrt{2}$	-12	$5\sqrt{2}$	2	$2\sqrt{2}$	0	$\sqrt{2}$	-1
1, 2, 3, 5	$\sqrt{3}$	1	$\sqrt{3}$	-1	$9\sqrt{3}$	-17	$-\sqrt{3}$	-1	$-\sqrt{3}$	-17	$9\sqrt{3}$	-1	$\sqrt{3}$	1	$\sqrt{3}$	-1
1, 2, 4, 5	$\sqrt{2}$	1	$\sqrt{2}$	7	$8\sqrt{2}$	-12	$-4\sqrt{2}$	-12	$-4\sqrt{2}$	-12	$8\sqrt{2}$	7	$\sqrt{2}$	1	$\sqrt{2}$	-1
1, 3, 4, 5	$\sqrt{3}$	0	$2\sqrt{3}$	4	$6\sqrt{3}$	-18	0	-6	0	-18	$6\sqrt{3}$	4	$2\sqrt{3}$	0	$\sqrt{3}$	-1
2, 3, 4, 5	$\sqrt{2}$	1	$2\sqrt{2}$	4	$8\sqrt{2}$	-16	$\sqrt{2}$	-10	$\sqrt{2}$	-16	$8\sqrt{2}$	4	$2\sqrt{2}$	1	$\sqrt{2}$	-1
1, 2, 3, 4, 5	$\sqrt{3}$	1	$2\sqrt{3}$	6	$9\sqrt{3}$	-22	0	-11	0	-22	$9\sqrt{3}$	6	$2\sqrt{3}$	1	$\sqrt{3}$	-1

$\{s_i\}$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}
1, 2	1	1														
1, 3	1	$2\sqrt{2}$	5	$5\sqrt{2}$												
2, 3	1	1	1	3												
1, 2, 3	1	$\sqrt{2}$	2	$4\sqrt{2}$												
1, 4	1	1	2	1	4	8	13	17								
2, 4	1	0	2	0	5	0	13	0								
3, 4	1	1	1	1	5	7	8	17								
1, 2, 4	1	1	3	3	11	21	49	93								
1, 3, 4	1	$\sqrt{2}$	2	$2\sqrt{2}$	11	$16\sqrt{2}$	41	$64\sqrt{2}$								
2, 3, 4	1	1	2	3	11	20	43	93								
1, 2, 3, 4	1	$\sqrt{2}$	3	$4\sqrt{2}$	20	$33\sqrt{2}$	113	$200\sqrt{2}$								
1, 5	1	$\sqrt{2}$	2	$2\sqrt{2}$	1	$4\sqrt{2}$	13	$16\sqrt{2}$	34	$29\sqrt{2}$	89	$128\sqrt{2}$	325	$377\sqrt{2}$	842	$1088\sqrt{2}$
2, 5	1	1	2	3	1	8	8	21	34	41	109	144	313	512	842	1659
3, 5	1	$\sqrt{2}$	1	$2\sqrt{2}$	1	$5\sqrt{2}$	13	$8\sqrt{2}$	37	$29\sqrt{2}$	89	$130\sqrt{2}$	233	$377\sqrt{2}$	841	$1024\sqrt{2}$
4, 5	1	1	2	1	1	8	13	17	16	41	109	176	233	377	842	1649
1, 2, 5	1	$\sqrt{2}$	3	$4\sqrt{2}$	5	$15\sqrt{2}$	43	$72\sqrt{2}$	216	$295\sqrt{2}$	989	$1500\sqrt{2}$	4733	$7267\sqrt{2}$	22065	$34416\sqrt{2}$
1, 3, 5	1	$\sqrt{3}$	2	$3\sqrt{3}$	5	$12\sqrt{3}$	49	$51\sqrt{3}$	214	$245\sqrt{3}$	989	$1296\sqrt{3}$	4733	$6125\sqrt{3}$	23110	$29427\sqrt{3}$
1, 4, 5	1	$\sqrt{2}$	3	$4\sqrt{2}$	4	$15\sqrt{2}$	49	$64\sqrt{2}$	153	$256\sqrt{2}$	947	$1470\sqrt{2}$	4187	$6223\sqrt{2}$	20172	$32512\sqrt{2}$
2, 3, 5	1	$\sqrt{2}$	2	$4\sqrt{2}$	4	$16\sqrt{2}$	41	$56\sqrt{2}$	214	$256\sqrt{2}$	947	$1408\sqrt{2}$	4187	$6929\sqrt{2}$	20168	$32368\sqrt{2}$
2, 4, 5	1	1	3	3	5	21	41	93	171	405	947	2079	4523	9799	22065	49011
3, 4, 5	1	$\sqrt{2}$	2	$2\sqrt{2}$	5	$16\sqrt{2}$	43	$56\sqrt{2}$	178	$295\sqrt{2}$	989	$1472\sqrt{2}$	4523	$7267\sqrt{2}$	23110	$36288\sqrt{2}$
1, 2, 3, 5	1	$\sqrt{3}$	3	$5\sqrt{3}$	11	$27\sqrt{3}$	113	$155\sqrt{3}$	729	$979\sqrt{3}$	4531	$6615\sqrt{3}$	28717	$42601\sqrt{3}$	185163	$272645\sqrt{3}$
1, 2, 4, 5	1	$\sqrt{2}$	4	$4\sqrt{2}$	11	$32\sqrt{2}$	113	$200\sqrt{2}$	652	$1199\sqrt{2}$	4531	$8192\sqrt{2}$	29275	$52319\sqrt{2}$	190124	$345200\sqrt{2}$
1, 3, 4, 5	1	$\sqrt{3}$	3	$3\sqrt{3}$	11	$27\sqrt{3}$	113	$147\sqrt{3}$	639	$979\sqrt{3}$	4531	$6561\sqrt{3}$	28627	$42601\sqrt{3}$	190608	$282387\sqrt{3}$
2, 3, 4, 5	1	$\sqrt{2}$	3	$4\sqrt{2}$	11	$33\sqrt{2}$	104	$184\sqrt{2}$	657	$1199\sqrt{2}$	4531	$8052\sqrt{2}$	29275	$53248\sqrt{2}$	194073	$353648\sqrt{2}$
1, 2, 3, 4, 5	1	$\sqrt{3}$	4	$5\sqrt{3}$	20	$48\sqrt{3}$	223	$355\sqrt{3}$	1732	$2880\sqrt{3}$	14641	$24000\sqrt{3}$	118717	$196463\sqrt{3}$	976880	$1618445\sqrt{3}$

Figure 2: The top table contains the b_i values: $\forall n > 2^{s_k-1}$, $a_n = \sum_{i=1}^{2^{s_k-1}} b_i a_{n-i}$. The bottom table contains the initial conditions a_n for $n \leq 2^{s_k-1}$. The b_i values for the $(s_1, s_2) = (2, 4)$ case are not reported since, as described in the text, $T(C_n^{2,4}) = 0$ for even n and $T(C_n^{2,4}) = T(C_n^{1,2})$ for odd n.

All of the above results were for non-directed graphs. Using Lemma 5 we can similarly find formulas for the number of spanning trees in any *directed* circulant graph. For example, $T(\vec{C}_n^{2,3}) = na_n$, where a_n satisfies the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2a_{n-3} - 4a_{n-4}$ with the initial conditions $a_1 = 1,\ a_2 = 1,\ a_3 = 1,\ a_4 = 5$; and $T(\vec{C}_n^{1,2,3}) = na_n$, where a_n satisfies the recurrence relation $a_n = 2a_{n-1} + 2a_{n-2} + 6a_{n-3} - 9a_{n-4}$ with the initial conditions $a_1 = 1,\ a_2 = 2,\ a_3 = 3,\ a_4 = 16$.

4 Asymptotic properties

Given the recurrence relations for a_n it is a simple matter to find the asymptotic growth rate of $T(C_n^{s_1,s_2,\cdots,s_k}) = na_n^2$ as n increases. In this section we present asymptotic formulas for a_n for all undirected circulant graphs $T(C_n^{s_1,s_2,\cdots,s_k})$ for which $s_k \leq 5$.

We first derive the recurrence relations and initial conditions for all of these a_n; These are listed in the tables in Figure 2. It is well known that if a_n satisfies a linear recurrence relation then the generating function $G(x) = \sum_i a_n x^n$ of the a_n can be expressed as a rational function $G(x) = \frac{F(x)}{Q(x)}$ where $F(x)$ and $Q(x)$ are polynomials. If $Q(x)$ has a root α with unique minimum modulus, such that α is a simple root, then $a_n \sim -\frac{F(\alpha)}{\alpha Q'(\alpha)}(\frac{1}{\alpha})^n$ [10]. From the note following Lemma 4 we know that *every* root of $Q(x)$ is simple. From Corollary 1, we know that if $\gcd(s_1, s_2, \cdots, s_k) = 1$ then the root with the minimum modulus has unique minimum modulus. We used Maple to find the minimum modulus root α of each of the equations that satisfy $\gcd(s_1, s_2, \cdots, s_k) = 1$. We then calculated $c = -\frac{F(\alpha)}{\alpha Q'(\alpha)}$ and $\phi = \frac{1}{\alpha}$. The results are displayed in the table below.

For the one case in which $\gcd(s_1, s_2, \cdots, s_k) \neq 1$, the case $s_1 = 2,\ s_2 = 4$, we did not calculate the roots. From Corollary 2 we know that for even n (the graph is disconnected so) $T(C_n^{2,4}) = 0$ and for odd n, $T(C_n^{2,4}) = T(C_n^{1,2})$. The $2,4$ row thus simply presents the same growth rate as the $1,2$ row but its growth rate only applies for odd n.

$\{s_k\}$	c	ϕ
$\{1,2\}$	0.4472136	1.618034
$\{1,3\}$	0.3162278	1.700016
$\{1,4\}$	0.2425356	1.736815
$\{1,5\}$	0.1961161	1.755602
$\{2,3\}$	0.2773501	1.722084
$\{2,4\}$	0.4472136	1.618034
$\{2,5\}$	0.1856934	1.759576
$\{3,4\}$	0.2000000	1.754878
$\{3,5\}$	0.1714986	1.764394
$\{4,5\}$	0.1561738	1.769046
$\{1,2,3\}$	0.2672612	2.102256
$\{1,2,4\}$	0.2182179	2.147396
$\{1,3,4\}$	0.1961161	2.165786
$\{2,3,4\}$	0.1856953	2.181935
$\{1,2,5\}$	0.1825742	2.183137
$\{1,3,5\}$	0.1690309	2.200510
$\{1,4,5\}$	0.1543035	2.194750
$\{2,3,5\}$	0.1622214	2.189798
$\{2,4,5\}$	0.1490712	2.211485
$\{3,4,5\}$	0.1414214	2.224979
$\{1,2,3,4\}$	0.1825742	2.509601
$\{1,2,3,5\}$	0.1601282	2.537090
$\{1,2,4,5\}$	0.1474420	2.555259
$\{1,3,4,5\}$	0.1400280	2.563612
$\{2,3,4,5\}$	0.1360828	2.572032
$\{1,2,3,4,5\}$	0.1348400	2.866404

5 Conclusion

In this paper we described a general method for determining a recurrence relation for $T(C_n^{s_1,s_2,\cdots,s_k})$, the number of spanning trees in the undirected circulant graph described by indices $s_1, s_2, \cdots, s_k$ and $T(\vec{C}_n^{s_1,s_2,\cdots,s_k})$, the same quantity for directed circulant graphs. Our method does not, though, provide a general recurrence relation or formula for general case $T(C_n^{s_1,s_2,\cdots,s_k})$ or $T(\vec{C}_n^{s_1,s_2,\cdots,s_k})$ when the s_i are permitted to vary. Finding such a more general relationship, if one exists, would be an interesting problem.

We conclude by pointing out that it is known [14] that $T(\vec{C}_n^{s_1,s_2,\cdots,s_k}) \sim \frac{nk^n}{\vec{f}_{s_1,s_2,\cdots,s_k}(1)}$. One might hope that a similar asymptotic property, independent of $s_1, s_2, \cdots, s_k$, would hold for undirected graphs. Unfortunately, the asymptotic results presented in the table seem to show that such a strong result does not apply. This also implies that, for the undirected case, the result presented in [14] is not true. An interesting open question would be to discover if there are some weaker asymptotic properties that are not dependent upon $s_1, s_2, \ldots, s_k$ but only upon k, or possibly on s_k (or s_1).

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