Multidimensional Divide-and-Conquer and Weighted Digital Sums (Extended Abstract)*

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Abstract

This paper studies two functions arising separately in the analysis of algorithms. The first function is the solution to the Multidimensional Divide-And-Conquer (MDC) Recurrence that arises when solving problems involving points in d-dimensional space. The second function concerns weighted digital sums. Let $n = (b_i b_{i-1} \cdots b_1 b_0)_2$ and $S_M(n) = \sum_{t=0}^{i} t(t+1)(t+2)\cdots(t+M-1)b_t 2^t$, and set $T_M(n) = \frac{1}{n}\sum_{j<n} S_M(j)$ be its average.

We show that both the MDC function and $T_M(n)$ (with $d = M-1$) have solutions of the form

$$T_d(n) = \alpha n \lg^{d-1} n + \sum_{m=0}^{d-2} n \lg^m n A_m(\lg n) + c_d.$$

The α, c_d are explicitly calculated constants and the $A_m(x)$ are periodic functions of period one given by explicitly stated Fourier series.

1 Introduction

In this paper we analyze two problems that, although they arise separately in the analysis of algorithms, have very similar solutions. The first problem, multidimensional divide-and-conquer, initially arose in the context of calculating maximal points in multidimensional space [1, 2]. Previous analyses [3] gave only first order asymptotics, showing that the running time for the d-dimensional version of the problem is ($\lg n \equiv \log_2 n$)

$$T_d(n) = \alpha n \lg^{d-1} n + o(n \lg^{d-1} n)$$

for some explicitly calculated constant α. We will extend Mellin transform techniques for solving divide-and-conquer problems originally developed in [4] (see [5] for a review of more recent innovations) to derive exact solutions, which will be in the form

$$T_d(n) = \alpha n \lg^{d-1} n + \sum_{m=0}^{d-2} n \lg^m n A_m(\lg n) + c_d, \tag{1.1}$$

where α, c_d are explicitly calculated constants and the $A_m(x)$ are periodic functions of period 1 given by absolutely convergent Fourier series.

The second problem studied will be certain weighted types of digital sums. Start by expressing integer n in binary as $n = (b_i b_{i-1} \cdots b_1 b_0)_2$, i.e., $n = \sum_{t=0}^{i} b_t 2^t$. Define

$$S_1(n) = \sum_{t=0}^{i} t b_t 2^t. \tag{1.2}$$

This sum arises naturally in the analysis of binomial queues where Brown [6] gave upper and lower bounds

$$\lceil n \lg n - 2n \rceil \le S_1(n) \le \lfloor n \lg n \rfloor.$$

We will be interested here in analyzing the more generalized version of this function, allowing weights to be polynomial in t:

$$S_M(n) = \sum_{t=0}^{i} t(t+1)(t+2)\cdots(t+M-1)b_t 2^t. \tag{1.3}$$

It turns out that $S_M(n)$ itself is not "smooth" but its average $T_M(n) = \frac{1}{n}\sum_{j<n} S_M(j)$ *is* and allows a closed formula. This formula will be precisely in the same form (1.1) seen above with $M = d+1$.

Note: In this extended abstract, proofs of some Theorems and lemmas have been omitted.

2 Background and Tools

The main tools used in this paper are Dirichlet generating functions, Mellin transforms and the Mellin-Perron Formula. For detailed discussion of these tools see, e.g., [7] and [8, pp.762-767]. We first state the key needed results:

Theorem 2.1. (The Mellin-Perron formula) *Let $\{\lambda_j\}$, $j = 1, 2, \ldots$ be a sequence and $c > 0$ lie in*

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the half-plane of absolute convergence of $\sum_{j=1}^{\infty} \lambda_j j^{-s}$. *Then for any* $m \ge 1$,

$$(2.4)\quad \frac{1}{m!}\sum_{j<n} \lambda_j \left(1-\frac{j}{n}\right)^m = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\sum_{j=1}^{\infty} \frac{\lambda_j}{j^s}\right) n^s \frac{ds}{s(s+1)(s+2)\cdots(s+m)}.$$

In particular, when $m = 1$,

$$(2.5)\quad \frac{1}{n}\sum_{j<n} \lambda_j (n-j) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\sum_{j=1}^{\infty} \frac{\lambda_j}{j^s}\right) n^s \frac{ds}{s(s+1)}.$$

2.1 Solving Divide and Conquer Recurrences

Our main technique will be a generalization of one developed in [4] to solve divide-and-conquer recurrences of the form

$$(2.6)\quad f_n = f_{\lfloor n/2 \rfloor} + f_{\lceil n/2 \rceil} + e_n$$

with initial condition $f_1 = 0$ and given "conquer" cost sequence $\{e_n\}$ where $e_0 = e_1 = 0$. Since we will have to generalize/modify the technique, we first quickly review how it worked.

Distinguishing between odd and even cases of (2.6), we find that for $j \ge 1$,

$$(2.7)\quad f_{2j} = 2f_j + e_{2j}, \quad f_{2j+1} = f_j + f_{j+1} + e_{2j+1}.$$

Let $\nabla g_n = g_n - g_{n-1}$ be the backward difference operator. Then, for $j \ge 1$,

$$(2.8)\quad \nabla f_{2j} = \nabla f_j + \nabla e_{2j}, \quad \nabla f_{2j+1} = \nabla f_{j+1} + \nabla e_{2j+1}.$$

Let $\Delta g_n = g_{n+1} - g_n$, be the forward difference operator, i.e.,

$$(2.9)\quad \begin{cases} \Delta\nabla f_n = \nabla f_{n+1} - \nabla f_n = f_{n+1} - 2f_n + f_{n-1} \\ \Delta\nabla e_n = \nabla e_{n+1} - \nabla e_n = e_{n+1} - 2e_n + e_{n-1}. \end{cases}$$

Then, from (2.8),

$$\Delta\nabla f_{2j} = \Delta\nabla f_j + \Delta\nabla e_{2j}, \quad \Delta\nabla f_{2j+1} = \Delta\nabla e_{2j+1}$$

for $j \ge 1$, with $\Delta\nabla f_1 = f_2 - 2f_1 = e_2 = \Delta\nabla e_1$.

Now, some basic calculation shows that, for any sequence f_n,

$$(2.10)\quad f_n - nf_1 = \sum_{j<n} (n-j)\Delta\nabla f_j.$$

Therefore, (2.5) gives that

$$f_n - nf_1 = \frac{n}{2\pi i}\int_{c-i\infty}^{c+i\infty} D_f(s) n^s \frac{ds}{s(s+1)},$$

where $D_f(s) = \sum_{j=1}^{\infty} \frac{\Delta\nabla f_j}{j^s}$ is the *Dirichlet Generating Function* (DGF) of $\Delta\nabla f_j$.

Calculation yields

$$\begin{aligned} & D_f(s) \\ = \; & \Delta\nabla f_1 + \sum_{j=1}^{\infty} \frac{\Delta\nabla f_{2j}}{(2j)^s} + \sum_{j=1}^{\infty} \frac{\Delta\nabla f_{2j+1}}{(2j+1)^s} \\ = \; & \Delta\nabla e_1 + \left(\sum_{j=1}^{\infty} \frac{\Delta\nabla f_j}{(2j)^s} + \sum_{j=1}^{\infty} \frac{\Delta\nabla e_{2j}}{(2j)^s}\right) + \sum_{j=1}^{\infty} \frac{\Delta\nabla e_{2j+1}}{(2j+1)^s} \\ = \; & \frac{1}{2^s}\sum_{j=1}^{\infty} \frac{\Delta\nabla f_j}{j^s} + \Delta\nabla e_1 + \sum_{j=1}^{\infty} \frac{\Delta\nabla e_{2j}}{(2j)^s} + \sum_{j=1}^{\infty} \frac{\Delta\nabla e_{2j+1}}{(2j+1)^s} \\ = \; & \frac{D_f(s)}{2^s} + \sum_{j=1}^{\infty} \frac{\Delta\nabla e_j}{j^s}. \end{aligned}$$

Solving for $D_f(s)$ gives

$$(2.11)\quad D_f(s) = \frac{1}{1-2^{-s}}\sum_{j=1}^{\infty} \frac{\Delta\nabla e_j}{j^s}.$$

Combining (2.5), (2.10) and (2.11) proves the following lemma from [4]:

Lemma 2.1. *The recurrence (2.6) with boundary conditions* $e_0 = e_1 = 0$ *and* $f_1 = 0$ *is solved by*

$$f_n = \frac{n}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{1-2^{-s}} \left(\sum_{j=1}^{\infty} \frac{\Delta\nabla e_j}{j^s}\right) n^s \frac{ds}{s(s+1)},$$

where c lies in the half-plane of absolute convergence of $\sum_{j=1}^{\infty} \frac{\Delta\nabla e_j}{j^s}$.

3 Multidimensional Divide-and-Conquer

3.1 Background *Multidimensional divide and conquer (MDC)* was first introduced by Bentley and Shamos [1, 2] in the context of solving multidimensional computational geometry problems. The generic idea is to solve a problem on n d-dimensional points by (i) first splitting the points into two almost equal subsets and solving the problem on each of them, then (ii) taking all n points, projecting them down to $(d-1)$ dimensional space and solving the problem on the projected set, and finally (iii) constructing a solution to the complete problem by intelligently combining the solutions to the 3 previously solved ones. The recursion bottoms out when the dimension $d = 2$, in which case a straightforward solution is given, or when $n = 1$, which has a trivial solution.

The methodology can be applied to give good solutions for many problems, including the Empirical Cumulative Distribution Function (ECDF) problem, maxima, range searching, closest pair, and the all nearest neighbour problem.

Of particular interest to us is the *all-points ECDF problem in* $\mathbb{R}^k$ (ECDF-k). For two points $x = (x_1, x_2, \cdots, x_k)$, $y = (y_1, y_2, \cdots, y_k) \in \mathbb{R}^k$, we say x *dominates* y if $x_i \geq y_i$ for all $1 \leq i \leq k$. Given a set S of n points in $\mathbb{R}^k$, the *rank* of a point x is the number of points in S dominated by x. The ECDF-k problem is to compute the rank of each point in S.

When $k = 2$, a slight modification of bottom-up mergesort will solve ECDF-2 in $O(n \log n)$ time. Monier [3] proposed a MDC algorithm for solving ECDF-k for larger k, which was based on the description of Bentley [2]. Monier analyzed the worst-case running time of this algorithm, $T(n, k)$, with the following recurrence:

(3.12)
$$T(n,k) = \begin{cases} \begin{array}{l} T\left(\lfloor \frac{n}{2} \rfloor, k\right) + T\left(\lceil \frac{n}{2} \rceil, k\right) \\ +T(n, k-1) + n \end{array} & \text{if } n > 1, k > 2, \\ 1 & \text{if } n = 1, k > 2, \\ n \lg n & \text{if } n \geq 1, k = 2. \end{cases}$$

By translation into a combinatorial path-counting problem he derived the first order aymptotics of $T(n, k)$. More specifically, he showed that, for fixed k,

$$T(n,k) = \frac{1}{(k-1)!} n \lg^{k-1} n + \Theta(n \lg^{k-2} n)$$

We will derive exact solutions for the ECDF-k running time using Lemma 2.1. To do so, we will have to slightly modify the case $k = 2$ to have a more precise initial condition. In what follows we will denote $T(n, k)$ by f_n^k. The recurrences corresponding to 3.12 will be:

$$(3.13) \qquad f_n^k = \begin{cases} f^k_{\lfloor n/2 \rfloor} + f^k_{\lceil n/2 \rceil} + e^k_n, & n \geq 2 \\ 0, & n = 1 \end{cases}$$

where

$$(3.14) \qquad e_n^k = \begin{cases} f_n^{k-1} + n - 1, & k \geq 3 \\ n - 1, & k = 2. \end{cases}$$

3.2 The Dirchlet Generating Function for Multidimensional Divide-and-Conquer

We can now solve recurrence (3.13) and (3.14) via Lemma 2.1. To do so, we need to calculate the DGF of the $\Delta\nabla e_k^n$, where the e_k^n are given by (3.13). One can work out directly that $\Delta\nabla e_1^2 = 1$ while, for $j \geq 2$, $\Delta\nabla e_j^2 = 0$.

Thus,

$$(3.15) \qquad D_{f_2}(s) = \frac{1}{1-2^{-s}} \sum_{j=1}^{\infty} \frac{\Delta\nabla e_j^2}{j^s} = \frac{1}{1-2^{-s}}.$$

For $k \geq 3$, we have

$$\Delta\nabla e_j^k = \begin{cases} \Delta\nabla f_j^{k-1}, & \text{for } j \geq 2 \\ e_2^k = f_2^{k-1} + 1 = \Delta\nabla f_1^{k-1} + 1, & \text{for } j = 1. \end{cases}$$

Hence

$$\begin{aligned} D_{f_k}(s) &= \frac{1}{1-2^{-s}} \sum_{j=1}^{\infty} \frac{\Delta\nabla e_j^k}{j^s} \\ &= \frac{1}{1-2^{-s}} \left(\Delta\nabla f_1^{k-1} + 1 + \sum_{j=2}^{\infty} \frac{\Delta\nabla f_j^{k-1}}{j^s} \right) \\ &= \frac{1}{1-2^{-s}} + \frac{D_{f_{k-1}}(s)}{1-2^{-s}}. \end{aligned}$$

Iterating the above recurrence, and applying (3.15) gives

(3.16)
$$D_{f_k}(s) = \frac{1}{1-2^{-s}} + \frac{1}{(1-2^{-s})^2} + \cdots + \frac{1}{(1-2^{-s})^{k-1}}.$$

By Lemma 2.1, we obtain a formula for f_n^k in terms of f_n^{k-1} and a complex integral:

(3.17)
$$\begin{aligned} f_n^k &= \frac{n}{2\pi i} \int_{3-i\infty}^{3+i\infty} \left(\sum_{d=1}^{k-1} \frac{1}{(1-2^{-s})^d} \right) n^s \frac{ds}{s(s+1)} \\ &= f_n^{k-1} + \frac{n}{2\pi i} \int_{3-i\infty}^{3+i\infty} \frac{1}{(1-2^{-s})^{k-1}} n^s \frac{ds}{s(s+1)}. \end{aligned}$$

We note that [4] explicitly solved this for the case $k = 2$ to derive

$$(3.18) \qquad f_n^2 = n \lg n + n A_0^2(\lg n) + 1$$

where, setting $\chi_j = \frac{2\pi i j}{\ln 2}$,

(3.19)
$$A_0^2(u) = \left(\frac{1}{2} - \frac{1}{\ln 2} \right) + \sum_{j \in \mathbb{Z} \setminus \{0\}} \frac{1}{(\ln 2)\chi_j(\chi_j + 1)} e^{2\pi i j u}.$$

3.3 Evaluation of the Integral We now see how to evaluate the integral on the right-hand-side of (3.17):

$$I_{f,k}(s) = \frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty} \frac{1}{(1-2^{-s})^{k-1}} n^s \frac{ds}{s(s+1)} \tag{3.20}$$

Fix some real $R > 0$ and consider the counter-clockwise rectangular contour $\Upsilon = \Upsilon_1 \bigcup \Upsilon_2 \bigcup \Upsilon_3 \bigcup \Upsilon_4$, where (see Figure 1)

(3.21)

$$\begin{aligned}
\Upsilon_1 &= \{3+iy : -R \le y \le R\} \\
\Upsilon_2 &= \{x+iR : -R \le x \le 3\} \\
\Upsilon_3 &= \{-R+iy : -R \le y \le R\} \\
\Upsilon_4 &= \{x-iR : -R \le x \le 3\}
\end{aligned}$$

Denote the kernel of the integral in (3.20) by $K_{f,k}(s)$:

$$K_{f,k}(s) = \frac{n^s}{s(s+1)(1-2^{-s})^{k-1}}. \tag{3.22}$$

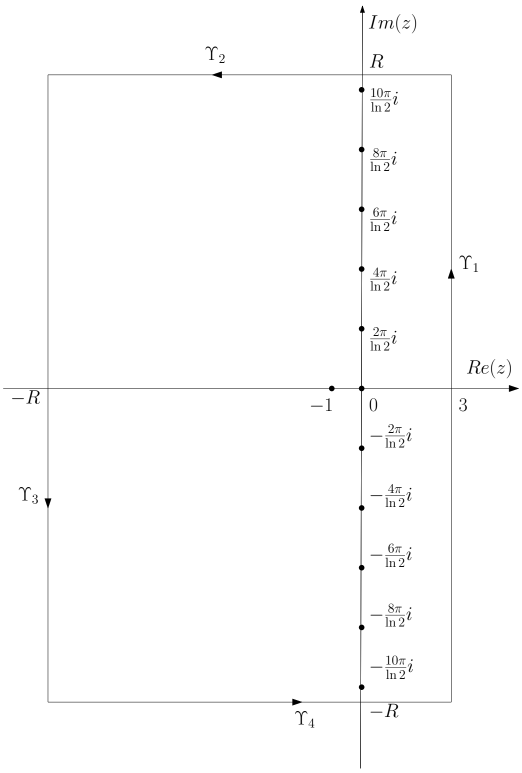

Figure 1: The figure is contour Υ defined in (3.21). The dots represent the poles of $K_{f,k}(s)$ inside Υ.

Note that (3.20) is just $\lim_{R\to\infty} \frac{1}{2\pi i}\int_{\Upsilon_1} K_{f,k}(s)ds$. The idea will essentially be to show that, for $q = 2,3,4$, $\lim_{R\to\infty}\int_{\Upsilon_q} K_{f,k}(s)ds = 0$. Thus, (3.20) is the limit, as $R \to \infty$, of the integral around the closed curve Υ. But, by the Cauchy residue theorem, this will be equal to the sum of the residues inside Υ as $R \to \infty$, which can be easily calculated.

We now provide details. Consider the horizontal paths $q = 2,4$ and the sequence $R = R_j = \frac{(2j+0.5)\pi}{\ln 2}$. Then

$$\begin{aligned}
&\left|\int_{\Upsilon_q} K_{f,k}(s)ds\right| \\
\le\ & \int_{-R_j \pm iR_j}^{3\pm iR_j} |K_{f,k}(s)|\, ds \\
\le\ & \max_{-R_j\le\sigma\le 3}\left|\frac{n^\sigma}{(1\pm 2^{-\sigma}i)^{k-1}}\right| \frac{1}{R_j(R_j+1)}\int_{-R_j}^{3} d\sigma \\
=\ & O(j^{-1}).
\end{aligned}$$

For the leftmost path we also see

$$\left|\int_{\Upsilon_3} K_{f,k}(s)ds\right| = O\left(\frac{1}{R_j(2^{k-1}n)^{R_j}}\right).$$

Hence, by the Cauchy residue theorem, $I_{f,k}(s)$ is the sum of the residues at the poles of $K_{f,k}(s)$ inside Υ as $j \to \infty$. The poles of $K_{f,k}(s)$ inside Υ are:

1. A simple pole at $s = -1$.
2. A pole of order k at $s = 0$.
3. Poles of order $(k-1)$ at $s = \chi_j$, where $j \in \mathbb{Z} \setminus \{0\}$

Standard techniques for finding residues, e.g., multiplying the respective Laurent series, give

$$\operatorname{Res}\left(K_{f,k}(s), s=-1\right) = \frac{(-1)^k}{n}, \tag{3.23}$$

$$\begin{aligned}
&\operatorname{Res}\left(K_{f,k}(s), s=0\right) \\
&= \frac{\lg^{k-1} n}{(k-1)!} + \left(\frac{k-1}{2} - \frac{1}{\ln 2}\right)\frac{\lg^{k-2} n}{(k-2)!} \\
&\qquad + \sum_{m=0}^{k-3} c_{0,m}^k \lg^m n
\end{aligned} \tag{3.24}$$

and

(3.25)

$$\operatorname{Res}\left(K_{f,k}(s), s=\chi_j\right) = \frac{e^{2\pi i j(\lg n)}}{\chi_j(\chi_j+1)}\sum_{m=0}^{k-2} c_{j,m}^k \lg^m n$$

where the $c_{j,m}^k$ are explicitly calculable constants.

We point out (omitting the proof in this extended abstract) that it is not difficult to show that, for fixed k and m, the $|c_{j,m}^k|$ are uniformly bounded as $j \to \infty$.

THEOREM 3.1. *For $k \geq 2$, the recurrence f_n^k, defined by (3.13) and (3.14), satisfies*

(3.26)
$$f_n^k = \frac{1}{(k-1)!} n \lg^{k-1} n + \sum_{m=0}^{k-2} n \lg^m n A_m^k(\lg n) + c_k,$$

where $A_m^k(u)$, $m = 0, 1, \cdots, k-2$, are periodic with period 1. Furthermore, the $A_m^k(u)$ are given by absolutely convergent Fourier series

$$A_m^k(u) = \sum_{j=-\infty}^{\infty} a_{m,k,j} e^{2\pi i j u}$$

whose coefficients $a_{m,k,j}$ can be determined explicitly. $c_k = 1$ if k is even, 0 if odd.

Proof. (Sketch) As previously mentioned, for $k = 2$ this theorem was already proved by Flajolet and Golin [4]. For $k \geq 3$ the proof follows from equation (3.17) and the residue calculations in (3.23), (3.24) and (3.25).

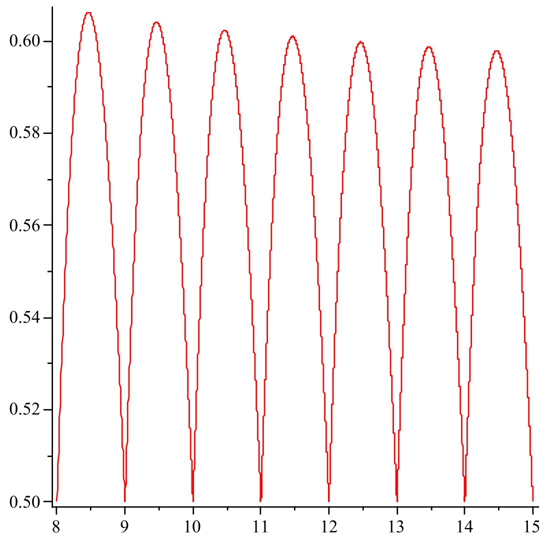

Figure 2: As an example, the graph shows $\frac{1}{n \lg n}(f_n^3 - \frac{1}{2} n \lg^2 n)$ plotted against $\lg n$. The graph illustrates the periodicity behaviour in the second order asymptotic terms of f_n^3.

4 Weighted Digital Sums

We now return to analyze $T_M(n) = \frac{1}{n}\sum_{j<n} S_M(j)$, where $S_M(j)$ is the weighted digital sum function as defined in (1.3).

Set $W_M(n) = \sum_{j<n} S_M(j)$. Note that $\Delta\nabla W_M(n) = \Delta S_M(n-1) = \nabla S_M(n)$ and $W_M(1) = 0$. Then from (2.10) and (2.5),

(4.27)
$$\begin{aligned} T_M(n) &= \frac{1}{n} W_M(n) \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\sum_{j=1}^{\infty} \frac{\nabla S_M(j)}{j^s} \right) n^s \frac{ds}{s(s+1)}. \end{aligned}$$

We now proceed, as in Section 3.2 to first derive a usable expression for the DGF $\sum_{j=1}^{\infty} \frac{\nabla S_M(j)}{j^s}$ and then, as in Section 3.3, use the Cauchy residue theorem to evaluate the integral in (4.27).

4.1 Deriving the DGF First note that $\forall M$, $S_M(1) = 0$. Next, observe that if $n = (b_i b_{i-1} \cdots b_1 b_0)_2$, then
$$2n = (b_i b_{i-1} \cdots b_1 b_0, 0)_2$$
and
$$2n+1 = (b_i b_{i-1} \cdots b_1 b_0, 1)_2.$$
This shows that $\forall M$,

(4.28)
$$S_M(2n+1) = S_M(2n),$$

(4.29)
$$\begin{aligned} S_1(2n) &= \sum_{i=0}^{k} (i+1) b_i 2^{i+1} \\ &= 2\sum_{i=0}^{k} i b_i 2^i + 2\sum_{i=0}^{k} b_i 2^i \\ &= 2S_1(n) + 2n \end{aligned}$$

and

(4.30)
$$S_M(2n) = 2S_M(n) + M S_{M-1}(2n).$$

Combining these in a similar calculation as performed in Section 3.2 yields

LEMMA 4.1.
$$A_M(s) = \sum_{j=1}^{\infty} \frac{\nabla S_M(j)}{j^s} = M! \frac{2^{(M-1)(s-1)}}{(2^{s-1}-1)^M} \zeta(s)$$

where $\zeta(s) = \sum_{n>0} n^{-s}$ is the Riemann Zeta function.

4.2 Evaluation of the Integral By (4.27) and Lemma 4.1, we have

(4.31)
$$T_M(n) = \frac{M!}{2\pi i} \int_{3-i\infty}^{3+i\infty} \frac{2^{(M-1)(s-1)}}{(2^{s-1}-1)^M} \zeta(s) n^s \frac{ds}{s(s+1)}$$

Similarly as in the analysis of f_n^k in Section 3.3, the integral in (4.31) can be evaluated by integrating (in the limit) over a counterclockwise contour. Fix some real $R > 0$. The contour for this case will be $\Gamma = \Gamma_1 \bigcup \Gamma_2 \bigcup \Gamma_3 \bigcup \Gamma_4$, where (see Figure 3)

(4.32)
$$\begin{aligned}\Gamma_1 &= \{3 + iy : -R \le y \le R\}\\ \Gamma_2 &= \{x + iR : -1/4 \le x \le 3\}\\ \Gamma_3 &= \{-1/4 + iy : -R \le y \le R\}\\ \Gamma_4 &= \{x - iR : -1/4 \le x \le 3\}\end{aligned}$$

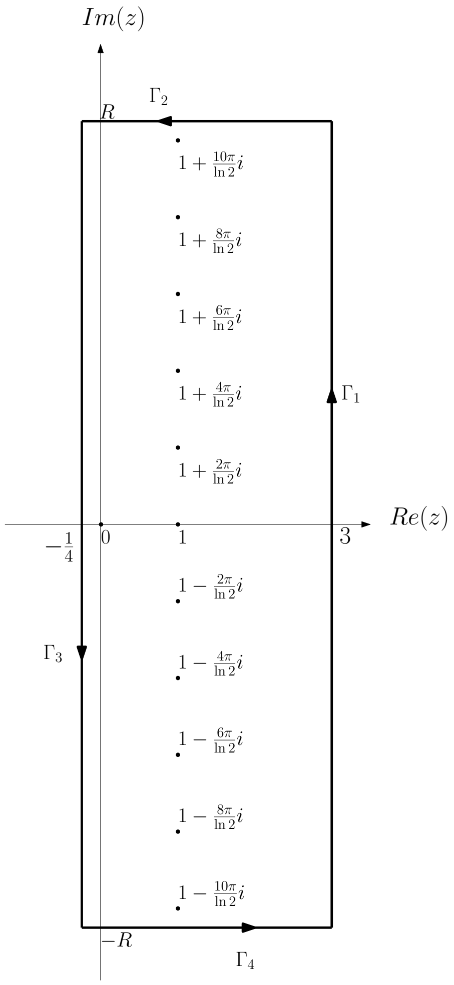

Figure 3: The figure is contour Γ defined in (4.32). The dots represent the poles of $K_{T_M}(s)$ inside Γ.

Denote the kernel of the integral in (4.31) by $K_{T_M}(s)$:

(4.33)
$$K_{T_M}(s) = \frac{2^{(M-1)(s-1)}}{(2^{s-1}-1)^M}\zeta(s)n^s\frac{1}{s(s+1)}$$

Note that the RHS of (4.31) is just $\lim_{R\to\infty} \frac{M!}{2\pi i}\int_{\Gamma_1} K_{T_M}(s)ds$. As before, we will start by showing that the integrals along Γ_2, Γ_3, Γ_4, go to 0 as $R \to \infty$. Therefore, the value of $T_M(n)$ will be $M!$ times the sum of the residues inside Γ, as $R \to \infty$. We therefore conclude by evaluating those residues.

To show that $\lim_{R\to\infty}\int_{\Gamma_j} K_{T_M}(s)ds = 0$ is tending to 0 for $j = 2, 3, 4$, we need the following two lemmas that follow easily from general properties of the Zeta function (proofs omitted in this extended abstract):

Lemma 4.2. *Consider integral*

(4.34)
$$I(R) = \int_{-a+iR}^{3+iR} f(s)\zeta(s)n^s ds$$

where $0 < a \le \frac{5}{4}$. *Furthermore, suppose that for* $s = \alpha + iB$ *with* $-a \le \alpha \le 3$, $|f(s)| = O(|B|^{-2})$. *Then, both as* $R \to \infty$ *and* $R \to -\infty$, *we have* $I(R) \to 0$.

Lemma 4.3. *Consider integral*

$$I = \int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty} g(s)\zeta(s)n^s\frac{ds}{s(s+1)}.$$

If $g(s)$ *can be expressed as a series which is uniformly convergent on* $-\frac{1}{4} + (-\infty, \infty)i$ *and is of the form*

$$\sum_{j=0}^{\infty} g_j(K_j)^s$$

for some real sequence $\{g_j\}$ *and integer sequence* $\{K_j\}$, *then* $I = 0$.

To evaluate the integrals along Γ_2 and Γ_4, consider the sequence $R_j = \frac{(2j+0.5)\pi}{\ln 2}$. Note that $\left|\frac{2^{(M-1)(s-1)}}{(2^{s-1}-1)^M}\right|$ is bounded as $j \to \infty$ and $\left|\frac{1}{s(s+1)}\right| = O(j^{-2})$. Thus, by Lemma 4.2, as $R_j \to \infty$,

$$\int_{\Gamma_2} K_{T_M}(s)ds \to 0, \quad \int_{\Gamma_4} K_{T_M}(s)ds \to 0.$$

To evaluate $\int_{\Gamma_3} K_{T_M}(s)ds$, note that along Γ_3, $Re(s) < 0$, so it is legitimate to write

$$\frac{1}{2^{s-1}-1} = -1 - \left(\frac{1}{2}\right)2^s - \left(\frac{1}{4}\right)4^s - \left(\frac{1}{8}\right)8^s \cdots.$$

The series is both absolutely convergent and uniformly convergent on $-\frac{1}{4} + (-\infty, \infty)i$, so we can safely write

$$\frac{2^{(M-1)(s-1)}}{(2^{s-1}-1)^M} = \sum_{j=0}^{\infty} a_j(2^{M+j-1})^s$$

for some $\{a_j\}$. This series is again uniformly convergent on $-\frac{1}{4} + (-\infty, \infty)i$, so by Lemma 4.3, we have

$$\int_{\Gamma_3} K_{T_M}(s)ds \to 0$$

as $R_j \to \infty$.

Hence by the Cauchy residue theorem, $T_M(n)$ is just $M!$ times the sum of the residues at the poles of $K_{T_M}(s)$ inside Γ after taking $j \to \infty$. It is well known that $\zeta(s)$ has a simple pole at $s = 1$ and it is analytic elsewhere. Hence, the poles of $K_{T_M}(s)$ inside Γ are:

1. A simple pole at $s = 0$.
2. A pole of order $(M+1)$ at $s = 1$.
3. Poles of order M at $s = \alpha_j := 1 + \frac{2\pi ij}{\ln 2}$, where $j \in \mathbb{Z} \setminus \{0\}$

The residues are listed below:

$$\text{Res}\,(K_{T_M}(s), s = 0) = (-1)^{M+1} \tag{4.35}$$

$$\begin{aligned} &\text{Res}\,(K_{T_M}(s), s = 1) \\ &= \frac{n\lg^M n}{2M!} + \left(\frac{2\gamma_0 - 3 + (M-2)\ln 2}{4(M-1)!\ln 2}\right) n\lg^{M-1} n \\ &\quad + \sum_{m=0}^{M-2} b_{0,m}^M n\lg^m n \end{aligned} \tag{4.36}$$

and

(4.37)

$$\text{Res}\,(K_{f,k}(s), s = \alpha_j) = e^{2\pi ij(\lg n)} \sum_{m=0}^{M-1} b_{j,m}^M n\lg^m n,$$

where the $b_{j,m}^M$ are explicitly calculable constants.

As in the MDC case, we can now derive a general closed formula:

Theorem 4.1.

(4.38)

$$T_M(n) = \frac{1}{2}n\lg^M n + \sum_{d=0}^{M-1} F_{M,d}(\lg n)n\lg^d n + (-1)^{M+1}M!$$

where

$$F_{M,d}(u) = \sum_{j\in\mathbb{Z}} a_{M,d,j}e^{2\pi iju} \tag{4.39}$$

is a function with period one. The Fourier coefficients $a_{M,d,j}$ can be determined explicitly and the Fourier series (4.39) are all absolutely convergent.

In particular, we can derive the following closed form for $T_1(n)$.

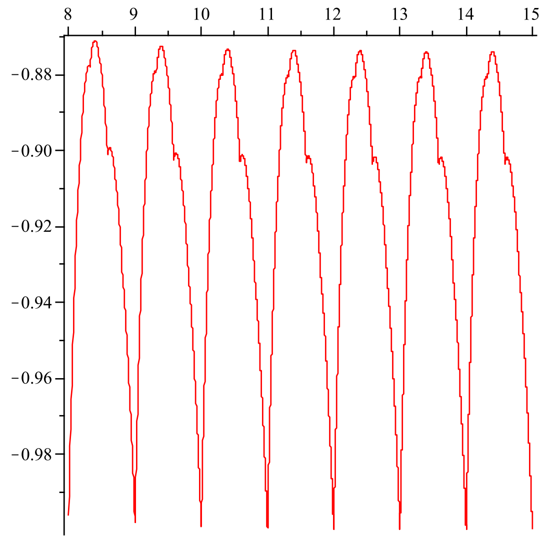

Figure 4: The graph shows $\frac{1}{n}(T_1(n) - \frac{1}{2}n\lg n)$ plotted against $\lg n$. The graph illustrate the periodicity behaviour in the second order asymptotic terms of $T_1(n)$.

Corollary 4.1.

$$T_1(n) = \frac{1}{2}n\lg n + nF_{1,0}(\lg n) + 1 \tag{4.40}$$

where

(4.41)

$$F_{1,0}(u) = \frac{2\gamma_0 - 3 - \ln 2}{4\ln 2} + \sum_{j\in\mathbb{Z}\setminus\{0\}} \frac{\zeta(\alpha_j)}{(\ln 2)\alpha_j(\alpha_j+1)}e^{2\pi iju}$$

which is a Fourier series with period one. $\alpha_j = 1 + \frac{2\pi ij}{\ln 2}$.

We conclude by noting that, using known properties of the Zeta function $\zeta(s)$, it is not difficult to prove that all of the Fourier series defined in (4.39) are absolutely convergent.

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