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Fast Nonsmooth Regularized Risk Minimization with Continuation

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Abstract

In regularized risk minimization, the associated optimization problem becomes particularly difficult when both the loss and regularizer are nonsmooth. Existing approaches either have slow or unclear convergence properties, are restricted to limited problem subclasses, or require careful setting of a smoothing parameter. In this paper, we propose a continuation algorithm that is applicable to a large class of nonsmooth regularized risk minimization problems, can be flexibly used with a number of existing solvers for the underlying smoothed subproblem, and with convergence results on the whole algorithm rather than just one of its subproblems. In particular, when accelerated solvers are used, the proposed algorithm achieves the fastest known rates of $O(1/T^2)$ on strongly convex problems, and $O(1/T)$ on general convex problems. Experiments on nonsmooth classification and regression tasks demonstrate that the proposed algorithm outperforms the state-of-the-art.

Introduction

In regularized risk minimization, one has to minimize the sum of an empirical loss and a regularizer. When both are smooth, it can be easily optimized by a variety of solvers (Nesterov 2004). In particular, a popular choice for big data applications is stochastic gradient descent (SGD), which is easy to implement and highly scalable (Kushner and Yin 2003). For many nonsmooth regularizers (such as the ℓ_1 and nuclear norm regularizers), the corresponding regularized risks can still be efficiently minimized by the proximal gradient algorithm and its accelerated variants (Nesterov 2013). However, when the regularizer is smooth but the loss is nonsmooth (e.g., the hinge loss and absolute loss), or when both the loss and regularizer are nonsmooth, proximal gradient algorithms are not directly applicable.

On nonsmooth problems, SGD can still be used, by simply replacing the gradient with subgradient. However, the information contained in the subgradient is much less informative (Nemirovski and Yudin 1983), and convergence is hindered. On general convex problems, SGD converges at a rate of $O(\log T/\sqrt{T})$, where T is the number of iterations; whereas on strongly convex problems, the rate is $O(\log T/T)$. In contrast, its smooth counterparts converge with the much faster $O(1/\sqrt{T})$ and $O(1/T)$ rates, respectively (Rakhlin, Shamir, and Sridharan 2012; Shamir and Zhang 2013). Recently, Shamir and Zhang (2013) recovered these rates by using a polynomial-decay averaging scheme on the SGD iterates. However, a major drawback is that it does not exploit properties of the regularizer. For example, when used with a sparsity-inducing regularizer, its solution obtained may not be sparse (Duchi and Singer 2009).

Nesterov (2005b) proposed to smooth the nonsmooth objective so that it can then be efficiently optimized. This smoothing approach is now popularly used for nonsmooth optimization. However, the optimal smoothness parameter needs to be known in advance. This restriction is later avoided by the (batch) excessive gap algorithm (Nesterov 2005a). In the stochastic setting, Ouyang and Gray (2012) combined Nesterov's smoothing with SGD. Though these methods achieve the fastest known convergence rates in the batch and stochastic settings respectively, they assume a Lipschitz-smooth regularizer, and nonsmooth regularizers (such as the sparsity-inducing regularizers) cannot be used.

Recently, based on the observation that the training set is indeed finite, a number of fast stochastic algorithms are proposed for both smooth and composite optimization problems (Schmidt, Roux, and Bach 2013; Johnson and Zhang 2013; Xiao and Zhang 2014; Mairal 2013; Defazio, Bach, and Lacoste-Julien 2014). They are based on the idea of variance reduction, and attain comparable convergence rates as their batch counterparts. However, they are not applicable when both the loss and regularizer are nonsmooth. To alleviate this, Shalev-Shwartz and Zhang (2014) suggested running these algorithms on the smoothed approximation obtained by Nesterov's smoothing. However, as in (Nesterov 2005b), it requires a careful setting of the smoothness parameter. Over-smoothing deteriorates solution quality, while under-smoothing slows down convergence.

The problem of setting the smoothness parameter can be alleviated by continuation (Becker, Bobin, and Candès 2011). It solves a sequence of smoothed problems, in which the smoothing parameter is gradually reduced from a large value (and the corresponding smoothed problem is easy to solve) to a small value (which leads to a solution closer to that of the original nonsmooth problem). Moreover, solution of the intermediate problem is used to warm-start the next smoothed problem. This approach is also similar to that

of gradually changing the regularization parameter in (Hale, Yin, and Zhang 2007; Wen et al. 2010; Mazumder, Hastie, and Tibshirani 2010). Empirically, continuation converges much faster than the use of a fixed smoothing parameter (Becker, Bobin, and Candès 2011). However, the theoretical convergence rate obtained in (Becker, Bobin, and Candès 2011) is only for one stage of the continuation algorithm (i.e., on the smoothed problem with a particular smoothing parameter), while the convergence properties for the whole algorithm are not clear. Recently, Xiao and Zhang (2012) obtained a linear convergence rate for their continuation algorithm, though only for the special case of ℓ_1-regularized least squares regression.

In this paper, we consider the general nonsmooth optimization setting, in which both the loss and regularizer may be nonsmooth. The proposed continuation algorithm can be flexibly used with a variety of existing batch/stochastic solvers in each stage. Theoretical analysis shows that the proposed algorithm, with this wide class of solvers, achieves the rate of $O(1/T^2)$ on strongly convex problems, and $O(1/T)$ on general convex problems. These are the fastest known rates for nonsmooth optimization. Note that these rates are for the whole algorithm, not just one of its stages as in (Becker, Bobin, and Candès 2011). Experiments on nonsmooth classification and regression models demonstrate that the proposed algorithm outperforms the state-of-the-art.

Notation. For $x, y \in \mathbb{R}^d$, $\|x\|_2 = \sqrt{\sum_{i=1}^d x_i^2}$ is its ℓ_2-norm, $\|x\|_1 = \sum_{i=1}^d |x_i|$ is its ℓ_1-norm, and $\langle x, y\rangle$ is the dot product between x, y. Moreover, ∂f denotes the subdifferential of a nonsmooth function f, if f is differentiable, then ∇f denotes its gradient. I is the identity matrix.

Related Work

Consider nonsmooth functions of the form

$$g(x) = \hat{g}(x) + \max_{u\in\mathcal{U}}[\langle Ax, u\rangle - Q(u)], \tag{1}$$

where $\hat{g}$ is convex, continuously differentiable with $\hat{L}$-Lipschitz-continuous gradient, $\mathcal{U} \subseteq \mathbb{R}^p$ is convex, $A \in \mathbb{R}^{p\times d}$, and Q is a continuous convex function. Nesterov (2005b) proposed the following smooth approximation:

$$\tilde{g}_\gamma(x) = \hat{g}(x) + \max_{u\in\mathcal{U}} \left[\langle Ax, u\rangle - Q(u) - \gamma\omega(u)\right], \tag{2}$$

where γ is a smoothness parameter, and ω is a nonnegative ζ-strongly convex function.

For example, consider the hinge loss $g(x) = \max(0, 1 - y_i z_i^T x)$, where x is the linear model parameter, and (z_i, y_i) is the ith training sample with $y_i \in \{\pm 1\}$. Using $\omega(u) = \frac{1}{2}\|u\|_2^2$, g can be smoothed to (Ouyang and Gray 2012)

$$\tilde{g}_\gamma(x) = \begin{cases} 0 & y_i z_i^T x \geq 1 \\ 1 - y_i z_i^T x - \frac{\gamma}{2} & y_i z_i^T x < 1 - \gamma \\ \frac{1}{2\gamma}(1 - y_i z_i^T x)^2 & \text{otherwise} \end{cases}. \tag{3}$$

Similarly, the ℓ_1 loss $g(x) = |y_i - z_i^T x|$ can be smoothed to

$$\tilde{g}_\gamma(x) = \begin{cases} y_i - z_i^T x - \frac{\gamma}{2} & y_i - z_i^T x \geq \gamma \\ -(y_i - z_i^T x) - \frac{\gamma}{2} & y_i - z_i^T x < -\gamma \\ \frac{1}{2\gamma}(y_i - z_i^T x)^2 & \text{otherwise} \end{cases}. \tag{4}$$

Other examples in machine learning include popular regularizers such as the ℓ_1, total variation (Becker, Bobin, and Candès 2011), overlapping group lasso, and graph-guided fused lasso (Chen et al. 2012).

Minimization of the smooth (and convex) $\tilde{g}_\gamma$ can be performed efficiently using first-order methods, including the so-called "optimal method" and its variants (Nesterov 2005b) that achieve the optimal convergence rate.

Nesterov Smoothing with Continuation

Consider the following nonsmooth minimization problem

$$\min_x P(x) \equiv f(x) + r(x), \tag{5}$$

where both f and r are convex and nonsmooth. In machine learning, x usually corresponds to the model parameter, f is the loss, and r the regularizer. We assume that the loss f on a set of n training samples can be decomposed as $f(x) = \frac{1}{n}\sum_{i=1}^n f_i(x)$, where f_i is the loss value on the ith sample. Moreover, each f_i can be written as in (1), i.e., $f_i(x) = \hat{f}_i(x) + \max_{u\in\mathcal{U}}[\langle A_i x, u\rangle - Q(u)]$. One can then apply Nesterov's smoothing, and $P(x)$ in (5) is smoothed to

$$\tilde{P}(x) = \tilde{f}_\gamma(x) + r(x), \tag{6}$$

where $\tilde{f}_\gamma(x) = \frac{1}{n}\sum_{i=1}^n \tilde{f}_i(x)$ and

$$\tilde{f}_i(x) = \hat{f}_i(x) + \max_{u\in\mathcal{U}} \left[\langle A_i x, u\rangle - Q(u) - \gamma\omega(u)\right]. \tag{7}$$

As for r, we assume that it is "simple", namely that its proximal operator, $\text{prox}_{\lambda r}(\cdot) \equiv \arg\min_x \frac{1}{2}\|x - \cdot\|^2 + \lambda r(x)$ for any $\lambda > 0$, can be easily computed (Parikh and Boyd 2014).

Strongly Convex Objectives

In this section, we assume that P is μ-strongly convex. This strong convexity may come from f (e.g., ℓ_2-regularized hinge loss) or r (e.g., elastic-net regularizer) or both.

Assumption 1. *P is μ-strongly convex, i.e., there exists $\mu > 0$ such that $P(y) \geq P(x) + \xi^T(y - x) + \frac{\mu}{2}\|y - x\|_2^2, \forall \xi \in \partial P(x)$ and $x, y \in \mathbb{R}^d$.*

The proposed algorithm is based on continuation. It proceeds in stages, and a smoothed problem is solved in each stage (Becker, Bobin, and Candès 2011). The smoothness parameter is gradually reduced across stages, so that the smoothed problem becomes closer and closer to the original one. In each stage, an iterative solver $\mathcal{M}$ is used to solve the smoothed problem. It returns an approximate solution, which is then used to warm-start the next stage.

In stage s, let the smoothness parameter be γ_s, the smoothed objective in (6) be $\tilde{P}_s(x)$, $x_s^* = \arg\min_x \tilde{P}_s(x)$, and $\tilde{x}_s$ be the solution returned by $\mathcal{M}$. As $\mathcal{M}$ is warm-started by $\tilde{x}_{s-1}$, the error before running $\mathcal{M}$ is $\tilde{P}_s(\tilde{x}_{s-1}) - \tilde{P}_s(x_s^*)$. At the end of stage s, we assume that the error is reduced by a factor of ρ_s. The expectation $\mathbb{E}$ below is over the stochastic choice of training samples for a stochastic solver. For a deterministic solver, this expectation can be dropped.

Assumption 2. $\mathbb{E}\tilde{P}_s(\tilde{x}_s) - \tilde{P}_s(x_s^*) \leq \rho_s(\tilde{P}_s(\tilde{x}_{s-1}) - \tilde{P}_s(x_s^*))$, *where* $\rho_s \in (0, 1)$.

Table 1: Examples of non-accelerated and accelerated solvers. Note that Prox-GD and APG are batch solvers while the others are stochastic solvers. Here, θ and p are parameters related to the stepsize, and are fixed across stages. In particular, $\theta \in (0, 0.25)$ and satisfies $(1-4\theta)\rho_s - 4\theta > 0$, and $p \in (0,1)$ and satisfies $\rho_s > \frac{p(2+p)}{1-p}$. Accelerated Prox-SVRG has $T_s = O(\sqrt{\kappa_s}\log(1/\rho_s))$ only when a sufficiently large mini-batch is used.

		T_s	$\phi(\rho_s)$	a	b	c
non-accelerated	Prox-GD (Nesterov 2013)	$4\kappa_s \log(1/\rho_s)$	$\log(1/\rho_s)$	4	0	0
	Prox-SVRG (Xiao and Zhang 2014)	$\frac{\theta}{(1-4\theta)\rho_s - 4\theta}(\kappa_s + 4)$	$\frac{1}{(1-4\theta)\rho_s - 4\theta}$	θ	4θ	0
	SAGA (Defazio, Bach, and Lacoste-Julien 2014)	$\frac{3n}{\rho_s}\left(\frac{3\kappa_s}{n} + 1\right)$	$\frac{1}{\rho_s}$	9	$3n$	0
	MISO (Mairal 2013)	$\frac{n\kappa_s}{\rho_s}$	$\frac{1}{\rho_s}$	n	0	0
accelerated	APG (Schmidt, Roux, and Bach 2011)	$\sqrt{\kappa_s}\log(2/\rho_s)$	$\log(2/\rho_s)$	1	0	0
	Accelerated Prox-SVRG (Nitanda 2014)	$\sqrt{\kappa_s}\frac{\sqrt{2}}{(1-p)}\log\left(\frac{1}{\frac{\rho_s}{2+p} - \frac{p}{1-p}}\right)$	$\log\left(\frac{1}{\frac{\rho_s}{2+p} - \frac{p}{1-p}}\right)$	$\frac{\sqrt{2}}{(1-p)}$	0	0

We consider two types of solvers, which differ in the number of iterations (T_s) it takes to satisfy Assumption 2.

1. Non-accelerated solvers: $T_s = a\kappa_s\phi(\rho_s) + b\phi(\rho_s) + c$;
2. Accelerated solvers: $T_s = a\sqrt{\kappa_s}\phi(\rho_s) + b\phi(\rho_s) + c$.

Here, κ_s is the condition number of the objective, $a, b, c \geq 0$ are constants not related to κ_s and $\phi(\rho_s)$. Moreover, ϕ satisfies (i) $\phi(\rho_s) > 0$ and non-increasing for $\rho_s \in (0,1)$; (ii) $\phi(\rho_s)$ is not related to κ_s. Note that when κ_s is large (as is typical when the smoothed problem approaches the original problem), non-accelerated solvers need a larger T_s than accelerated solvers. Table 1 shows some non-accelerated and accelerated solvers popularly used in machine learning.

Algorithm 1 shows the proposed procedure, which will be called CNS (Continuation for NonSmooth optimization). It is similar to that in (Becker, Bobin, and Candès 2011), which however does not have convergence results. Moreover, a small but important difference is that Algorithm 1 specifies how T_s should be updated across stages, and this is essential for proving convergence. Note the different update options for non-accelerated and accelerated solvers.

Algorithm 1 CNS algorithm for strongly convex problems.

Input: number of iterations T_1 and smoothness parameter γ_1 for stage 1, and shrinking parameter $\tau > 1$.
Initialize: $\tilde{x}_0$.
for $s = 1, 2, \ldots$ **do**
  $\tilde{P}_s \leftarrow$ smooth P with smoothing parameter γ_s;
  $\tilde{x}_s \leftarrow$ minimize $\tilde{P}_s(x)$ by running $\mathcal{M}$ for T_s iterations;
  $\gamma_{s+1} = \gamma_s/\tau$;
  Option I (non-accelerated solvers): $T_{s+1} = \tau T_s$;
  Option II (accelerated solvers): $T_{s+1} = \sqrt{\tau} T_s$;
end for
Output: $\tilde{x}_s$.

The following Lemma shows that when T_1 is large enough, error reduction can be guaranteed across all stages.

Lemma 1. *For both non-accelerated and accelerated solvers, if T_1 is large enough such that $\rho_1 \leq 1/\tau^2$, then $\rho_s \leq 1/\tau^2$ for all $s > 1$.*

If κ_1 is known, a sufficiently large T_1 can be obtained from Table 1; otherwise, we can obtain T_1 by ensuring $\tilde{P}_1(\tilde{x}_1) \leq \tilde{P}_1(\tilde{x}_0)/\tau^2$, which then implies $\tilde{P}_1(\tilde{x}_1) - \tilde{P}_1(x_1^*) \leq (\tilde{P}_1(\tilde{x}_0) - \tilde{P}_1(x_1^*))/\tau^2$.

Convergence when Non-Accelerated Solver is used Let $x^* = \arg\min_x P(x)$, and $D_u = \max_{u \in \mathcal{U}} \omega(u)$. The following Lemma shows that if x is an ϵ-accurate solution of the $\tilde{P}_s$ (i.e., $\tilde{P}_s(x) - \tilde{P}_s(x_s^*) \leq \epsilon$), it is also an $(\epsilon + \gamma_s D_u)$-accurate solution of the original objective P.

Lemma 2. $\tilde{P}_s(x) - \tilde{P}_s(x_s^*) - \gamma_s D_u \leq P(x) - P(x^*) \leq \tilde{P}_s(x) - \tilde{P}_s(x_s^*) + \gamma_s D_u$.

Since Lemma 2 holds for any x, it also holds in expectation, i.e., $\mathbb{E}\tilde{P}_s(\tilde{x}_s) - \tilde{P}_s(x_s^*) - \gamma_s D_u \leq \mathbb{E}P(\tilde{x}_s) - P(x^*) \leq \mathbb{E}\tilde{P}_s(\tilde{x}_s) - \tilde{P}_s(x_s^*) + \gamma_s D_u$.

Theorem 1. *Assume that T_1 in Algorithm 1 is large enough so that $\rho_1 \leq 1/\tau^2$. When non-accelerated solvers are used,*

$$\mathbb{E}P(\tilde{x}_S) - P(x^*) \leq \left(\prod_{s=1}^{S} \rho_s\right)(P(\tilde{x}_0) - P(x^*)) + O\left(\frac{\gamma_1 D_u}{T}\right), \quad (8)$$

where S is the number of stages, $T = \sum_{s=1}^{S} T_s$, and $\prod_{s=1}^{S} \rho_s = O(1/T^2)$.

The first term on the RHS of (8) reflects the cumulative decrease of the objective after S stages, while the second term is due to smoothing. The condition $\rho_1 \leq 1/\tau^2$ is used to obtain the $O(1/T)$ rate in the last term of (8). If we instead require that $\rho_1 \leq 1/\tau$, it can be shown that the rate will be slowed to $O(\log T/T)$; if $\rho_1 \leq 1/\sqrt{\tau}$, it degrades further to $O(1/\sqrt{T})$. On the other hand, if $\rho_1 \leq 1/\tau^c$ with $c > 2$, the rate will not be improved.

Corollary 1. *Together with Lemma 1, we have*

$$\mathbb{E}P(\tilde{x}_S) - P(x^*) \leq \frac{P(\tilde{x}_0) - P(x^*)}{\tau^{2S}} + O\left(\frac{\gamma_1 D_u}{T}\right), \quad (9)$$

where $1/\tau^{2S} = O(1/T^2)$.

Existing stochastic algorithms such as SGD, FOBOS and RDA have a convergence rate of $O(\log T/T)$ (Rakhlin, Shamir, and Sridharan 2012; Duchi and Singer 2009; Xiao 2009), while here we have the faster $O(1/T)$ rate. Recent

works in (Shamir and Zhang 2013; Ouyang and Gray 2012) also achieve a $O(1/T)$ rate. However, Shamir and Zhang (2013) use stochastic subgradient, and do not exploit properties of the regularizer (such as sparsity). This can lead to inferior performance (Duchi and Singer 2009; Xiao 2009; Mazumder, Hastie, and Tibshirani 2010). On the other hand, (Ouyang and Gray 2012) is restricted to $r \equiv 0$ in (5).

Next, we compare with the case where continuation is not used (i.e., γ_s is a constant). Equivalently, this corresponds to setting $\tau = 1$ in Algorithm 1.

Proposition 1. *When continuation is not used, let $\rho \in (0,1)$ be the error reduction factor at each stage, and $\gamma > 0$ be the fixed smoothing parameter. When either an accelerated or non-accelerated solver is used,*

$$\mathbb{E}P(\tilde{x}_S) - P(x^*) \leq \rho^S (P(\tilde{x}_0) - P(x^*)) + (1+\rho^S)\gamma D_u. \quad (10)$$

Proposition 2. *Assume that the two terms on the RHS of (9) and (10) are equal to $\alpha\epsilon$ and $(1-\alpha)\epsilon$, respectively, where $\alpha > 0$ and $\epsilon > 0$. Let $\rho_1 = \rho = 1/\tau^2$ in (8) and (10). Assume that Algorithm 1 needs a total of T iterations to obtain an ϵ-accurate solution, while its fixed-γ_s variant takes T' iterations. Then,*

$$T \geq \frac{\tau^S - 1}{\tau - 1}\left(a\kappa_1\phi\left(\frac{1}{\tau^2}\right) + b\phi\left(\frac{1}{\tau^2}\right) + c\right),$$

$$T' \geq S\left(a\left(\frac{\tau^{2S}+1}{\tau^{S+1}+\tau^S}\kappa_1 + C\right)\phi\left(\frac{1}{\tau^2}\right) + b\phi\left(\frac{1}{\tau^2}\right) + c\right),$$

where $S \geq \log\left(\frac{\alpha\epsilon}{(P(\tilde{x}_0)-P(x^*))}\right) / \log\left(\frac{1}{\tau^2}\right)$, $C = \left(1 - \frac{\tau^{2S}+1}{\tau^{S+1}+\tau^S}\right)\frac{K}{\mu}$, and K is a constant,

T and T' are usually dominated by the $a\kappa_1\phi(1/\tau^2)$ term, and T' is roughly S times that of T. This is also consistent with empirical observations that continuation is much faster than fixed smoothing (Becker, Bobin, and Candès 2011).

Convergence when Accelerated Solver is used

Theorem 2. *Assume that T_1 in Algorithm 1 is large enough so that $\rho_1 \leq 1/\tau^2$. When accelerated solvers are used,*

$$\mathbb{E}P(\tilde{x}_S) - P(x^*) \leq \left(\prod_{s=1}^{S}\rho_s\right)(P(\tilde{x}_0) - P(x^*)) + O\left(\frac{\gamma_1 D_u}{T^2}\right).$$

where $T = \sum_{s=1}^{S} T_s$, and $\prod_{s=1}^{S}\rho_s = O(1/T^4)$.

As the ρ_s's for non-accelerated and accelerated solvers are different, the $\prod_{s=1}^{S}\rho_s$ term here is different from that in Theorem 1. Moreover, the last term is improved from $O(1/T)$ in Theorem 1 to $O(1/T^2)$ with accelerated solvers. This is also better than the rates of existing stochastic algorithms ($O(\log T/T)$ in (Duchi and Singer 2009; Xiao 2009) and $O(1/T)$ in (Rakhlin, Shamir, and Sridharan 2012; Shamir and Zhang 2013; Ouyang and Gray 2012)). Besides, the black-box lower bound of $O(1/T)$ for strongly convex problems (Agarwal et al. 2009) does not apply here, as we have additional assumptions that the objective is of the form in (1) and the number of training samples is finite. Though the (batch) excessive gap algorithm (Nesterov 2005a) also has a $O(1/T^2)$ rate, it is limited to $r \equiv 0$ in (5).

As in Proposition 2, the following shows that if continuation is not used, the algorithm is roughly S times slower.

Proposition 3. *With the same assumptions in Proposition 2,*

$$T \geq \frac{\sqrt{\tau}^S - 1}{\sqrt{\tau} - 1}\left(a\sqrt{\kappa_1}\phi\left(\frac{1}{\tau^2}\right) + b\phi\left(\frac{1}{\tau^2}\right) + c\right),$$

$$T' \geq S\left(a\sqrt{\frac{\tau^{2S}+1}{\tau^{S+1}+\tau^S}\kappa_1 + C}\,\phi\left(\frac{1}{\tau^2}\right) + b\phi\left(\frac{1}{\tau^2}\right) + c\right),$$

where S, C are as defined in Proposition 2.

General Convex Objectives

When P is not strongly convex, we add to it a small ℓ_2 term (with weight λ_s). We then gradually decrease γ_s and λ_s simultaneously to approach the original problem. The revised procedure is shown in Algorithm 2.

Algorithm 2 CNS algorithm for general convex problems.

Input: number of iterations T_1, smoothness parameter γ_1 and strong convexity parameter λ_1 for stage 1, and shrinking parameter $\tau > 1$.
Initialize: $\tilde{x}_0$.
for $s = 1, 2, \ldots$ **do**
  $\tilde{P}_s \leftarrow$ smooth P with smoothing parameter γ_s;
  $\tilde{x}_s \leftarrow$ minimize $\tilde{P}_s(x) + \frac{\lambda_s}{2}\|x\|_2^2$ by running $\mathcal{M}$ for T_s iterations;
  $\gamma_{s+1} = \gamma_s/\tau$; $\lambda_{s+1} = \lambda_s/\tau$;
  Option I (non-accelerated solvers): $T_{s+1} = \tau^2 T_s$;
  Option II (accelerated solvers): $T_{s+1} = \tau T_s$;
end for
Output: $\tilde{x}_s$.

We assume that there exists $R > 0$ such that $\|x^*\|_2 \leq R$, and $\|x_s^*\|_2 \leq R$ for all s. Define $H_s(x) = \tilde{P}_s(x) + \frac{\lambda_s}{2}\|x\|_2^2$, and let $x_s^* = \arg\min_x H_s(x)$. The following assumption is similar to that for strongly convex problems.

Assumption 3. $\mathbb{E}H_s(\tilde{x}_s) - H_s(x_s^*) \leq \rho_s(H_s(\tilde{x}_{s-1}) - H_s(x_s^*))$, *where $\rho_s \in (0,1)$.*

Theorem 3. *Assume that T_1 in Algorithm 2 is large enough so that $\rho_1 \leq 1/\tau^2$. When non-accelerated solvers are used,*

$$\mathbb{E}P(\tilde{x}_S) - P(x^*) \leq \left(\prod_{s=1}^{S}\rho_s\right)\left(P(\tilde{x}_0) - P(x^*) + \frac{\lambda_1}{2}\|\tilde{x}_0\|_2^2\right) + O\left(\frac{\lambda_1 R^2/2}{\sqrt{T}}\right) + O\left(\frac{\gamma_1 D_u}{\sqrt{T}}\right), \quad (11)$$

where $\prod_{s=1}^{S}\rho_s = O(\frac{1}{T})$. For accelerated solvers,

$$\mathbb{E}P(\tilde{x}_S) - P(x^*) \leq \left(\prod_{s=1}^{S}\rho_s\right)\left(P(\tilde{x}_0) - P(x^*) + \frac{\lambda_1}{2}\|\tilde{x}_0\|_2^2\right) + O\left(\frac{\lambda_1 R^2/2}{T}\right) + O\left(\frac{\gamma_1 D_u}{T}\right), \quad (12)$$

where $\prod_{s=1}^{S}\rho_s = O(\frac{1}{T^2})$.

For non-accelerated solvers, the $O(1/\sqrt{T})$ convergence rate in (11) is only as good as that obtained in (Xiao 2009; Duchi and Singer 2009; Ouyang and Gray 2012; Shamir and

Zhang 2013). Hence, they will not be studied further in the sequel. However, for accelerated solvers, the $O(1/T)$ convergence rate in (12) is faster than the $O(1/\sqrt{T})$ rate in (Xiao 2009; Duchi and Singer 2009; Ouyang and Gray 2012; Shamir and Zhang 2013) and the $O(\frac{1}{T^2} + \frac{\log T}{T})$ rate in (Orabona, Argyriou, and Srebro 2012). The $O(1/T)$ convergence rate is also obtained in (Nesterov 2005a; 2005b), but again only for $r \equiv 0$ in (5).

When continuation is not used, the following results are analogous to those obtained in the previous section.

Proposition 4. *Let $\tilde{x}_0 = 0$. When continuation is not used, let ρ be the error reduction factor at each stage. When either an accelerated or non-accelerated solver is used,*

$$\mathbb{E}P(\tilde{x}_S) - P(x^*) \quad \leq \quad \rho^S(P(\tilde{x}_0) - P(x^*)) + (1+\rho^S)\gamma D_u + \frac{\lambda}{2}R^2. \quad (13)$$

Proposition 5. *Let $\tilde{x}_0 = 0$. Suppose that the three terms on the RHS of (13) are equal to $\alpha\epsilon$, $\beta\epsilon$ and $\zeta\epsilon$, respectively, where $\alpha, \beta, \zeta > 0$ and $\alpha + \beta + \zeta = 1$. Let $\rho_1 = \rho = 1/\tau^2$ in (12) and (13). Assume that Algorithm 2 (with accelerated solver) needs a total of T iterations to obtain an ϵ-accurate solution, while its fixed-γ_s variant takes T' iterations. Then,*

$$T \geq \frac{\tau^S - 1}{\tau - 1}\left(a\sqrt{\kappa_1}\phi\left(\frac{1}{\tau^2}\right) + b\phi\left(\frac{1}{\tau^2}\right) + c\right),$$

$$T' \geq S\left(a\sqrt{\frac{\tau^{2S}+1}{(\tau+1)^2}\kappa_1 + C}\,\phi\left(\frac{1}{\tau^2}\right) + b\phi\left(\frac{1}{\tau^2}\right) + c\right),$$

where $S \geq \log\left(\frac{\alpha\epsilon}{(P(\tilde{x}_0)-P(x^*))}\right) / \log\left(\frac{1}{\tau^2}\right)$, $C = \left(1 - \frac{\tau^{2S}+1}{(\tau+1)^2}\right) + \left(\frac{\tau^S}{1+\tau} - \frac{\tau^{2S}+1}{(\tau+1)^2}\right)\frac{K}{\lambda_1}$ and K is a constant.

A summary of the convergence results is shown in Table 2. As can be seen, the convergence rates of the proposed CNS algorithm match the fastest known rates in nonsmooth optimization, but CNS is less restrictive and can exploit the composite structure of the optimization problem.

Table 2: Comparison with the fastest known convergence rates for nonsmooth optimization problem (1). The fastest known batch solver is restricted to $r \equiv 0$, while the fastest known stochastic solver does not exploit properties of r.

strongly convex	batch solver	stochastic solver	CNS (batch/stochastic) non-accel.	accel.
yes	$1/T^2$	$1/T$	$1/T$	$1/T^2$
no	$1/T$	$1/\sqrt{T}$	$1/\sqrt{T}$	$1/T$

Experiments

Because of the lack of space, we only report results on two data sets (Table 3) from the LIBSVM archive: (i) the popularly used classification data set *rcv1*; and (ii) *YearPredictionMSD*, the largest regression data in the LIBSVM archive, and is a subset of the Million Song data set. We use the hinge loss for classification, and ℓ_1 loss for regression. Both can be smoothed using Nesterov's smoothing (to (3) and (4), respectively). As for the regularizer, we use the

1. elastic-net regularizer $r(x) = \nu_1\|x\|_1 + \frac{\nu_2}{2}\|x\|_2^2$ (Zou and Hastie 2005), and problem (5) is strongly convex; and
2. ℓ_1 regularizer $r(x) = \nu_1\|x\|_1$, and (5) is (general) convex.

Here, ν_1, ν_2 are tuned by 5-fold cross-validation. Obviously, all losses and regularizers are convex but nonsmooth. We use mini-batch for all methods. The mini-batch size b is 50 for *rcv1*, and 100 for *YearPredictionMSD*.

Table 3: Data sets used in the experiments.

	#train	#test	#features
rcv1	20,242	677,399	47,236
YearPredictionMSD	463,715	51,630	90

The following stochastic algorithms are compared:

1. Forward-backward splitting (FOBOS) (Duchi and Singer 2009), a standard baseline for nonsmooth stochastic composite optimization.
2. SGD with polynomial-decay averaging (Poly-SGD) (Shamir and Zhang 2013), the state-of-art for nonsmooth optimization.
3. Regularized dual averaging (RDA) (Xiao 2009): This is another state-of-the-art for sparse learning problems.
4. The proposed CNS algorithm: We use proximal SVRG (PSVRG) (Xiao and Zhang 2014) as the underlying non-accelerated solver, and accelerated proximal SVRG (ACC-PSVRG) (Nitanda 2014) as the accelerated solver. The resultant procedures are denoted CNS-NA and CNS-A, respectively. We set $\gamma_1 = 0.01, \tau = 2$, and $T_1 = \lceil n/b \rceil$. Empirically, this ensures $\rho_1 \leq 1/\tau^2$ (in Theorems 1 and 3) on the two data sets.

Note that FOBOS, RDA and the proposed CNS can effectively make use of the composite structure of the problem, while Poly-SGD cannot. For each method, the stepsize is tuned by running on a subset containing 20% training data for a few epochs (for the proposed method, we tune η_1). All algorithms are implemented in Matlab.

Strongly Convex Objectives

Figure 1 shows convergence of the objective and testing performance (classification error for *rcv1* and ℓ_1-loss for *YearPredictionMSD*). The trends are consistent with Theorem 1. CNS-A is the fastest (with a of $O(1/T^2)$). This is followed by CNS-NA and Poly-SGD, both with $O(1/T)$ rate (from Theorem 1 and (Shamir and Zhang 2013)). The slowest are FOBOS and RDA, which converge at a rate of $O(\log T/T)$ (Duchi and Singer 2009; Xiao 2009).

Figure 2 compares with the case where continuation is not used. Two fixed smoothness settings, $\gamma = 10^{-2}$ and $\gamma = 10^{-3}$, are used. As can be seen, they are much slower (Propositions 2 and 3). Moreover, a smaller γ leads to slower convergence but better solution, while a larger γ leads to

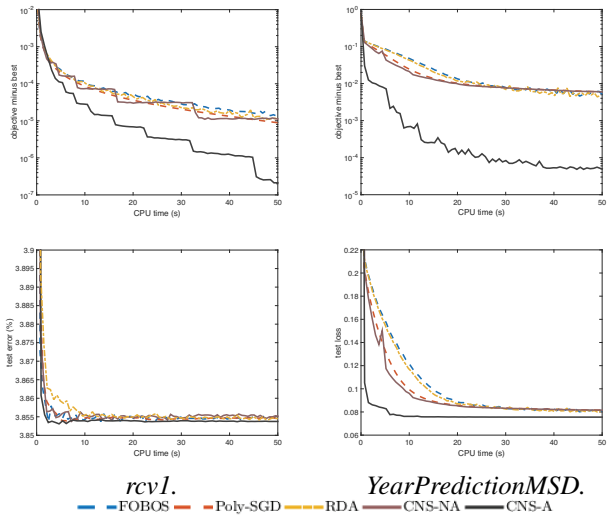

Figure 1: Objective (top) and testing performance (bottom) vs CPU time (in seconds) on a strongly convex problem.

faster convergence but worse solution. This is also consistent with Proposition 1, as using a fixed γ only allows convergence to the optimal solution with a tolerance of $(1+\rho^S)\gamma D_u$. Moreover, a smaller γ leads to a larger condition number, and convergence becomes slower.

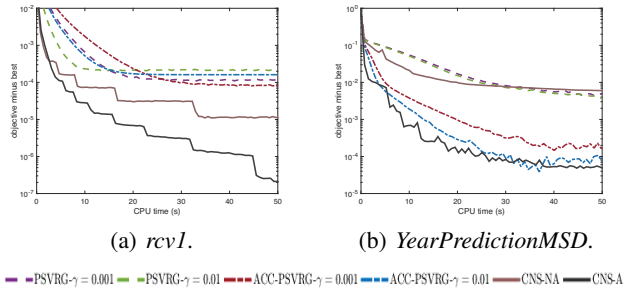

Figure 2: Effect of continuation (strongly convex problem).

General Convex Objectives

We set λ_1 in Algorithm 2 to 10^{-5} for *rcv1*, and 10^{-7} for *YearPredictionMSD*. As can be seen from Figure 3, the trends are again consistent with Theorem 3. CNS-A is the fastest ($O(1/T)$ convergence rate), while the others all have a rate of $O(1/\sqrt{T})$ (Duchi and Singer 2009; Xiao 2009; Shamir and Zhang 2013). Also, RDA shows better performance than FOBOS and Poly-SGD. Recall that Poly-SGD outperforms FOBOS and RDA on strongly convex problems. However, on general convex problems, Poly-SGD is the worst as its rate is only as good as others, and it does not exploit the composite structure of the problem.

Figure 4 compares with the case where continuation is not used. As in the previous section, CNS-NA and CNS-A show faster convergence than its fixed-smoothing counterparts.

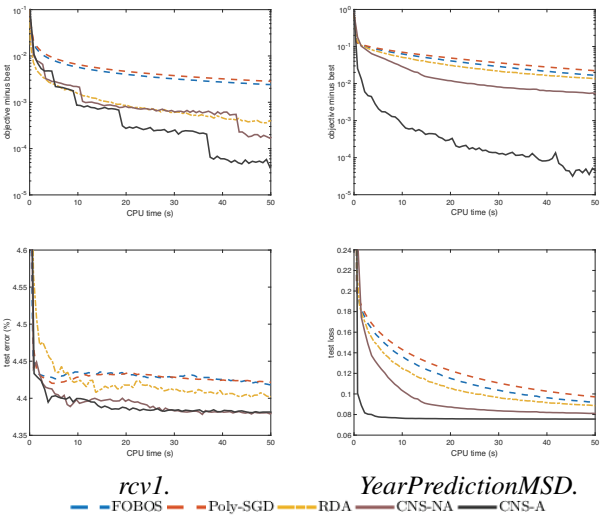

Figure 3: Objective (top) and testing performance (bottom) vs CPU time (in seconds) on a general convex problem.

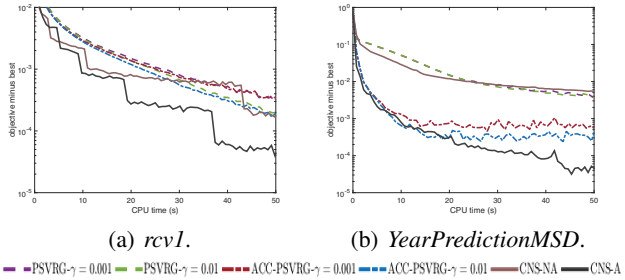

Figure 4: Effect of continuation (general convex problem).

Conclusion

In this paper, we proposed a continuation algorithm (CNS) for regularized risk minimization problems, in which both the loss and regularizer may be nonsmooth. In each of its stages, the smoothed subproblem can be easily solved by either existing accelerated or non-accelerated solvers. Theoretical analysis establishes convergence results on the whole continuation algorithm, not just one of its stages. In particular, when accelerated solvers are used, the proposed CNS algorithm achieves the rate of $O(1/T^2)$ on strongly convex problems, and $O(1/T)$ on general convex problems. These are the fastest known rates for nonsmooth optimization. However, CNS is advantageous in that it allows the use of a regularizer (unlike the fastest batch algorithm) and can exploit the composite structure of the optimization problem (unlike the fastest stochastic algorithm). Experiments on nonsmooth classification and regression models demonstrate that CNS outperforms the state-of-the-art.

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