SRC TR 88-74

The Preprocessing of Search Spaces for Branch and Bound Search

by

Q. Yang and D.S. Nau

The Preprocessing of Search Spaces for Branch and Bound Search

Qiang Yang* Dana S. Nau†

University of Maryland

Abstract

Heuristic search procedures are useful in a large number of problems of practical importance. Such procedures operate by searching several paths in a search space at the same time, expanding some paths more quickly than others depending on which paths look most promising. Often large amounts of time are required in keeping track of the information control knowledge.

For some problems, this overhead can be greatly reduced by preprocessing the problem in appropriate ways. In particular, we discuss a data structure called a *threaded decision graph*, which can be created by preprocessing the search space for some problems, and which captures the control knowledge for problem solving. We show how this can be done, and we present an analysis showing that by using such a method, a great deal of time can be saved during problem solving processes.

1 Introduction

Heuristic search procedures are useful in a large number of problem domains. Most heuristic search procedures (for example, A*, SSS*, B*, AO*, and alpha-beta) have been shown to be special cases of best-first Branch-and-Bound search [3,4]. These procedures search several solution paths at the same time, expanding some paths more quickly than others depending on which paths look most promising.

One source of computational overhead during such search is the time spent keeping track of the alternate partial solutions that the procedure is examining. Typically, the search algorithm stores these partial solutions on a list called the *agenda*, *open list*, or *active list*, in order of their estimated cost. Any time a new partial solution is generated, its estimated cost must be compared with the estimated costs of the other partial solutions already on the list, in order to find the appropriate place to put it on the list. Thus, significant computational overhead is required just to maintain the active list.

*Computer Science Department. *E-mail address: yang@brillig.umd.edu.*

†Computer Science Department, Institute for Advanced Computer Studies, and Systems Research Center. This work supported in part by an NSF Presidential Young Investigator award, with matching funds provided by Texas Instruments and General Motors Research Laboratories. *E-mail address: nau@mimsy.umd.edu.*

Certain problem domains have special properties which allow us to eliminate this overhead. In particular, in some problem domains it is possible to do automatic preprocessing or "compiling" of the search space, to extract control knowledge which can be used to do the heuristic search without having to maintain the active list explicitly. In this paper, we discuss what kind of domains allow us to do such preprocessing, how to do the preprocessing, how to use the information gathered from the preprocessing, and how much time can be saved by doing this preprocessing. We also present an example from a problem domain of particular interest to us: generative process planning for the manufacture of machined parts.

2 Problem Characteristics

Preprocessing of the search space is possible whenever the following conditions are satisfied:

1. With the exception of feasibility/infeasibility of nodes, the search space has the same shape regardless of the particular problem instance being considered. Thus, if P and P' are two problem instances, then there is a one-to-one mapping between the nodes of their search spaces S and S', such that if some feasible node a in S has n children, then its corresponding node a' in S' will either have n children or be infeasible.

2. Corresponding nodes need not have the same cost—but if two nodes a and b in S_1 have $\text{cost}(a) > \text{cost}(b)$, then for the corresponding nodes a' and b' in S', we must have $\text{cost}(a') > \text{cost}(b')$.

Nearly all heuristic search problems satisfy the first condition above. One example of particular interest to us is generative process planning for the manufacture of machined parts. Our ongoing work on the development of SIPP (a generative process planning system written in Prolog) and SIPS (a more sophisticated system written in Lisp) is described in [5,6,7,9,8,10,11]. Both SIPP and SIPS consider a machinable part to be a collection of machinable features. For each feature, they find an optimal plan for that feature via a best-first Branch-and-Bound search. For example, Figure 1 shows the search tree developed by SIPS for the problem of machining a hole. The nodes labeled with an "X" are nodes which were found to be infeasible and thus were not considered further; the nodes labeled with a "P" are the nodes along the solution path found by SIPS, and the nodes with no labels were nodes which SIPS did not get around to examining because they were too costly.

Fewer heuristic search problems satisfy the second condition, but there are still many problems which do satisfy it—and process planning is such a problem. SIPS's knowledge base consists of information on a large number of different machining processes, organized in a taxonomic hierarchy. For two different machinable features, different machining processes may have different costs–but if process a costs more than process b on feature f, it generally also costs more than process b on feature f' as well. In such cases, preprocessing is possible.

3 Threaded Decision Graphs

Suppose that for a given problem instance the search space is as shown in Figure 2. The goal is to search for the cheapest leaf node for which all the nodes along the path from the root are feasible. In Figure 2, suppose the problem solver is checking the conditions associated with node c. If c succeeds, we know that the next node to check will be f, since c must be the current minimum costly node to be checked, and f has a cost the same as c's. Similarly, if c fails, the next node to check in the search space must be j. This is because at the time c is being checked, nodes a, b, d, e, h and k must all have been checked, since they all have costs less than that of c's. Control information like this is independent of the particular goals to be achieved, and can be gathered before the problem solving process starts. Therefore, we can assign a *success link* from node c to node f, and *failure link* from c to j. During the course of finding the cheapest actions to achieve a given goal, once we reach node c, if all the conditions are satisfied, we can follow the success link of c to get to the next node to be checked. On the other hand, if they are not satisfied, then the failure link at c can be followed to get to the next node to be checked. Figure 3 shows such a structure.

If all the deterministic control information is gathered, it is no longer necessary to maintain an active list for storing control information. Before problem solving starts, the information can be used to construct a data structure, which contains one or more occurrences of each action, and for each action occurrence there is a success pointer and a failure pointer. We will call such a data structure a *threaded decision graph* of the original search space. Problem solving can then be completely guided by its threaded decision graph. Figure 4 shows the threaded decision graph for the search space in Figure 2.

Two special nodes in Figure 4 are worth noting. The node marked "success" is called a *success node*. If this node is reached through the success links, the search process terminates with success. The path from the root node to success in the threaded decision graph contains the solution path which should be returned. The node marked "fail" is called a *failure node*, which marks the termination of the search process without success.

In the following, we first discuss in detail how problem solving is done with the help of threaded decision graphs. We then consider how such a data structure could be automatically constructed.

4 Problem Solving With the Threaded Decision Graphs

Given a search space, assume that we have constructed its threaded decision graph. A request to solve the problem would correspond to finding out the cheapest path in the tree. For the given threaded decision graph, this is done by checking the conditions at each node, starting from the root node. If the node being tested is feasible, then the next node to test would be the one led to by the success link of the current one. If, on the other hand, the node is infeasible, then its failure link will be followed. If a success node is reached via a success link, then the search ends with success. If a failure node is reached through failure link, then no actions satisfying the criteria exist. In this case, search is terminated with failure.

If, in the process of searching through the threaded decision graph, a non-leaf node fails, then before getting to the next node by following the failure link, all of its descendents in the original search space are marked as fail. Such a process is called *failure marking*. If any of marked nodes are reached later, no checking for condition need be made, the failure link is automatically followed.

5 The Construction of Threaded Decision Graphs

In this section, we consider the problem of how the threaded decision graphs are constructed. For simplicity, we assume the search space is in the shape of a tree and is finite. The same technique should also be applicable to a search space in the shape of a graph which is not a tree. Later we will discuss the cases where the search space is not finite.

With the above restrictions, the following algorithm generates the threaded decision graph. This algorithm is a modified version of best-first search.

```
procedure Construct
   A := { a0 }         (A is the active list, and a0 is the root of the search space)
   While A is not empty do
   begin
         n := pop(A).
         If A is empty then Failure-link(n):=Failure-node
         else Failure-link(n):= head(A);
         If n is a leaf node, then Success-link(n):=Success-node
         else begin
                  Success-link(n):=the cheapest node among {Successors(n), head(A)};
                  A:=insert(A, Successors(n));
         end{else}
   end{while}
end Construct
```

Several properties of this algorithm should be noted. First of all, since any node in the search space appears at the head of the search space once and only once, the resultant threaded decision graph must have the same number of nodes as the original search space. Since every node has only two edges, the success link and the failure link, the number of edges of the threaded decision graph is twice the number of nodes as that of the search space. The same argument also guarantees the absence of cycles in the graph.

As discussed before, we require failure marking during search. There exist special cases when such a marking is not necessary. Let $n_1, n_2, \ldots, n_k$ be the successors of the root of the search space tree in the increasing order of their cost. If the costs of all the nodes in the subtree of n_i is no greater than the cost of n_{i+1}, then no failure-marking is needed. This is because when all of the siblings in the tree are sorted in ascending order of their cost from left to right, then the above requirement on cost ensures that all the failure links in the threaded-decision-graph point from left to right. Therefore, once we leave a node via failure link, there is no chance for us to come back to any of the successors of this node via failure links, and thus no failure-marking is needed.

In the case when the search space is a graph, i.e., some nodes in the search space have more than one parent, we can make a slight modification to our procedure *construct* in order to guarantee the linearity of the graph. Notice that if node n appears in the active-list more than once, then the nodes between the two occurrences of n must have the same cost as n. If we move all these nodes to the front of n, the two n's can be merged to be one. This guarantees the uniqueness of n in the resultant threaded decision graph. With such a modification, all of the previous results apply to search spaces which are graphs.

6 Infinite Search Space

When the search space is infinite, we can preprocess a finite portion of it. This partial threaded decision graph can be created by running the algorithm *Construct* for a finite number of iterations, until one or more goal states appear in the graph.

During the problem solving process, the threaded decision graph can be used in the same manner as described in the previous sections, until a node which has no success and failure links is reached. At this point, Branch-and-Bound search can be utilized as follows: An active list is constructed by including all of the successor nodes of the current node and the feasible nodes along the path back to the root node in the state space. This list is sorted according to the costs of its nodes, and least costly of these is expanded.

7 Analysis

By gathering all the possible static control information beforehand, preprocessing of the search space can greatly improve problem solving efficiency. In order to see this claim more quantitively, let's look at a simplified example. Assume that the search space is in the shape of a tree with a branching factor of m and depth k(the root of the tree has a depth of 1). We would like to see how much time the problem solver spends on manipulating the list of incomplete paths. To do this, let's consider two extreme cases in the following.

In the best case, the search procedure expands only one path down the tree. Along this path, the number of nodes inserted into the active list is km. The time for manipulating the list is the same as the time taken to sort this number of elements, which is $O(km\log(km))$.

In the worst case, the search procedure expands the tree in a breadth first manner. The total time spent on manipulating the list in this case is bounded by $m \times \sum_{i=1}^{m^{(k-1)}} \log(i \times m) = O(m^k \log m^k)$.

To sum up, the time saved by using the threaded decision graph to guide the search procedure can be between $O(km\log(km))$ and $O(m^k \log m^k)$, where k is the depth of the tree and m is the branching factor.

8 Discussion

In this paper we presented a technique for preprocessing search space in order to gather control information for problem solving. We also discussed a number of conditions under which the method works. Through complexity analysis, we demonstrated that by using preprocessing, a great deal of computational effort is saved during the search process.

Our idea of preprocessing the search spaces has some similarity to that of threading of binary trees for tree traversal. However, the way such threading is done, and the way it is used, are both clearly different from our threaded decision graphs.

As described in the paper, our technique only works on state-space graphs. An extension of the work is to allow the search space to be an AND/OR graph. When AND branches are allowed, the problem can be very complicated depending on how the costs are assigned its nodes. In the worst case, the threaded decision graph can contain an exponential number of the occurrences of nodes in the original search space. An example of this situation is given in figure 5. The problem is that the i^{th} and $i + 1^{th}$ best paths, for $i = 1, 2, \ldots$, are not next to each other, but on the left and right subtrees respectively. One way to tackle this problem is to *partially* preprocess the search space. For example, we can assign success and failure links only to the nodes which are in the first k best paths.

If the search space is very large, the time it takes to preprocess it may be prohibitive. In such a case, we could build the search space with the following learning strategy: During the use of the search space, every time a node is tested we will look for its success or failure links. If they do not exist, we will use the search procedure to find out which node we should test next. The success or failure links can then be assigned to this node. In other words, the threaded decision graph is built on the job.

Another problem which needs more attention is that the relative costs of some nodes in different problems may not be the same, but may change with different situations. However, if the cost of nodes change with different situations in some predicatable manner, and if the number of such changes is finite, the preprocessing technique can still be made to work. For example, we may be able to differentiate the situations into separate classes, each with a different threaded decision graph. Many domains satisfy this property, including process planning in automated manufacturing.

References

[1] E. Horowitz and S. Sahni, "Fundamentals of Computer Algorithms," Computer Science Press, Potomac, MD, 1978.

[2] D. E. Knuth, *The Art of Computer Programming, Volume 1: Fundamental Algorithms*, Addison-Wesseley, Reading, Mass., 1968.

[3] V. Kumar and L. Kanal, "A General Branch and Bound Formulation for Understanding and Synthesizing And/Or Tree Search Procedures," *Artificial Intelligence* (21), 1983, 179–198.

[4] D. S. Nau, V. Kumar and L. Kanal, "General Branch and Bound, and Its Relation to A* and AO*," *Artificial Intelligence* (23), 1984, 29–58.

[5] D. S. Nau and T.-C. Chang, "A Knowledge-Based Approach to Generative Process Planning," *Proc. Computer-Aided/Intelligent Process Planning at ASME Winter Annual Meeting*, Miami Beach, FL, November 1985, 65–71.

[6] D. S. Nau and T.-C. Chang, "Hierarchical Representation of Problem-Solving Knowledge in a Frame-Based Process Planning System," *Jour. Intelligent Systems* 1:1, 1986, 29–44.

[7] D. S. Nau and M. Gray, "SIPS: An Application of Hierarchical Knowledge Clustering to Process Planning," *Proc. Symposium on Integrated and Intelligent Manufacturing at ASME Winter Annual Meeting*, Anaheim, CA, Dec. 1986, 219–225.

[8] D. S. Nau and M. Luce, "Knowledge Representation and Reasoning Techniques for Process Planning: Extending SIPS to do Tool Selection," *Proc. 19th CIRP International Seminar on Manufacturing Systems*, June 1987, 91–98.

[9] D. S. Nau and M. Gray, "Hierarchical Knowledge Clustering: A Way to Represent and Use Problem-Solving Knowledge," *in* J. Hendler, ed., *Expert Systems: The User Interface*, Ablex, 1987, 81–98.

[10] D. S. Nau, "Hierarchical Abstraction for Process Planning" *Second Internat. Conf. Applications of Artificial Intelligence in Engineering*, 1987.

[11] D. S. Nau, "Automated Process Planning Using Hierarchical Abstraction," *TI Technical Journal*, 1987, 39–46. Award winner, Texas Instruments 1987 Call for Papers on Industrial Automation.

[12] D. S. Nau, R. Karinthi, G. Vanecek, and Q. Yang, "Integrating AI and Solid Modeling for Design and Process Planning," *Second IFIP Working Group 5.2 Workshop on Intelligent CAD*, University of Cambridge, Cambridge, UK, Sept. 1988, to appear.

[13] D. S. Nau, N. Ide, R. Karinthi, G. Vanecek, and Q. Yang, "Solid Modeling and Geometric Reasoning for Design and Process Planning," *AAAI Workshop on Manufacturing Planning and Scheduling*, St. Paul, MN, Aug. 1988.

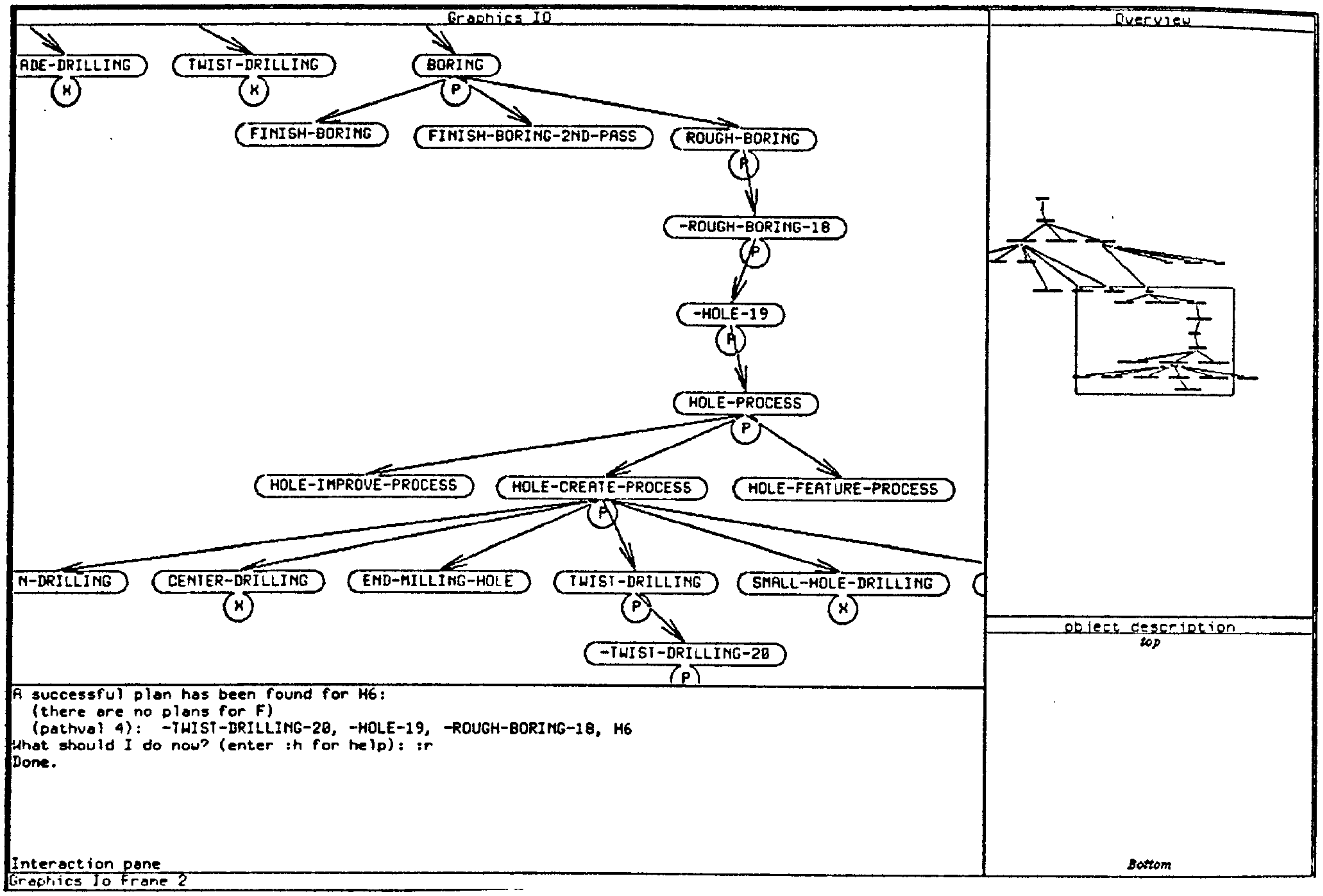

Fig 1. A search space in SIPS.

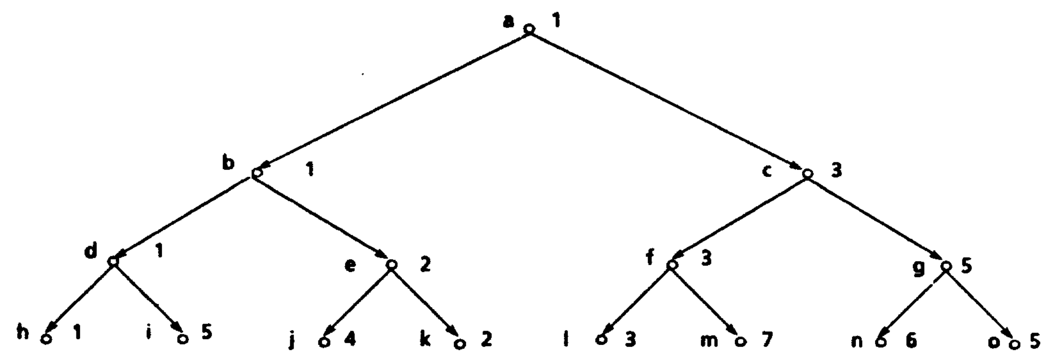

Figure 2: An example search space. The numbers associated with the nodes are their cost values.

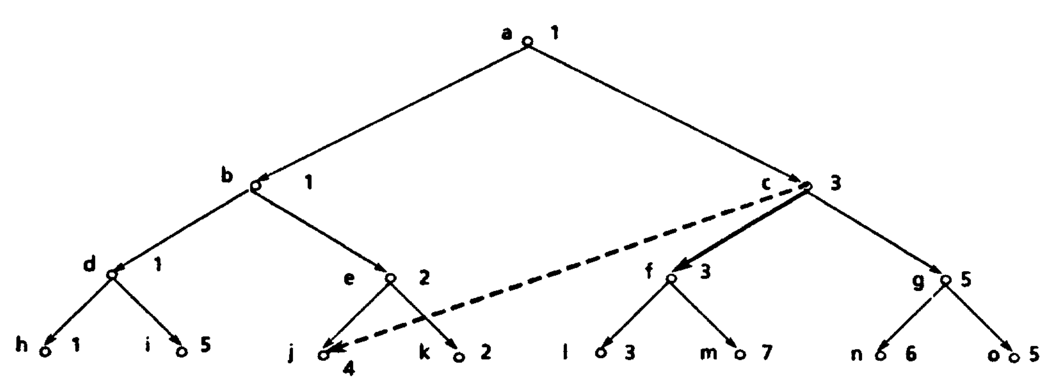

Figure 3: The search space in figure 2 with threads inserted at node c. The thick solid line is c's success-link, and the thick dotted line its failure-link.

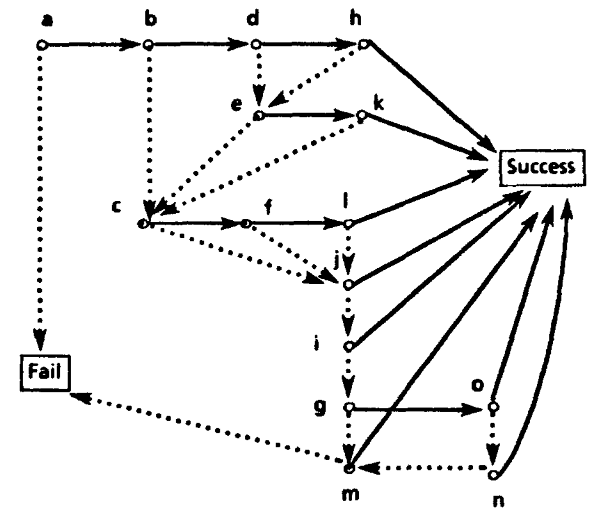

Figure 4: The threaded decision graph for the search space in figure 2. In this figure, solid lines represent success links, while the dotted ones the failure ones.

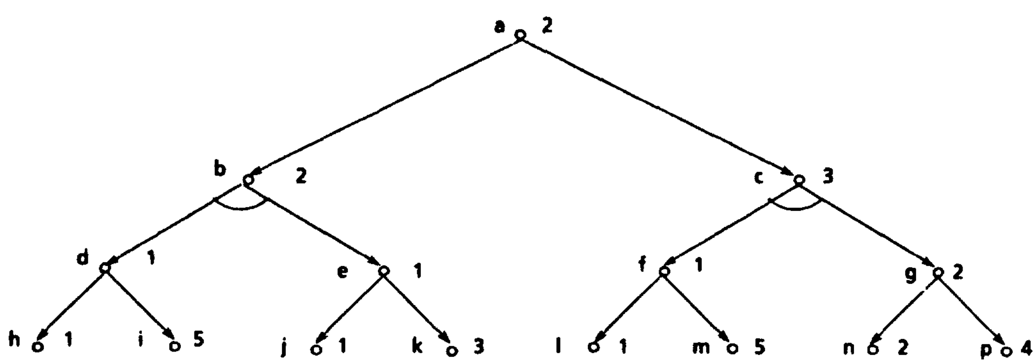

Figure 5: An example AND/OR tree in the worst case situation.

SRC TR 88-75

Hessenberg Varieties and Generalized Eulerian Numbers for Semisimple Lie Groups: The Classical Cases

by

F. De Mari and M.A. Shayman

HESSENBERG VARIETIES AND GENERALIZED EULERIAN NUMBERS FOR SEMISIMPLE LIE GROUPS: THE CLASSICAL CASES *

Filippo De Mari

Forschungsschwerpunkt Dynamische Systeme

Universität Bremen

Postfach 330 440

2800 Bremen 33, West Germany

Mark A. Shayman

Electrical Engineering Department and Systems Research Center

University of Maryland

College Park, Maryland 20742

July 15, 1988

* Research partially supported by the National Science Foundation under Grant ECS-8696108.

Abstract: Certain subvarieties of flag manifolds arise from the study of Hessenberg and banded forms for matrices. For a matrix $A \in gl(n,\mathbb{C})$ (or $sl(n,\mathbb{C})$) and a nonnegative integer p, the p^{th} *Hessenberg variety of* A is the subvariety of the (complete) flag manifold consisting of those flags $(S_1, \ldots, S_{n-1})$ satisfying the condition $AS_i \subset S_{i+p}, \quad \forall i$. The definition of these varieties extends to an arbitrary connected complex semisimple Lie group G with Lie algebra $\mathbf{g}$ using the root-space decomposition. We investigate the topology of these varieties for the classical linear Lie algebras. If A is a regular element, then for $p \geq 1$, the p^{th} Hessenberg variety is smooth and connected. The odd Betti numbers vanish, while the even Betti numbers represent (apparently new) generalizations of the classical Eulerian numbers which are determined by the height function on the root system of the Lie algebra. In particular, for $\mathbf{g} = sl(n,\mathbb{C})$, they yield a family of symmetric unimodal sequences which link the classical Eulerian numbers ($p = 1$) to the classical Mahonian numbers ($p = n-1$), while if $\mathbf{g} = sp(n,\mathbb{C})$ and $p = 1$, they are f-Eulerian numbers in the sense of R.P. Stanley.

List of Notations

G	connected complex semisimple Lie group
$\mathbf{g}$	Lie algebra of G
B	Borel subgroup of G
$\mathbf{b}$	Lie algebra of B (Borel subalgebra)
Borel($\mathbf{g}$)	variety of Borel subalgebras of $\mathbf{g}$
$\mathbf{s}$	Cartan subalgebra of $\mathbf{g}$
$\mathbf{g} = \mathbf{s} + \sum_{\alpha\in\Phi} \mathbf{g}^\alpha$	root-space decomposition
$h_p(\mathbf{b}, \mathbf{g})$, $h_p(\mathbf{g})$	p^{th} Hessenberg subspace of $\mathbf{g}$ (relative to $\mathbf{b}$)
Reg($\mathbf{g}$)	regular elements of $\mathbf{g}$
$\mathbf{H}_p(k)$	$k \times k$ height-p Hessenberg matrices ($h_{ij} = 0$, $\forall\, i - j > p$)
$L^+(k)$	$k \times k$ unipotent lower-triangular matrices
$v(k)$	$k \times k$ upper-triangular matrices
$V(k)$	$k \times k$ nonsingular upper-triangular matrices
$S(k)$	$k \times k$ symmetric matrices
$so(k,\mathbb{C})$	$k \times k$ skew-symmetric matrices

If e_j is the j^{th} standard unit column vector in $\mathbb{C}^n$, $\theta \doteq [e_n, \ldots, e_1]$, $\Omega \doteq \begin{pmatrix} I & 0 \\ 0 & \theta \end{pmatrix}$, and $\bar{X} \doteq \Omega X \Omega$. (For any $2n \times 2n$ matrix or set of such matrices, overbar denotes conjugation by Ω.)

If G is a group and H is a subgroup, $\langle X \rangle_H$ denotes the left coset XH.

If $X \in L^+(k)$, $X = (x_{ij})$, $\rho = diag(\rho_1, \ldots, \rho_k)$ and $1 \leq \beta < \alpha \leq k$, then

$$f_{\alpha\beta}^{(\rho)}(X) \doteq (\rho_\alpha - \rho_\beta)x_{\alpha\beta} + \sum_{t=1}^{\alpha-\beta-1} \sum_{\alpha > \Gamma_t > \beta} (-1)^t (\rho_{\gamma_t} - \rho_\beta) x_{\alpha\gamma_1} \cdots x_{\gamma_t\beta},$$

where $\Gamma_t = (\gamma_1, \ldots, \gamma_t)$ and $\gamma_1 > \ldots > \gamma_t$.

I. Introduction

The purpose of this paper is to study the topology of certain subvarieties of flag manifolds which arise from fundamental algorithms in numerical linear algebra–primarily the QR-algorithm for matrix eigenvalue problems. These varieties, which we refer to collectively as *Hessenberg varieties*, are of interest for at least three reasons: Firstly, there is a close relationship between their topology and the convergence properties of the numerical algorithms. Secondly, these varieties can be viewed as generalizations of the varieties of fixed flags. Thirdly, the topology of these varieties yields (apparently new) generalizations of the classical Eulerian numbers in combinatorics.

The QR-algorithm is the most commonly used method for finding the eigenvalues and invariant subspaces of a matrix. (See e.g., [1].) The QR-algorithm applied to a given $A \in GL(n, \mathbb{C})$ can be interpreted as the discrete dynamical system $\tilde{A}$ induced by A on the manifold $\mathrm{Flag}(n)$ of complete flags in $\mathbb{C}^n$ [2,3,4,5]. Consequently, the convergence behavior of the algorithm is closely related to the topology of the flag manifold, especially to the Bruhat decomposition [4].

However, in actual practice, the QR-algorithm is rarely applied directly to a given matrix A. Instead, the computational complexity is reduced (from $O(n^3)$ to $O(n^2)$ operations per iteration) by first reducing A to *Hessenberg form* using a finite sequence of elementary unitary transformations. (An $n \times n$ matrix C is in Hessenberg form if $c_{ij} = 0$ for $i - j > 1$.) This corresponds to the restriction of the dynamical system $\tilde{A}$ to the invariant subvariety

$$\mathrm{Hess}(1, A) \doteq \{(S_1, \ldots, S_{n-1}) \in \mathrm{Flag}(n) \mid AS_i \subset S_{i+1},\ \forall i\}.$$

(For details, see [3,6].) Consequently, the properties of the QR-algorithm as applied to Hessenberg matrices are closely related to the topology of $\mathrm{Hess}(1, A)$.

This motivation led us to investigate the topology of $\mathrm{Hess}(1, A)$ and of its natural generalization

$$\mathrm{Hess}(p, A) \doteq \{(S_1, \ldots, S_{n-1}) \in \mathrm{Flag}(n) \mid AS_i \subset S_{i+p},\ \forall i\}$$

(which corresponds to initializations for the QR-algorithm which are zero below the p^{th} diagonal). We refer to $\mathrm{Hess}(p, A)$ as the p^{th} *Hessenberg variety of A.*

If A is nilpotent, the subvariety of invariant flags, Hess$(0, A)$, has a very complicated geometric structure which has received considerable attention in recent years. (See e.g., [7-15].) In contrast to the case of Hess$(0, A)$, the varieties Hess(p, A), $p \geq 1$ have interesting topological structure even when A has distinct eigenvalues. It was shown in [16,17] that for such an A, Hess(p, A) is smooth and connected. Its odd Betti numbers vanish, while its even Betti numbers are an apparently unstudied generalization of the classical Eulerian numbers. (For $p = 1$, they are the classical Eulerian numbers, while for $p = n - 1$, they are the classical Mahonian numbers.) These results are summarized in Section II.

Since the key step in the QR-algorithm is the QR-factorization, and since this factorization essentially corresponds to the Iwasawa decomposition of G, a QR-like algorithm can be defined for any connected complex semisimple Lie group, or, using the exponential map, for its Lie algebra. Recently, there has been considerable interest in such algorithms, especially for $sp(n, \mathbb{C})$ [3,18] (in order to solve, for example, algebraic Riccati equations arising from optimal control and filtering problems). Consequently, it is of interest to understand the structure of "Hessenberg varieties" for G a connected complex semisimple Lie group. This is the purpose of the present paper, although we focus mostly on the classical cases.

Let G be a connected complex semisimple Lie group with Lie algebra $\mathbf{g}$. Let B be a Borel subgroup of G, and let $\mathbf{b}$ be the corresponding Borel subalgebra. Let $\mathbf{s}$ denote any Cartan subalgebra which is contained in $\mathbf{b}$, and let $\mathbf{g} = \mathbf{s} + \sum_{\alpha \in \Phi} \mathbf{g}^\alpha$ be the root-space decomposition for $\mathbf{g}$ relative to $\mathbf{s}$. For each nonnegative integer p, we define the p^{th} *Hessenberg subspace of* $\mathbf{g}$ *relative to* $\mathbf{b}$ to be

$$(1.1) \qquad h_p(\mathbf{b}, \mathbf{g}) \doteq \mathbf{s} + \sum_{\alpha \in \Phi, h(\alpha) \geq -p} \mathbf{g}^\alpha, \qquad \text{(direct sum)}$$

where $h(\alpha)$ denotes the height of the root α relative to the unique base Δ for which $h_0(\mathbf{b}, \mathbf{g}) = \mathbf{b}$. It is easily verified (see Section III) that $h_p(\mathbf{b}, \mathbf{g})$ is well-defined—i.e., independent of the choice of $\mathbf{s}$. Thus, to each Borel subalgebra $\mathbf{b}$ there is associated a partial flag in $\mathbf{g}$ which contains $\mathbf{b}$, namely

$$\mathbf{b} = h_0(\mathbf{b}, \mathbf{g}) \subset h_1(\mathbf{b}, \mathbf{g}) \subset \ldots \subset h_N(\mathbf{b}, \mathbf{g}) = \mathbf{g},$$

where $N = \max_{\alpha\in\Phi} h(\alpha)$.

Fix B, and let $A \in \mathbf{g}$. The p^{th} *Hessenberg variety of* A is the subvariety of the flag manifold G/B defined by

$$\text{(1.2)} \qquad \operatorname{Hess}(p, A) \doteq \left\{ \langle X\rangle_B \in G/B \,\middle|\, (\operatorname{Ad} X^{-1})(A) \in h_p(\mathbf{b}, \mathbf{g}) \right\}.$$

It is easily shown (Section III) that $h_p(\mathbf{b}, \mathbf{g})$ is $\operatorname{Ad}(B)$-invariant, so $\operatorname{Hess}(p, A)$ is well-defined. If G/B is identified with Borel($\mathbf{g}$) (the variety of all Borel subalgebras of $\mathbf{g}$) by identifying $\langle X\rangle_B$ with $\operatorname{Ad}(X)\mathbf{b}$, then $\operatorname{Hess}(p, A)$ is identified with the subvariety of Borel($\mathbf{g}$) consisting of those Borel subalgebras $\mathbf{b}'$ for which the p^{th} Hessenberg subspace $h_p(\mathbf{b}', \mathbf{g})$ contains A.

In the present paper, we generalize the results in [17] to the case where $\mathbf{g}$ is any classical linear Lie algebra—i.e., of type A_{n-1}, B_n, C_n or D_n. (The results for A_{n-1} and C_n are also contained in the Ph.D. thesis of the first listed author [16].) Let Φ be a reduced root system with Weyl group W, and let Φ^+ (respectively, Φ^-) be the set of positive (respectively, negative) roots with respect to some fixed basis Δ. Let $h(\cdot)$ be the height function on Φ with respect to Δ, and let $w \in W$. Define the p^{th} *Eulerian dimension of* w to be

$$\text{(1.3)} \qquad E_p^{\Phi}(w) \doteq \mathit{card}\left\{ \alpha \in \Phi^+ \,\middle|\, h(\alpha) \le p, \quad w(\alpha) \in \Phi^- \right\},$$

and define the *generalized Eulerian numbers of height* p *on* Φ to be

$$\text{(1.4)} \qquad \Phi(p, k) \doteq \mathit{card}\left\{ w \in W \,\middle|\, E_p^{\Phi}(w) = k - 1 \right\}.$$

Let $\mathbf{g}$ be a linear Lie algebra of classical type with root system Φ, and let A be a regular element of $\mathbf{g}$. We show that $\operatorname{Hess}(p, A)$ is smooth and connected. Its odd Betti numbers vanish, while its even Betti numbers are given by the generalized Eulerian numbers of height p on Φ. Since the numbers $\{\Phi(p, k)\}$ occur as the even Betti numbers of a nonsingular irreducible projective variety, it then follows from results of Stanley [19] that (for fixed p), they form a sequence which is both unimodal and symmetric.

As mentioned previously, in the special case where $\Phi = A_{n-1}$ (i.e., $\mathbf{g} = sl(n, \mathbb{C})$), $p = 1$ (respectively, $p = n - 1$), the generalized Eulerian numbers correspond to the classical Eulerian (respectively, Mahonian) numbers. We also analyze in detail the generalized Eulerian numbers in the case where $\Phi = C_n$ (i.e., $\mathbf{g} = sp(n, \mathbb{C})$) and $p = 1$. We determine the recurrence relation, exact values and generating function, and show that these numbers are f-Eulerian numbers in the sense of Stanley [20], with $f(s) = (2s+1)^n$.

Several generalizations of the Eulerian numbers have been considered in the combinatorics literature. (See e.g., [20-25].) However, we are unaware of any studies in which the Eulerian numbers are generalized via the combinatorics of root systems and the height function. Although we have limited our analysis to the classical cases, the nature of both the techniques (e.g., the Bruhat decomposition) and the results (e.g., combinatorial properties of the height function on the root system) suggest that the results may hold in more generality.

The organization of this paper is as follows: In Section II, we review the results for $G = GL(n, \mathbb{C})$ contained in [16,17]. In Section III, we extrapolate from the cases $G = SL(n, \mathbb{C})$, p arbitrary, and $G = Sp(n, \mathbb{C})$, p=1, to obtain the general definitions (1.1), (1.2) given above for Hessenberg subspaces and Hessenberg varieties. In Sections IV, V and VI, we study the topology of $\text{Hess}(p, A)$ for the linear Lie algebras of types C_n, D_n and B_n, respectively. Finally, in Section VII, we show that the (even) Betti numbers computed in the previous sections are in fact generalized Eulerian numbers (of height p) on the appropriate root systems, and we examine in detail the symplectic Eulerian numbers of height one.

II. Review of Results for $G = GL(n, \mathbb{C})$

In this section, we review the definitions and results from [16,17] which are needed in the sequel. Let $G = GL(n, \mathbb{C})$ and let $\mathbf{g}$ be the Lie algebra of G. Let $\text{Reg}(\mathbf{g})$ denote the set of regular elements in $\mathbf{g}$ consisting of those matrices which have distinct eigenvalues. Let $\text{Flag}(n)$ denote the variety of *complete* flags in $\mathbb{C}^n$. This can be identified with G/B where B is the Borel subgroup consisting of upper-triangular matrices. In this context, for $X \in G$, $\langle X \rangle_B$ denotes the image of X under the projection $G \to G/B$ and is interpreted as the flag $S_1 \subset \ldots \subset S_{n-1}$, where S_i is the subspace of $\mathbb{C}^n$ spanned by the first i columns of X.

A covering system of analytic charts for G/B is given by

$$ch(\sigma) \doteq \left\{ \langle \sigma X \rangle_B \mid X \in L^+(n) \right\}, \qquad \sigma \in \Sigma(n),$$

where $L^+(n)$ denotes the group of unipotent lower triangular $n \times n$ complex matrices, and $\Sigma(n)$ denotes the symmetric group of permutation matrices. The correspondence $\langle \sigma X \rangle_B \to X$ is a bijection of $ch(\sigma)$ onto $L^+(n)$, this latter being identified with $\mathbb{C}^{\binom{n}{2}}$. Local calculations will be performed in these coordinates.

(II.1) Definition: Let $A \in \mathbf{g}$, $0 \leq p \leq n-1$. A *Hessenberg flag of height* p for A is a flag $S_1 \subset \ldots \subset S_{n-1}$ such that

$$AS_i \subset S_{i+p} \qquad i = 1, \ldots, n-p-1. \tag{2.1}$$

The set of all such flags is denoted Hess(p, A) and called the p^{th} *Hessenberg variety of* A.

Remarks: The set Hess$(0, A)$ is the set Invar(A) of flags invariant under A. One has the inclusions

$$\text{Invar}(A) \subset \text{Hess}(1, A) \subset \ldots \subset \text{Hess}(n-1, A) = \text{Flag}(n).$$

Notice that $\langle X \rangle_B \in \text{Hess}(p, A)$ if and only if

$$AX = XH_p, \tag{2.2}$$

where H_p is a matrix in *Hessenberg form of height* p, namely

$$H_p = \begin{pmatrix} h_{11} & \cdots & \cdots & \cdots & h_{1n} \\ \vdots & & & & \vdots \\ h_{p+1,1} & & & & \vdots \\ \vdots & \ddots & & & \vdots \\ 0 & \cdots & h_{n,n-p} & \cdots & h_{nn} \end{pmatrix}. \tag{2.3}$$

In other words, H_p is in Hessenberg form of height p provided $h_{ij} = 0$ whenever $i - j > p$. The vector space of all such matrices will be denoted $h_p(gl(n, \mathbb{C}))$, or more simply $\mathbf{H}_p(n)$.

We denote by $h_p(sl(n, \mathbb{C}))$ its intersection with the special linear Lie algebra. The set $\mathrm{Hess}(p, A)$ is a projective algebraic variety.

(II.2) Theorem: Let $A \in \mathrm{Reg}(\mathbf{g})$. For $1 \le p \le n-1$, $\mathrm{Hess}(p, A)$ is a compact, connected and smooth submanifold of G/B of complex dimension $p(2n-p-1)/2$. In particular, it is an irreducible variety.

The above result can be obtained from the local algebraic equations expressing the Hessenberg conditions (2.2) in the charts $\{ch(\sigma)\}$, $\sigma \in \Sigma(n)$. It is shown in [16,17] that if $A = diag(\lambda_1, \dots, \lambda_n)$ and $X = (x_{ij}) \in L^+(n)$, then $\langle \sigma X \rangle_B \in ch(\sigma)$ is an element in $\mathrm{Hess}(p, A)$ if and only if for $\alpha - \beta > p$,

$$(2.4) \qquad f_{\alpha\beta}(X) \doteq \Lambda(\alpha, \beta) x_{\alpha\beta} + \sum_{t=1}^{\alpha-\beta-1} \sum_{\alpha > \gamma_1 > \dots > \gamma_t > \beta} (-1)^t \Lambda(\gamma_t, \beta) x_{\alpha\gamma_1} \cdots x_{\gamma_t\beta} = 0,$$

where $\Lambda(\mu, \nu) \doteq (\lambda_{\sigma(\mu)} - \lambda_{\sigma(\nu)})$.

Let $G = \coprod_{\sigma \in \Sigma(n)} B\sigma B$ be the Bruhat decomposition of G, and let

$$G/B = \coprod_{\sigma \in \Sigma(n)} \langle B\sigma \rangle_B$$

be the induced Bruhat decomposition of G/B. We will write $B_\sigma \doteq \langle B\sigma \rangle_B$ and we observe now for later use that

$$(2.5) \qquad B_\sigma = \langle \sigma L^+_\sigma(n) \rangle_B$$

where

$$(2.6) \qquad L^+_\sigma(n) \doteq \{X = (x_{ij}) \in L^+(n) |\, x_{ij} = 0 \text{ if } \sigma(i) > \sigma(j)\}.$$

It is known (see e.g. [26]) that $B_\sigma \cong \mathbb{C}^{l(\sigma)}$, where $l(\sigma)$ is the length of σ–i.e., $card\{(i,j) |\, 1 \le j < i \le n \text{ and } \sigma(i) < \sigma(j)\}$.

(II.3) Theorem: Let $A \in \mathrm{Reg}(\mathbf{g})$, $\sigma \in \Sigma(n)$, $1 \le p \le n-1$. Then $B_\sigma \cap \mathrm{Hess}(p, A)$ is analytically isomorphic to $\mathbb{C}^{E_p(\sigma)}$, where

$$E_p(\sigma) \doteq card\{(i,j) |\, 1 \le i, j \le n,\ 1 \le i - j \le p,\ \sigma(i) < \sigma(j)\}.$$

The number $E_p(\sigma)$ is called the *p^{th} Eulerian dimension of σ*, for reasons that will be discussed in Section VII.

The decomposition of $\text{Hess}(p, A)$ into the disjoint union of the "cells" $B_\sigma \cap \text{Hess}(p, A)$ is not cellular, in that the boundary of a cell of dimension k can contain points of a cell of the same dimension. However, a recent theorem of A.H. Durfee [27] enables us to compute the Betti numbers from this decomposition, yielding the following result:

(II.4) Theorem: Let $A \in \text{Reg}(\mathbf{g})$, $1 \leq p \leq n-1$. Then

$$b_{2k+1}(\text{Hess}(p, A)) = 0,$$

$$b_{2k}(\text{Hess}(p, A)) = A_{n-1}(p, k+1)$$

where $b_\nu(\text{Hess}(p, A))$ is the ν^{th} Betti number of $\text{Hess}(p, A)$ and

$$A_{n-1}(p, k) \doteq card\{\sigma \in \Sigma(n) | \ E_p(\sigma) = k-1\}. \tag{2.7}$$

The numbers $\{A_{n-1}(p, k)\}$ are called *generalized Eulerian numbers of height p*. Again we refer to Section VII for more details.

Remark: The results described in this section specialize without essential changes to the case where $GL(n, \mathbb{C})$ is replaced by its subgroup $SL(n, \mathbb{C})$.

Remark: The existence of a decomposition of $\text{Hess}(p, A)$ into affine spaces also follows from a general theorem of Bialynicki-Birula [28]. This result shows that if any algebraic torus acts on a smooth projective variety with isolated fixed points, then the variety admits a natural decomposition into locally closed subvarieties each of which is isomorphic to a vector space and contains exactly one fixed point. In the case at hand, the centralizer of A in the unitary group $U(n)$ is an algebraic torus which acts naturally on $\text{Hess}(p, A)$. The fixed points of this action are precisely the $n!$ elements of $\text{Hess}(0, A)$, namely $\{\langle\sigma\rangle_B \mid \sigma \in \Sigma(n)\}$. The resulting decomposition coincides with the decomposition obtained by intersecting the Bruhat cells with $\text{Hess}(p, A)$.

III. Hessenberg Flags for a Semisimple Lie Group

The geometric approach to matrix eigenvalue problems extends to the case when the given matrix A satisfies the additional property of being, for instance, an element in the symplectic group $G = Sp(n, \mathbb{C})$ or its Lie algebra $\mathbf{g} = sp(n, \mathbb{C})$. (See e.g., [3].) In particular, the appropriate homogeneous space and QR-type factorization can be obtained from the Iwasawa decomposition of $Sp(n, \mathbb{C})$. This means that if KAN is such a decomposition, then $Q \in K$ and $R \in AN$. Explicitly, $Q \in Sp(n) \doteq U(n) \cap Sp(n, \mathbb{C})$ and

$$R \in \left\{ \begin{pmatrix} u & um \\ 0 & {}^t u^{-1} \end{pmatrix} \;\middle|\; u \in V(n),\ m \in S(n) \right\} \doteq B,$$

where $V(n)$ denotes the set of $n \times n$ nonsingular upper-triangular (complex) matrices, and $S(n)$ denotes the set of $n \times n$ symmetric matrices. (To make the factorization unique, one can require that the diagonal entries of u be positive real numbers.) One obtains then a symplectic QR-algorithm which can be interpreted as a linear-induced dynamical system on G/B, which can be identified with the so-called *Lagrange-flag manifold*

$$\mathrm{JFlag}(n) \doteq \{S_1 \subset \ldots \subset S_{2n-1} \in \mathrm{Flag}(2n) |\ S_n = S_n^{\perp} \text{ and } S_{n+i} = S_{n-i}^{\perp}\},$$

where $\perp$ denotes orthogonality with respect to the symplectic form

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

If $X \in Sp(n, \mathbb{C})$, then $\langle X \rangle_B$ is identified with the Lagrangian flag $(S_1, \ldots, S_{2n-1})$, where for $i \leq n$, S_i is the subspace spanned by the first i columns of X, while for $i > n$, S_i is the subspace spanned by the first n and last $i - n$ columns of X. Using the exponential map, the QR-algorithm for $Sp(n, \mathbb{C})$ extends to $sp(n, \mathbb{C})$. (See e.g., [3,18].)

An analogue of height-one Hessenberg form for matrices in $sp(n, \mathbb{C})$ was introduced by Byers [18] and referred to as *Hamiltonian-Hessenberg form.* A matrix H_1 in such form is of the type

$$H_1 = \begin{pmatrix} h & m \\ z & -{}^t h \end{pmatrix},$$

where $h \in \mathrm{H}_1(n)$, $m \in S(n)$ and $z = (z_{ij})$ is such that $z_{ij} = 0$ unless $i = j = n$.

It follows that initial reduction of A to Hamiltonian-Hessenberg form–i.e., choosing P such that $P^{-1}AP = H_1$–simply means that

$$\langle P \rangle_B \in \mathrm{JHess}(1, A) \doteq \{T_1 \subset \ldots \subset T_{2n-1} \in \mathrm{JFlag}(n) |\ AT_i \subset T_{i+1},\ \forall i\}.$$

Thus, we are led to investigate the topology of the varieties JHess(p, A) whose definition is the obvious one–namely,

$$\text{JHess}(p, A) \doteq \{T_1 \subset \dots T_{2n-1} \in \text{JFlag}(n) |\ AT_i \subset T_{i+p},\ \ \forall i\}.$$

Notice that the elements $\langle X \rangle_B$ in JHess(p, A) are those for which an equation of the type

$$AX = XH_p \tag{3.1}$$

holds, where

$$H_p = \begin{pmatrix} h & m \\ z & -{}^t h \end{pmatrix}, \tag{3.2}$$

with $h \in \mathbf{H}_p(n)$, $m \in S(n)$ and $z = (z_{ij})$ is such that $z_{ij} = 0$ unless $i + j \geq 2n + 1 - p$. We let $h_p(sp(n, \mathbb{C}))$ denote the vector space of such matrices H_p, which we refer to as being in *Hessenberg form of height p*.

The above discussion suggests that a QR-type algorithm can be defined for all connected semisimple Lie groups. For these groups, the Iwasawa decomposition holds and $B = AN$ is known to be a solvable Lie subgroup of G. Reduction of A to Hessenberg-type form, and consequently a good definition of p^{th} *Hessenberg variety*, depends on the existence of a good Lie-theoretic interpretation of the Hessenberg-type matrices H_p. Such an interpretation comes, in the opinion of the authors, from the root-space decomposition of the Lie algebra $\mathbf{g}$ of G.

In each of the cases $G = SL(n, \mathbb{C})$, $Sp(n, \mathbb{C})$ discussed above, a specific choice of Borel subgroup B was made, and then a nested sequence of subspaces of $\mathbf{g}$ was defined, namely

$$\mathbf{b} = h_0(\mathbf{g}) \subset h_1(\mathbf{g}) \subset \dots \subset h_N(\mathbf{g}) = \mathbf{g},$$

where $\mathbf{b}$ denotes the Borel subalgebra associated with B, and N is an appropriate positive integer. It is obvious that B could be replaced by gBg^{-1} for $g \in G$, in which case $h_p(\mathbf{g})$ must be replaced by Ad(g) $h_p(\mathbf{g})$. In other words, *any* Borel subgroup of G can be chosen as B, and this choice determines the sequence $\{h_p(\mathbf{g})\}$ of Hessenberg subspaces. When we wish to emphasize the dependence of $h_p(\mathbf{g})$ on B, we will use the symbol $h_p(\mathbf{b}, \mathbf{g})$.

We would like to extrapolate the examples of $sl(n, \mathbb{C})$ and $sp(n, \mathbb{C})$ considered above to obtain a general definition of $h_p(\mathbf{b}, \mathbf{g})$ where $\mathbf{g}$ is an arbitrary complex semisimple Lie algebra, and $\mathbf{b}$ is any Borel subalgebra of $\mathbf{g}$. To do this, we examine more closely the mathematical relationship between $\mathbf{b}$ and $h_p(\mathbf{b}, \mathbf{g})$ in the two examples. For the specific choices of B (and hence $\mathbf{b}$) in the cases of $sl(n, \mathbb{C})$ and $sp(n, \mathbb{C})$, we choose the following Cartan subalgebra's (CSA's) which are contained in $\mathbf{b}$:

$$\mathbf{s} = \begin{cases} \sum_{i=1}^{n-1} \mathbb{C}(E_{ii} - E_{i+1,i+1}) & \text{for } sl(n, \mathbb{C}) \\ \sum_{i=1}^{n} \mathbb{C}(E_{ii} - E_{n+i,n+i}) & \text{for } sp(n, \mathbb{C}) \end{cases}$$

where E_{ij} denotes the square matrix of the appropriate dimension whose entries are all 0 except for the ij^{th}, which is 1. These choices yield the standard root-space decompositions for $sl(n, \mathbb{C})$ and $sp(n, \mathbb{C})$. In particular, for the root systems

(3.3) $\quad A_{n-1}: \quad \{e_j - e_i |\ 1 \le i, j \le n, \quad i \ne j\}$

(3.4) $\quad C_n: \quad \{e_j - e_i |\ 1 \le i, j \le n, \quad i \ne j\} \cup \{\pm(e_j + e_i) |\ 1 \le j \le i \le n\}$

thereby arising, we have the following well-known data for the corresponding root-spaces and heights of the roots with respect to the unique base with the property that $\mathbf{b}$ is the sum of $\mathbf{s}$ and the root-spaces for the positive roots: For A_{n-1}, the root $e_j - e_i$ has root-space $\mathbb{C}E_{ji}$ and height $i - j, \quad i \ne j$. For C_n, the root $e_j - e_i$ has root-space $\mathbb{C}(E_{ji} - E_{n+i,n+j})$ and height $i - j, \quad i \ne j$; the root $e_j + e_i$ has root-space $\mathbb{C}(E_{j,n+i} + E_{i,n+j})$ and height $2n + 1 - (i + j), \quad j \le i$; the root $-e_j - e_i$ has root-space $\mathbb{C}(E_{n+j,i} + E_{n+i,j})$ and height $(i + j) - 2n - 1, \quad j \le i$.

It follows immediately from (2.3) and (3.2) that for both $sl(n, \mathbb{C})$ and $sp(n, \mathbb{C})$ with the standard choices of Borel subalgebra, H_p is in Hessenberg form of height p if and only if

$$H_p \quad \in \quad \mathbf{s} + \sum_{\alpha \in \Phi,\ h(\alpha) \ge -p} \mathbf{g}^{\alpha}. \qquad \text{(direct sum)}$$

Here, Φ denotes the set of roots of $\mathbf{g}$ relative to $\mathbf{s}$; $h(\alpha)$ denotes the height of the root α; and $\mathbf{g}^{\alpha}$ is the root-space of α.

This motivates the following

(III.1) Definition: Let G be a connected complex semisimple Lie group with Lie algebra $\mathbf{g}$. Let B be a Borel subgroup of G, and let $\mathbf{b}$ be the corresponding Borel subalgebra. Let $\mathbf{s}$ be any CSA contained in $\mathbf{b}$, and let $\mathbf{g} = \mathbf{s} + \sum_{\alpha\in\Phi} \mathbf{g}^\alpha$ be the root-space decomposition for $\mathbf{g}$ relative to $\mathbf{s}$. The p^{th} *Hessenberg subspace of* $\mathbf{g}$ *relative to* $\mathbf{b}$ is defined as

$$(3.5) \qquad h_p(\mathbf{b},\mathbf{g}) \doteq \mathbf{s} + \sum_{\alpha\in\Phi,\ h(\alpha)\geq -p} \mathbf{g}^\alpha. \qquad \text{(direct sum)}$$

where $h(\alpha)$ denotes the height of the root α relative to the unique base Δ for which $\mathbf{b} = h_0(\mathbf{b},\mathbf{g})$. We will often suppress the dependence on $\mathbf{b}$ and use the notation $h_p(\mathbf{g})$.

Remark: It is easy to see that $h_p(\mathbf{b},\mathbf{g})$ is well-defined—i.e., independent of the choice of the CSA $\mathbf{s}$. Let $h_p(\mathbf{s},\mathbf{b},\mathbf{g})$ denote the righthand side of (3.5). It follows from the choice of Δ that $\mathbf{b} = h_0(\mathbf{s},\mathbf{b},\mathbf{g})$, regardless of the choice of $\mathbf{s}$. It follows trivially from this that $h_p(\mathbf{s},\mathbf{b},\mathbf{g})$ is ad($\mathbf{b}$)-invariant, and hence Ad(B)-invariant. Also, if η is any automorphism of $\mathbf{g}$, then $h_p(\eta\mathbf{s},\eta\mathbf{b},\mathbf{g}) = \eta h_p(\mathbf{s},\mathbf{b},\mathbf{g})$. Let $\mathbf{s}$ and $\mathbf{s}'$ be a pair of CSA's in $\mathbf{b}$. Then there exists η in the subgroup of Aut($\mathbf{g}$) generated by Ad(B) such that $\eta\mathbf{s} = \mathbf{s}'$. Then $h_p(\mathbf{s}',\mathbf{b},\mathbf{g}) = h_p(\eta\mathbf{s},\eta\mathbf{b},\mathbf{g}) = \eta h_p(\mathbf{s},\mathbf{b},\mathbf{g}) = h_p(\mathbf{s},\mathbf{b},\mathbf{g})$.

With this definition available, one can read equations (2.2) and (3.1) as adjoint actions, thereby obtaining the next

(III.2) Definition: Let G be a connected complex semisimple Lie group with Lie algebra $\mathbf{g}$, and let B be a Borel subgroup with Borel subalgebra $\mathbf{b}$. Let $A \in \mathbf{g}$. The p^{th} *Hessenberg variety of* A is

$$(3.6) \qquad \mathrm{Hess}(p,A) \doteq \{\langle X\rangle_B \in G/B \mid (\mathrm{Ad}X^{-1})(A) \in h_p(\mathbf{b},\mathbf{g})\}.$$

Since $h_p(\mathbf{b},\mathbf{g})$ is Ad(B)-invariant, Hess(p,A) is well-defined.

Remark: The flag manifold G/B is often identified with Borel($\mathbf{g}$), the variety of all Borel subalgebras of $\mathbf{g}$. The flag $\langle X\rangle_B$ is identified with Ad(X)$\mathbf{b}$. The condition that $\langle X\rangle_B \in$ Hess($0,A$) is equivalent to the condition that $A \in$ Ad(X)$\mathbf{b}$. Thus, Hess($0,A$) is identified with the subvariety of Borel($\mathbf{g}$) consisting of those Borel subalgebras which contain A. More generally, Hess(p,A) can be identified with a subvariety of Borel($\mathbf{g}$) as follows: The condition that $\langle X \rangle_B \in \mathrm{Hess}(p, A)$–i.e., $\mathrm{Ad}(X^{-1})A \in h_p(\mathbf{b}, \mathbf{g})$–is equivalent to the condition that $A \in \mathrm{Ad}(X)h_p(\mathbf{b}, \mathbf{g}) = h_p(\mathrm{Ad}(X)\mathbf{b}, \mathbf{g})$. Thus, *$Hess(p, A)$ is identified with the subvariety of $Borel(\mathbf{g})$ consisting of those Borel subalgebras $\mathbf{b}'$ for which the p^{th} Hessenberg subspace $h_p(\mathbf{b}', \mathbf{g})$ contains A.*

In this section, we have demonstrated how Hessenberg elements H_p and Hessenberg varieties $\mathrm{Hess}(p, A)$ can be defined for an arbitrary connected complex semisimple Lie group using the root space decomposition and height function. In the case $G = SL(n, \mathbb{C})$, the Betti numbers of $\mathrm{Hess}(p, A)$ (for a regular element A) are described in terms of the p^{th} Eulerian dimension $E_p(\sigma)$ of a permutation $\sigma \in \Sigma(n)$, the Weyl group of $SL(n, \mathbb{C})$. Consequently, one might hope for a root-system type interpretation for $E_p(\sigma)$. This is indeed the case:

(III.3) Proposition: For $\sigma \in \Sigma(n)$,

$$E_p(\sigma) = card\{\alpha \in \Phi^+ \mid h(\alpha) \leq p \text{ and } \sigma(\alpha) \in \Phi^-\}.$$

Proof: Let $\alpha \doteq e_j - e_i \in \Phi^+$ be such that $h(\alpha) = i - j \leq p$. Then $\sigma(\alpha) = e_{\sigma(j)} - e_{\sigma(i)} \in \Phi^-$ if and only if $\sigma(i) < \sigma(j)$. Conversely, if (i, j) is a pair with $1 \leq i - j \leq p$ and $\sigma(i) < \sigma(j)$, then $e_j - e_i \in \Phi^+$, $h(e_j - e_i) \leq p$ and $\sigma(e_j - e_i) \in \Phi^-$. ■

IV. C_n-type Hessenberg Varieties

In this section, $\mathbf{g} \doteq sp(n, \mathbb{C})$ and $G \doteq Sp(n, \mathbb{C})$. We make the standard choice of Borel subgroup

$$B \doteq \left\{ \begin{pmatrix} u & us \\ 0 & {}^t u^{-1} \end{pmatrix} \,\middle|\, u \in V(n), s \in S(n) \right\}.$$

B stabilizes the Lagrangian flag p_0 defined by

$$sp\{e_1\} \subset \ldots \subset sp\{e_1, \ldots, e_n\} \subset sp\{e_1, \ldots, e_n, e_{2n}\} \subset \ldots \subset sp\{e_1, \ldots, e_n, e_{2n}, \ldots, e_{n+2}\},$$

and can be written as

$$B = \Omega V(2n) \Omega \cap G.$$

The (C_n-type) p^{th} Hessenberg variety $\mathrm{JHess}(p, A)$ for $A \in \mathbf{g}$ can be written

$$\mathrm{JHess}(p, A) = \mathrm{JFlag}(n) \cap \mathrm{Hess}(p, A),$$

a projective algebraic variety. From now on we make the assumption that $A \in \text{Reg}(\mathbf{g}) \cap \mathbf{s}$, the set of regular elements in the Cartan subalgebra $\mathbf{s}$ chosen in Section III, namely

$$A = \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix}, \qquad \Lambda = diag(\lambda_1, \ldots, \lambda_n), \qquad \lambda_i \pm \lambda_j \neq 0 \text{ if } i \neq j.$$

If $g = [g_1, \ldots, g_{2n}] \in G$, then the flag $\eta(\langle g \rangle_B)$ corresponding to $\langle g \rangle_B \in G/B$ under the isomorphism $\eta : G/B \to \text{JFlag}(n)$, $\eta(\langle g \rangle_B) \doteq g p_0$ is

$$sp\{g_1\} \subset \ldots \subset sp\{g_1, \ldots, g_n\} \subset sp\{g_1, \ldots, g_n, g_{2n}\} \subset sp\{g_1, \ldots, g_n, g_{2n}, \ldots, g_{n+2}\}.$$

On the other hand, if $a = [a_1, \ldots, a_{2n}] \in GL(2n, \mathbb{C})$, then the flag corresponding to $\langle a \rangle_V \in GL(2n, \mathbb{C})/V(2n)$, say $\delta(\langle a \rangle_V)$, is $sp\{a_1\} \subset \ldots \subset sp\{a_1, \ldots, a_{2n-1}\}$. Thus, if $j : G/B \to GL(2n, \mathbb{C})/V(2n)$ denotes the injection $j(\langle g \rangle_B) = \langle g\Omega \rangle_V$, then $\delta \circ j = \eta$. Therefore, $\langle Y \rangle_B \in \text{JHess}(p, A)$ if and only if $A(Y\Omega) = (Y\Omega)H_p$, where $H_p \in \mathbf{H}_p(2n)$. In other words,

$$AY = Y\bar{H}_p. \tag{4.1}$$

(Overbar denotes conjugation by Ω.) Observe that $\bar{H}_p \in h_p(sp(n, \mathbb{C}), \mathbf{b})$, where $\mathbf{b}$ is the Borel subalgebra associated with the Borel subgroup B.

We want to derive (local) algebraic equations from (4.1). A covering system of analytic charts for G/B can be obtained as follows: Let Z_n denote the n-fold Cartesian product of the two-element group $\{\pm 1\}$. To each $\epsilon \doteq (\epsilon_1, \ldots, \epsilon_n) \in Z_n$, we associate two $n \times n$ matrices

$$\begin{aligned} S_+(\epsilon) &\doteq diag(\delta_{1,\epsilon_1}, \ldots, \delta_{1,\epsilon_n}) \\ S_-(\epsilon) &\doteq diag(\delta_{-1,\epsilon_1}, \ldots, \delta_{-1,\epsilon_n}) \end{aligned} \tag{4.2}$$

and we form

$$\sigma_\epsilon \doteq \begin{pmatrix} S_+(\epsilon) & S_-(\epsilon) \\ -S_-(\epsilon) & S_+(\epsilon) \end{pmatrix}, \qquad \sigma_\epsilon^+ \doteq \begin{pmatrix} S_+(\epsilon) & S_-(\epsilon) \\ S_-(\epsilon) & S_+(\epsilon) \end{pmatrix}. \tag{4.3}$$

It is immediate to check that $\sigma_\epsilon \in G$, while $\sigma_\epsilon^+ \notin G$. Next, to each $\tau \in \Sigma(n)$, we associate

$$\tau^\natural \doteq \begin{pmatrix} \tau & 0 \\ 0 & \tau \end{pmatrix}, \tag{4.4}$$

and we consider the sets $W_G \doteq \{\sigma_\epsilon \tau^\natural \mid \epsilon \in Z_n,\ \tau \in \Sigma(n)\} \subset G$ and

$$W_G^+ \doteq \{\sigma_\epsilon^+ \tau^\natural \,\Big|\, \epsilon \in Z_n, \quad \tau \in \Sigma(n)\} \subset \Sigma(2n). \tag{4.5}$$

The set W_G^+ endowed with the group multiplication law

$$(\sigma_\epsilon^+ \tau^\natural)(\sigma_\eta^+ \nu^\natural) \doteq \sigma^+_{\epsilon\tau^{-1}(\eta)}(\tau\nu)^\natural, \qquad \tau^{-1}(\eta) \doteq (\eta_{\tau^{-1}(1)}, \ldots, \eta_{\tau^{-1}(n)}),$$

is isomorphic to the Weyl group of G, namely the semidirect product of $\Sigma(n)$ with Z_n. Observe that the above multiplication does not agree with the matrix multiplication in $\Sigma(n)$.

Finally, put

$$L \doteq \overline{L^+(2n)} \cap G = \left\{ \begin{pmatrix} l & 0 \\ {}^t l^{-1} g & {}^t l^{-1} \end{pmatrix} \,\Big|\, l \in L^+(n),\ g \in S(n) \right\}, \tag{4.6}$$

a closed Lie subgroup of G of dimension n^2. Then, the sets

$$ch(\pi) \doteq \left\{ \langle \pi Z \rangle_B \,\Big|\, Z \in L \right\}, \qquad \pi \in W_G,$$

give rise to a system of analytic charts for G/B, where L is topologically identified with $\mathbb{C}^{n^2}$.

Let now $Y = \pi \bar{X},\ \pi \in W_G,\ \bar{X} \in L$. Then (4.1) becomes

$$(\overline{\pi^{-1} A \pi}) X = X H_p, \qquad H_p \in \overline{h_p(C_n)} \subset \mathrm{H}_p(2n). \tag{4.7}$$

Now, if $\pi = \sigma_\epsilon \tau^\natural$, then

$$(\overline{\pi^{-1} A \pi}) = diag(\mu_1, \ldots, \mu_{2n}) \doteq \mu, \tag{4.8}$$

where $\mu_i = \epsilon_{\tau(i)} \lambda_{\tau(i)}$ for $i \leq n$, and $\mu_i = -\epsilon_{\tau(d(i))} \lambda_{\tau(d(i))}$ for $i > n$, with

$$d(\alpha) \doteq 2n + 1 - \alpha. \tag{4.9}$$

Then the following result is an immediate consequence of (2.4):

(IV.1) Proposition: Let $X \in L^+(2n)$ be such that $\bar{X} \in L$ and let $\pi \in W_G$. Then $\langle \pi \bar{X} \rangle_B \in \mathrm{JHess}(p, A)$ if and only if $f^{(\mu)}_{\alpha\beta}(X) = 0$, $\forall\, \alpha - \beta > p$.

It turns out that in the set of equations $\{f^{(\mu)}_{\alpha\beta}(X) = 0, \quad \alpha - \beta > p\}$, there is great redundancy, due to the fact that X satisfies $\bar{X} \in L$. For this reason, we now analyze the equations that such a condition yields.

Let $X \doteq \begin{pmatrix} X_1 & 0 \\ Z & X_2 \end{pmatrix} \in L^+(2n)$, so that $X_1, X_2 \in L^+(n)$ and Z is an arbitrary $n \times n$ matrix. Then

$$^t\bar{X} J \bar{X} = \begin{pmatrix} {}^tX_1 \theta Z - {}^tZ \theta X_1 & {}^tX_1 \theta X_2 \theta \\ -\theta\, {}^tX_2 \theta X_1 & 0 \end{pmatrix}.$$

Therefore, $\bar{X} \in L$ if and only if

$$(J1) \qquad {}^tX_1 \theta X_2 \theta = I$$

$$(J2) \qquad {}^tX_1 \theta Z \in S(n).$$

Writing $X = (x_{ij})$ and computing the above products, one gets

(IV.2) Proposition: The matrix $X = (x_{ij}) \in L^+(2n)$ is such that $\bar{X} \in L$ if and only if for $1 \le j < i \le n$

$$(J1)(i,j) \qquad x_{ij} + x_{d(j)d(i)} + \sum_{t=1}^{i-j-1} x_{i-t,j} x_{d(i-t)d(i)} = 0$$

$$(J2)(i,j) \qquad x_{d(i)j} + \sum_{t=1}^{n-i} x_{i+t,i} x_{d(i+t)j} = x_{d(j)i} + \sum_{t=1}^{n-j} x_{j+t,j} x_{d(j+t)i}.$$

In the next two lemmas, we give equivalent formulations of conditions (J1) and (J2). For this purpose, we introduce the following notation:

$$(4.10) \qquad g_{\alpha\beta}(X) \doteq x_{\alpha\beta} + \sum_{t=1}^{\alpha-\beta-1} \sum_{\alpha > \Gamma_t > \beta} (-1)^t x_{\alpha\gamma_1} \cdots x_{\gamma_t \beta},$$

where $X = (x_{ij}) \in L^+(2n), \quad 1 \le \beta < \alpha \le 2n.$

(IV.3) Lemma: Let $X = (x_{ij}) \in L^+(2n)$. Then (J1) holds if and only if $x_{ij} = -g_{d(j)d(i)}(X)$, $1 \leq j < i \leq n$, which holds if and only if $x_{d(j)d(i)} = -g_{ij}(X)$, $1 \leq j < i \leq n$.

Proof: We only prove the first equivalence. The proof of the second is analogous.

Assume first that (J1) holds. We prove by induction on $i-j$ that $x_{ij} = -g_{d(j)d(i)}(X)$. If $i-j=1$, this is (J1)(i,i-1). Assume now the result for all pairs (μ,ν) with $1 \leq \mu-\nu \leq s$. Fix $i > s+1$. By (J1)(i,i-s-1):

$$x_{i,i-s-1} = -x_{d(i-s-1)d(i)} - \sum_{u=1}^{s} x_{i-u,i-s-1}x_{d(i-u)d(i)}.$$

The pair $(i-u, i-s-1)$ is such that $(i-u)-(i-s-1) \leq s$ for all $u = 1,\ldots,s$. Hence, by induction

$$x_{i,i-s-1} = -x_{d(i-s-1)d(i)} - \sum_{u=1}^{s} g_{d(i-s-1)d(i-u)}x_{d(i-u)d(i)} = -g_{d(i-s-1)d(i)}(X).$$

Conversely, assume $x_{ij} = -g_{d(j)d(i)}(X)$ for all pairs (i,j) with $1 \leq j < i \leq n$. Then the above string of equalities gives (J1)(i,j) for the same set of pairs. ■

(IV.4) Lemma: Let $X = (x_{ij}) \in L^+(2n)$ and assume that (J1) holds. Then (J2) holds if and only if $x_{d(i)j} = g_{d(j)i}(X)$, $1 \leq j < i \leq n$, which holds if and only if $x_{d(j)i} = g_{d(i)j}(X)$, $1 \leq j < i \leq n$. Moreover, if both (J1) and (J2) hold, then

$$x_{d(i)i} = g_{d(i)i}(X), \qquad 1 \leq i \leq n, \tag{4.11}$$

which is *not* a condition on $x_{d(i)i}$.

Proof: Again we only prove the first equivalence. Assume first that (J2) holds. We will use induction on $d(i)-j$. If $d(i)-j=2$, then $(i,j)=(n,n-1)$ and (J2)(n,n-1) gives

$$x_{d(n),n-1} = x_{n+2,n} + x_{n,n-1}x_{n+1,n} = x_{n+2,n} + x_{n+1,n}(-x_{n+2,n+1}),$$

where $x_{n,n-1} = -x_{n+2,n+1}$ is simply (J1)(n,n-1).

Fix now (i,j) with $1 \leq j < i \leq n$ and assume that for all pairs (μ,ν) such that $1 \leq \nu < \mu \leq n$ and $d(\mu) - \nu < d(i) - j$ we have $x_{d(\mu)\nu} = g_{d(\nu)\mu}(X)$. Thus, from (J2)(i,j), Lemma (IV.3) and the induction assumption,

$$\begin{aligned} x_{d(i)j} &= x_{d(j)i} + \sum_{t=1}^{n-j} x_{j+t,j} x_{d(j+t)i} - \sum_{t=1}^{n-i} x_{i+t,i} x_{d(i+t)j} \\ &= x_{d(j)i} - \sum_{t=1}^{n-j} g_{d(j)d(j+t)}(X) x_{d(j+t)i} - \sum_{t=1}^{n-i} g_{d(j),i+t}(X) x_{i+t,i} \\ &= g_{d(j)i}(X). \end{aligned}$$

Conversely, if $x_{d(i)j} = g_{d(j)i}(X)$ for all pairs (i,j) with $1 \leq j < i \leq n$, then decomposing $g_{d(j)i}(X)$ as above, we derive (J2)(i,j) for the same set of pairs.

Assume finally that both (J1) and (J2) hold. Then

$$\begin{aligned} \sum_{t=1}^{d(i)-i-1} \sum_{d(i)>\Gamma_t>i} (-1)^t x_{d(i)\gamma_1} \cdots x_{\gamma_t i} &= -\sum_{t=1}^{n-i} g_{d(i)d(i+t)}(X) x_{d(i+t)i} - \sum_{t=1}^{n-i} g_{d(i),i+t}(X) x_{i+t,i} \\ &= 0, \end{aligned}$$

by Lemma (IV.3) and the first part of this lemma. ■

(IV.5) Proposition: Let $X \in L^+(2n)$ be such that $\bar{X} \in L$. Then

(4.12) $$f_{ij}^{(\mu)}(X) = -f_{d(j)d(i)}^{(\mu)}(X), \qquad 1 \leq j < i \leq n$$

(4.13) $$f_{d(i)j}^{(\mu)}(X) = +f_{d(j)i}^{(\mu)}(X), \qquad 1 \leq j < i \leq n.$$

Proof: First of all we observe that, by virtue of (4.8), the coefficients $M(\alpha,\beta) = (\mu_\alpha - \mu_\beta)$ satisfy $M(\alpha,\beta) = M(d(\beta),d(\alpha))$.

We first prove (4.12) using induction on $i-j$. If $i-j=1$, then (J1)(i,i-1) immediately gives the desired result. Fix now (i,j) and assume that (4.12) is true for all pairs (μ,ν)

with $1 \leq \nu < \mu \leq n$ and $\mu - \nu < i - j$. By Lemma (IV.3) and the induction assumption,

$$\begin{aligned} f_{ij}^{(\mu)}(X) &= M(i,j)x_{ij} - \sum_{s=1}^{i-j-1} x_{i,i-s} f_{i-s,j}^{(\mu)}(X) \\ &= -M(d(j), d(i)) g_{d(j)d(i)}(X) - \sum_{s=1}^{i-j-1} g_{d(i-s)d(i)}(X) f_{d(j)d(i-s)}^{(\mu)}(X). \end{aligned}$$

It should be clear that this last expression can be written as

$$-\left\{ M(d(j), d(i)) x_{d(j)d(i)} + \sum_{t=1}^{i-j-1} \sum_{d(j) > \Gamma_t > d(i)} (-1)^t c(i,j,\Gamma_t) x_{d(j)\gamma_1} \cdots x_{\gamma_t d(i)} \right\}.$$

Thus, all we have to show is that $c(i,j,\Gamma_t) = M(\gamma_t, d(i))$. In order to do so, we look at the coefficients corresponding to the various monomials, according to their degrees.

(1) Monomials of second degree. For any fixed $\Gamma_1 = (\gamma_1)$, we have

$$c(i,j,\Gamma_1) = M(d(j), d(i)) - M(d(j), \gamma_1) = M(\gamma_1, d(i)).$$

(2) Monomials of third degree. For any fixed $\Gamma_2 = (\gamma_1, \gamma_2)$, we have

$$-c(i,j,\Gamma_2) = -M(d(j), d(i)) + M(d(j), \gamma_1) + M(\gamma_1, \gamma_2) = -M(\gamma_2, d(i)).$$

(3) Monomials of degree > 3. For any fixed $\Gamma_t = (\gamma_1, \ldots, \gamma_t)$, $\quad t > 2$, we have:

$$\begin{aligned} -c(i,j,\Gamma_t) &= -M(d(j), d(i)) + M(d(j), \gamma_1) + M(\gamma_{t-1}, \gamma_t) + \sum_{s=2}^{t-1} M(\gamma_{s-1}, \gamma_s) \\ &= M(d(i), \gamma_1) + M(\gamma_{t-1}, \gamma_t) + M(\gamma_1, \gamma_{t-1}) \\ &= M(d(i), \gamma_t), \end{aligned}$$

as desired. This concludes the proof of (4.12).

We now prove (4.13) using induction on $d(i) - j$. If $d(i) - j = 2$, then $i = n$ and $j = n - 1$. By Lemmas (IV.3) and (IV.4), we have $x_{d(n-1)d(n)} = -g_{n,n-1}(X)$. I.e.,

$x_{n+2,n+1} = -x_{n,n-1}$, and $x_{d(n-1),n} = -g_{d(n),n-1}(X)$, which implies that $x_{n+1,n-1} = x_{n+2,n} + x_{n+1,n}x_{n,n-1}$. Hence,

$$\begin{aligned} f^{(\mu)}_{d(n),n-1}(X) &= M(n+1,n-1)x_{n+1,n-1} - M(n,n-1)x_{n+1,n}x_{n,n-1} \\ &= M(n+1,n-1)x_{n+2,n} + \{M(n+1,n-1) - M(n,n-1)\}x_{n+1,n}x_{n,n-1} \\ &= M(n+2,n)x_{n+2,n} - M(n+1,n)x_{n+2,n+1}x_{n+1,n}. \end{aligned}$$

Assume now (4.13) for all pairs (μ,ν) with $1 \le \nu < \mu \le n$ and $d(\mu) - \nu < d(i) - j$. Then

$$f^{(\mu)}_{d(i)j}(X) = M(d(i),j)x_{d(i)j} - \Big(\sum_{s=1}^{n-i} + \sum_{s=n-i+1}^{2n-i-j}\Big)\big(x_{d(i),d(i)-s}f^{(\mu)}_{d(i)-s,j}(X)\big).$$

Now, if $1 \le s \le n-i$, then $1 \le j < i+s \le n$ and so (i) the induction hypothesis implies that $f^{(\mu)}_{d(i+s)j}(X) = f^{(\mu)}_{d(j),i+s}(X)$; (ii) lemma (IV.3) implies that $x_{d(i)d(i+s)} = -g_{i+s,i}(X)$. Next, if $n-i+1 \le s \le 2n-i-j$, then $j < d(i+s) \le n$ and so (iii) (4.12) implies that $f^{(\mu)}_{d(i+s)j}(X) = -f^{(\mu)}_{d(j),i+s}(X)$; (iv) lemma (IV.4) implies that $x_{d(i)d(i+s)} = g_{i+s,i}(X)$. Observe that in (iv), we have used (4.11). From (i), (ii), (iii) and (iv), we see that for $s = 1,\ldots,2n-i-j$:

$$x_{d(i)d(i+s)}f^{(\mu)}_{d(i+s),j}(X) = -f^{(\mu)}_{d(j),i+s}(X)g_{i+s,i}(X).$$

Therefore, using once more Lemma (IV.4),

$$\begin{aligned} f^{(\mu)}_{d(i)j}(X) &= M(d(i),j)x_{d(i)j} + \sum_{s=1}^{2n-i-j} f^{(\mu)}_{d(j),i+s}(X)g_{i+s,i}(X) \\ &= M(d(j),i)g_{d(j)i}(X) + \sum_{s=1}^{2n-i-j} f^{(\mu)}_{d(j),i+s}(X)g_{i+s,i}(X). \end{aligned}$$

At this point one concludes the result via a term by term analysis which is completely analogous to that carried out for the proof of (4.12). ■

From Lemmas (IV.3) and (IV.4), we deduce that each $X \in L^+(2n)$ such that $\bar{X} \in L$ has free entries $x_{\alpha\beta}$, where

$$(4.14) \qquad \begin{aligned} (\alpha,\beta) \in I^J_n &\doteq \Big\{(i,j),\ (d(i),k)\ \Big|\ 1 \le j < i \le n, \quad 1 \le k \le i \le n\Big\} \\ &= \Big\{(i,j)\ \Big|\ 1 \le j < i \le 2n, \quad i+j \le 2n+1\Big\}. \end{aligned}$$

Put now

$$(4.15) \qquad I^J_{n,p} \doteq \left\{ (i,j) \in I^J_n \;\middle|\; i-j>p \right\}.$$

An easy calculation shows that the cardinality of $I^J_{n,p}$ is $n^2 - np + [p^2/4]$, where $[p^2/4]$ denotes the largest integer $\leq p^2/4$.

(IV.6) Corollary: Let $X \in L^+(2n)$ be such that $\bar{X} \in L$, and let $\pi \in W_G$. Then $\langle \pi \bar{X} \rangle_B \in \text{JHess}(p, A)$ if and only if $f^{(\mu)}_{\alpha\beta}(X) = 0, \quad \forall (\alpha, \beta) \in I^J_{n,p}$.

(IV.7) Theorem: If $A \in \text{Reg}(\mathbf{g})$, then for $p \geq 1$, $\text{JHess}(p, A)$ is a smooth submanifold of $\text{JFlag}(n)$ of dimension $np - [p^2/4]$.

Proof: Analogous to the proof of [17, Theorem (III.4)].

(IV.8) Lemma: For $p \geq 1$ and $\pi \in W_G$, $ch(\pi) \cap \text{JHess}(p, A)$ is contractible.

Proof: Let $X = (x_{\mu\nu}) \in L^+(2n)$ be such that $\langle \pi \bar{X} \rangle_B \in \text{JHess}(p, A)$. This is equivalent to saying that (i) $x_{d(j)d(i)} = -g_{ij}(X), \quad (i,j) \in I^J_n, \ 1 \leq j < i \leq n$; (ii) $x_{d(j)i} = g_{d(i)j}(X), \quad (i,j) \in I^J_n, \ 1 \leq j < i \leq n$; (iii) $f^{(\mu)}_{\alpha\beta}(X) = 0, \quad (\alpha, \beta) \in I^J_{n,p}$. For $u \in [0,1]$, put

$$y_{ij}(u, X) \doteq u^{i-j} x_{ij}, \qquad Y(u, X) \doteq (y_{ij}(u, X)).$$

Then $Y(u, X) \in L^+(2n)$ for $u \in [0,1]$, and $Y(0, X) = I$. Moreover, for $1 \leq j < i \leq n$ and $(\alpha, \beta) \in I^J_{n,p}$, we have

$$-g_{ij}(Y(u,X)) = -u^{i-j} g_{ij}(X) = u^{d(j)-d(i)} x_{d(j)d(i)} = y_{d(j)d(i)}(u, X)$$

$$g_{d(i)j}(Y(u,X)) = y_{d(j)i}(u, X)$$

$$f^{(\mu)}_{\alpha\beta}(Y(u,X)) = u^{\alpha-\beta} f^{(\mu)}_{\alpha\beta}(X) = 0.$$

Therefore, $\langle \pi \overline{Y(u,X)} \rangle_B \in \text{JHess}(p, A)$ for all $u \in [0,1]$. Thus, the continuous map $H : [0,1] \times ch(\pi) \cap \text{JHess}(p, A) \to ch(\pi) \cap \text{JHess}(p, A)$ given by $H(u, \langle \pi \bar{X} \rangle_B) \doteq \langle \pi \overline{Y(u,X)} \rangle_B$ is a homotopy between the constant map $H(0, \cdot)$ and the identity map $H(1, \cdot)$ on $ch(\pi) \cap \text{JHess}(p, A)$. ■

(IV.9) Corollary: For $p \geq 1$ and $\pi \in W_G$, $ch(\pi) \cap \text{JHess}(p, A)$ is connected.

(IV.10) Theorem: If $A \in \text{Reg}(\mathfrak{g})$, then for $p \geq 1$, JHess(p, A) is connected.

Proof: By (IV.9), it is enough to show that for any $\pi, \sigma \in W_G$, there exist $w_1, \ldots, w_m \in W_G$ such that

$$(4.16) \qquad \begin{aligned} &\pi w_1 \cdots w_m = \sigma \qquad \text{(matrix product in } G) \\ &ch(\pi w_1 \cdots w_j) \cap ch(\pi w_1 \cdots w_{j-1}) \cap \text{JHess}(p, A) \neq \emptyset, \quad j = 1, \ldots, m. \end{aligned}$$

On the other hand, since JHess$(1, A) \subset$ JHess(p, A) for all $p > 1$, it is enough to show (4.16) for $p = 1$. Moreover, every element in W_G can be written as a finite product (in G) of elements in $\{\sigma_{\epsilon(n)}, \ \tau_i^\natural \mid i = 1, \ldots, n-1\}$, where $\epsilon(n) = (1, \ldots, 1, -1)$ and $\tau_i \in \Sigma(n)$ is the adjacent transposition which interchanges i and $i+1$. Therefore, it is enough to show that for all $\pi \in W_G$,

$$(4.17) \qquad ch(\pi) \cap ch(\pi\sigma_{\epsilon(n)}) \cap \text{JHess}(1, A) \neq \emptyset$$

$$(4.18) \qquad ch(\pi) \cap ch(\pi\tau_i^\natural) \cap \text{JHess}(1, A) \neq \emptyset, \qquad i = 1, \ldots, n-1.$$

We prove (4.17) first. Let $X = (x_{ij}) \in L^+(2n)$ be given by

$$x_{ij} = \begin{cases} 1 & \text{if } (i,j) = (n+1, n) \\ 0 & \text{if } (i,j) \neq (n+1, n), \quad 1 \leq j < i \leq 2n. \end{cases}$$

It is immediate to check that $\langle \pi\bar{X} \rangle_B \in$ JHess$(1, A)$ for all $\pi \in W_G$. The matrix $\sigma_{\epsilon(n)}\bar{X}$ differs from $\bar{X}$ only at rows n and $2n$, and these two rows are, respectively, ${}^t(e_n + e_{2n})$ and ${}^t(-e_n)$.

Define now a $2n \times 2n$ matrix $b(\epsilon(n))$ by stipulating that its j^{th} column $b(\epsilon(n))_j$ is given by

$$b(\epsilon(n))_j = \begin{cases} e_{2n} - e_n & \text{if } j = 2n \\ e_j & \text{if } j \neq 2n. \end{cases}$$

Clearly, $b(\epsilon(n)) \in B$. Moreover, the symplectic matrix $Z = (z_{ij}) \doteq \sigma_{\epsilon(n)}\bar{X}b(\epsilon(n))$ differs from $\sigma_{\epsilon(n)}\bar{X}$ only at column $2n$. However, $z_{i,2n} = 0$ for $1 \leq i \leq n$ and $z_{2n,2n} = 1$. Thus, Z is of the form $\bar{Y}$ for some $Y \in L^+(2n)$. Finally,

$$\langle \pi\bar{X} \rangle_B = \langle \pi\bar{X}b(\epsilon(n)) \rangle_B = \langle \pi\sigma_{\epsilon(n)}\bar{Y} \rangle_B \in ch(\pi\sigma_{\epsilon(n)}) \cap \text{JHess}(1, A),$$

and since $\langle \pi \bar{X} \rangle_B \in ch(\pi)$, (4.17) is proved.

To prove (4.18), fix $i \in \{1, \ldots, n-1\}$, and let $X = (x_{lj}) \in L^+(2n)$ be given by

$$x_{lj} = \begin{cases} 1 & \text{if } (l,j) = (i+1, i) \\ -1 & \text{if } (l,j) = (d(i), d(i+1)) \\ 0 & \text{otherwise, } 1 \le j < l \le 2n. \end{cases}$$

It is immediate to check that $\langle \pi \bar{X} \rangle_B \in \text{JHess}(1, A)$ for all $\pi \in W_G$, and that the flag $\langle \bar{X} \rangle_B$ is fixed by $\tau_i^\natural$. I.e., $\langle \tau_i^\natural \bar{X} \rangle_B = \langle \bar{X} \rangle_B$. Thus, $\langle \pi \bar{X} \rangle_B = \langle \pi \tau_i^\natural \bar{X} \rangle_B$, proving (4.18). ■

Let $G = \coprod_{\pi \in W_G} B\pi B$ be the Bruhat decomposition of G and $G/B = \coprod_{\pi \in W_G} \langle B\pi \rangle_B$ the induced decomposition of G/B. Let $\sigma \in \Sigma(2n)$. We have seen in (2.5),(2.6) that $\langle V\sigma \rangle_V = \langle \sigma L_\sigma^+(2n) \rangle_V$, where $V = V(2n)$. This equality can actually be extended to

$$\langle V\tilde{\sigma} \rangle_V = \langle \tilde{\sigma} L_\sigma^+(2n) \rangle_V, \tag{4.19}$$

where $\tilde{\sigma}$ is any matrix obtained from σ by changing the signs of some of its entries.

Define now a map $i : G/B \to GL(2n, \mathbb{C})/V(2n)$ by $i(\langle g \rangle_B) = \langle \bar{g} \rangle_V$. This is immediately checked to be injective. Let $\pi = \sigma_\epsilon \tau^\natural \in W_G$ and $\pi^+ = \sigma_\epsilon^+ \tau^\natural \in W_G^+ \subset \Sigma(2n)$. Then $i(\langle \pi(\Omega L_{\pi^+}^+(2n)\Omega \cap G) \rangle_B) = \langle \Omega\pi(\Omega L_{\pi^+}^+(2n)\Omega \cap G)\Omega \rangle_V = \langle \bar{\pi} L_{\pi^+}^+(2n) \rangle_V \cap i(G/B) = \langle V\bar{\pi} \rangle_V \cap i(G/B) = \langle \Omega\Omega V \Omega \pi \Omega \rangle_V \cap i(G/B) = \langle \Omega(\Omega V \Omega \cap G)\pi\Omega \rangle_V = i(\langle B\pi \rangle_B)$. Thus,

$$\langle B\pi \rangle_B = \langle \pi(\overline{L_{\pi^+}^+(2n)} \cap G) \rangle_B. \tag{4.20}$$

This shows that the Bruhat cell $B_\pi \doteq \langle B\pi \rangle_B$ is the slice of $ch(\pi)$ obtained by setting equal to zero those coordinates x_{ij} for which $\bar{\pi}^+(i) > \bar{\pi}^+(j)$.

(IV.10) Proposition: $B_\pi \cap \text{JHess}(p, A)$ is a (quasiprojective) subvariety of $\text{JHess}(p, A)$ which is analytically isomorphic to a $\mathbb{C}$-affine space of dimension

$$E_p^J(\pi) \doteq card\left\{(i,j) \in I_n^J - I_{n,p}^J \;\middle|\; \bar{\pi}^+(i) < \bar{\pi}^+(j)\right\}. \tag{4.21}$$

Proof: The proof is essentially identical to that of [17, Proposition (V.1)]. Therefore, we only give an outline of it.

Consider the mapping $\{\tilde{f}^{(\mu)}_{\alpha\beta}\}_{(\alpha,\beta)\in I^J_{n,p}}$ obtained by restricting $\{f^{(\mu)}_{\alpha\beta}\}_{(\alpha,\beta)\in I^J_{n,p}}$ from $ch(\pi)$ to $B_\pi = \langle B\pi\rangle_B$. As in [17], one sees that $\tilde{f}^{(\mu)}_{\alpha\beta}$ vanishes identically if and only if $\bar{\pi}^+(\alpha) > \bar{\pi}^+(\beta)$. Next, if $X = (x_{ij}) \in L^+(2n)$ is such that $\langle \pi\bar{X}\rangle_B \in B_\pi$, equations $\tilde{f}^{(\mu)}_{\alpha\beta}(X) = 0, \quad \bar{\pi}^+(\alpha) < \bar{\pi}^+(\beta), \quad (\alpha,\beta) \in I^J_{n,p}$, determine all the $x_{\mu\nu}$'s for which $(\mu,\nu) \in I_{n,p}$ and $\bar{\pi}^+(\mu) < \bar{\pi}^+(\nu)$ as polynomial functions of the x_{ij}'s for which $(i,j) \in I^J_n - I^J_{n,p}$ and $\bar{\pi}^+(i) < \bar{\pi}^+(j)$. ■

We refer to $E^J_p(\pi)$ as the p^{th} *symplectic Eulerian dimension of* π.

(IV.11) Definition: For $p < 2n$ and $k = 1, \ldots, np - [p^2/4] + 1$, we define the *generalized symplectic Eulerian numbers of height* p by

$$C_n(p,k) \doteq card\left\{\pi \in W_G \;\middle|\; E^J_p(\pi) = k-1\right\}. \tag{4.22}$$

For justification of this terminology, see Section VII.

For the next theorem, we use the following result of A.H. Durfee [27]:

Theorem (Durfee): Let X be a smooth complex projective variety. Suppose that X is a finite disjoint union $X_1 \cup \ldots \cup X_m$, where the X_i are smooth contractible quasiprojective subvarieties. Then

$$b_k(X) = card\{X_i \mid 2(\dim X_i) = k\},$$

where $b_k(X)$ denotes the k^{th} Betti number of X.

(IV.12) Theorem: Let $A \in \mathrm{Reg}(sp(n,\mathbb{C}))$. Then for $p \geq 1$,

$$b_{2k+1}(\mathrm{JHess}(p,A)) = 0$$

$$b_{2k}(\mathrm{JHess}(p,A)) = C_n(p,k+1).$$

I.e., the odd Betti numbers of JHess(p,A) vanish, while the even Betti numbers are generalized symplectic Eulerian numbers of height p.

Proof: Readily follows from (IV.10) and Durfee's Theorem. ■

V. D_n-type Hessenberg Varieties

Let G denote the identity component of the group of linear transformations preserving the symmetric bilinear form on $\mathbb{C}^{2n}$ defined by the matrix

$$L = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Thus,

$$G = \left\{ X \in GL(2n, \mathbb{C}) \,\middle|\, {}^tXLX = L, \quad \det X = 1 \right\}, \tag{5.1}$$

and the Lie algebra $\mathbf{g}$ of G is

$$\begin{aligned} \mathbf{g} &= \left\{ X \in gl(2n, \mathbb{C}) \,\middle|\, {}^tXL + LX = 0 \right\} \\ &= \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \,\middle|\, B, C \in so(n, \mathbb{C}) \right\}, \end{aligned} \tag{5.2}$$

a semisimple Lie algebra of the classical type D_n. An explicit isomorphism $\omega : so(2n, \mathbb{C}) \to \mathbf{g}$ is given by $\omega(X) = S^{-1}XS$, where

$$S = \begin{pmatrix} I & I \\ iI & -iI \end{pmatrix}.$$

We make the standard choice of Borel subalgebra, namely

$$\begin{aligned} \mathbf{b} &\doteq \Omega v(2n)\Omega \cap \mathbf{g} \\ &= \left\{ \begin{pmatrix} u & s \\ 0 & -{}^tu \end{pmatrix} \,\middle|\, u \in v(n), \quad s \in so(n, \mathbb{C}) \right\}. \end{aligned} \tag{5.3}$$

A Cartan subalgebra contained in $\mathbf{b}$ is

$$\mathbf{s} \doteq \sum_{i=1}^{n} \mathbb{C}(E_{ii} - E_{n+i,n+i}). \tag{5.4}$$

The corresponding root-system is

$$D_n: \qquad \{e_j - e_i \mid 1 \le j, i \le n, \ \ i \ne j\} \cup \{\pm(e_j + e_i) \mid 1 \le j < i \le n\}. \tag{5.5}$$

The root-spaces and heights of the roots (with respect to the base determined by **b**) are as follows: The root $e_j - e_i$ has root-space $\mathbb{C}(E_{ji} - E_{n+i,n+j})$ and height $i-j$; the root $e_j + e_i$ has root-space $\mathbb{C}(E_{i,n+j} - E_{j,n+i})$ and height $2n-(i+j)$; the root $-e_j - e_i$ has root-space $\mathbb{C}(E_{n+i,j} - E_{n+j,i})$ and height $(i+j)-2n$.

It follows immediately from Definition (III.1) that the p^{th} Hessenberg subspace of **g** relative to **b** is given by

$$h_p(\mathbf{b},\mathbf{g}) = \left\{ \begin{pmatrix} h & s \\ z & -{}^t h \end{pmatrix} \,\middle|\, h \in \mathbf{H}_p(n),\ \ s \in so(n,\mathbb{C}),\ \ z \in \mathbf{g}_p^- \right\}, \tag{5.6}$$

where

$$\mathbf{g}_p^- \doteq \left\{ Z = (z_{ij}) \in so(n,\mathbb{C}) \,\middle|\, z_{ij} = 0 \text{ if } i+j < 2n-p \right\}.$$

Observe that if $H_p \in h_p(\mathbf{b},\mathbf{g})$, then $\bar{H}_p \in \mathbf{H}_{p+1}(2n)$.

Let B be the connected Lie subgroup of G with Lie algebra **b**. I.e.,

$$\begin{aligned} B &= \Omega V(2n)\Omega \cap G \\ &= \left\{ \begin{pmatrix} u & us \\ 0 & {}^t u^{-1} \end{pmatrix} \,\middle|\, u \in V(n),\ \ s \in so(n,\mathbb{C}) \right\}. \end{aligned} \tag{5.7}$$

Let LFlag(n) denote the subvariety of Flag$(2n)$ consisting of those flags which are isotropic with respect to L–i.e., for which $S_i^\perp = S_{2n-i}$, $i = 1,\ldots,n$, where orthogonals are taken with respect to L. Let LFlag$_0(n)$ denote the component of LFlag(n) containing the element $\langle\Omega\rangle_{V(2n)}$. G acts transitively on LFlag$_0(n)$, and B is the stabilizer of $\langle\Omega\rangle_{V(2n)}$. Thus, G/B is identified with LFlag$_0(n)$. In this identification, $\langle g\rangle_B$ is identified with the flag $T_1 \subset \ldots \subset T_n \subset T_{n+1} \subset \ldots \subset T_{2n-1}$, where T_i is the span of the first i columns of g if $i \le n$, and the span of the first n and last $i-n$ columns of g if $i > n$.

The condition $AX = XH_p$, $A \in \mathbf{g}$, $H_p \in h_p(\mathbf{b},\mathbf{g})$, can be rewritten as $A(X\Omega) = (X\Omega)\bar{H}_p$, and so we see that $(T_1,\ldots,T_{2n-1}) \in$ LFlag$_0(n)$ is an element in the D_n-type Hessenberg variety of height p, LHess(p,A), if and only if

$$AT_i \subset T_{i+p(i)}, \qquad p(i) = \begin{cases} p+1 & \text{if } \max(1,n-p) \le i \le n \\ p & \text{otherwise.} \end{cases} \tag{5.8}$$

Remark: The fact that the "shift" p is now varying reflects the structure of the D_n-type root-system and its height function.

The Weyl group of G is isomorphic to the group of conjugations of $\mathbf{s}$ by elements of the form $\sigma_\epsilon^+\tau^\natural$ (see (4.3),(4.4)), where now $\epsilon \in Z_n^e$, the subgroup of Z_n consisting of those elements with an even number of negative signs. We put

$$(5.9) \qquad W_G \doteq \left\{\sigma_\epsilon^+\tau^\natural \,\middle|\, \epsilon \in Z_n^e, \quad \tau \in \Sigma(n)\right\} \subset G \cap \Sigma(2n).$$

Let now

$$(5.10) \qquad L \doteq \overline{L^+(2n)} \cap G = \left\{\begin{pmatrix} l & 0 \\ \eta^{-1}s & \eta^{-1} \end{pmatrix} \,\middle|\, l \in L^+(n), \quad s \in so(n,\mathbb{C})\right\}.$$

Then the sets

$$(5.11) \qquad ch(\pi) \doteq \left\{\langle \pi Y\rangle_B \,\middle|\, Y \in L\right\}, \qquad \pi \in W_G$$

give rise to a system of analytic charts for G/B.

From now on we assume that A is a regular element in $\mathbf{s}$. This amounts to saying that A has the form

$$(5.12) \qquad A = diag(\lambda_1,\ldots,\lambda_n,-\lambda_1,\ldots,-\lambda_n), \quad \lambda_i \pm \lambda_j \neq 0, \ \ \forall i \neq j.$$

Let $X \in L^+(2n)$ be such that $\bar{X} \in L$. Then if $\pi \in W_G$, $\langle \pi\bar{X}\rangle_B \in \text{LHess}(p,A)$ if and only if

$$(5.13) \qquad (\overline{\pi^{-1}A\pi})X = X\bar{H}_p, \qquad \bar{H}_p \in \overline{h_p(\mathbf{g})} \subset \mathbf{H}_{p+1}(2n).$$

If $\pi = \sigma_\epsilon^+\tau^\natural$, then

$$(5.14) \qquad (\overline{\pi^{-1}A\pi}) = diag(\mu_1,\ldots,\mu_{2n}) \doteq \mu,$$

where $\mu_i = \epsilon_{\tau(i)}\lambda_{\tau(i)}$ for $i \le n$, and $\mu_i = -\epsilon_{\tau(d(i))}\lambda_{\tau(d(i))}$ for $i > n$. (As in Section IV, $d(\alpha) \doteq 2n+1-\alpha$.) If $\bar{H}_p = (h_{ij})$, $\ \ H_p \in h_p(\mathbf{g})$, then

$$h_{ij} = 0 \text{ if } (i,j) \in M \doteq \left\{(\mu,\nu) \,\middle|\, 1 \le \nu < \mu \le 2n \text{ and } i+j = 2n+1 \text{ or } i-j > P(i,j)\right\}$$

where

$$(5.15)\qquad P(i,j)=\begin{cases} p+1 & \text{if } 1\le j\le n \text{ and } n+1\le i\le 2n \\ p & \text{otherwise.}\end{cases}$$

Thus we obtain the following

(V.1) Proposition: Let $X\in L^+(2n)$ be such that $\bar{X}\in L$, and let $\pi\in W_G$. Then $\langle\pi\bar{X}\rangle_B\in \mathrm{LHess}(p,A)$ if and only if $f^{(\mu)}_{\alpha\beta}(X)=0$, $\forall\,(\alpha,\beta)\in M$.

Let now

$$X \doteq \begin{pmatrix} X_1 & 0 \\ Z & X_2\end{pmatrix}\in L^+(2n).$$

Then $\bar{X}\in L$ if and only if

$$(L1)\qquad {}^tX_1\theta X_2\theta = I$$

$$(L2)\qquad {}^tX_1\theta Z\in so(n,\mathbb{C}).$$

These conditions are equivalent to the set of equations

$$(L1)(i,j)\qquad x_{ij}+x_{d(j)d(i)}+\sum_{t=1}^{i-j-1}x_{i-t,j}x_{d(i-t)d(i)}=0$$

$$(L2)(i,j)\qquad \left(x_{d(j)i}+\sum_{t=1}^{n-j}x_{j+t,j}x_{d(j+t)i}\right)+\left(x_{d(i)j}+\sum_{t=1}^{n-i}x_{i+t,i}x_{d(i+t)j}\right)=0$$

for all pairs (i,j) with $1\le j<i\le n$, and

$$(L2)'(i)\qquad x_{d(i)i}+\sum_{t=1}^{n-i}x_{i+t,i}x_{d(i+t)i}=0,\qquad 1\le i\le n.$$

As in Section IV, we obtain equivalent formulations of (L1),(L2), namely:

(V.2) Lemma: Let $X=(x_{ij})\in L^+(2n)$. Then

(i) (L1) holds if and only if $x_{ij}=-g_{d(j)d(i)}(X)$, $\quad 1\le j<i\le n$, which holds if and only if $x_{d(j)d(i)}=-g_{ij}(X)$, $\quad 1\le j<i\le n$.

(ii) If (L1) holds, then (L2) holds if and only if $x_{d(i)j}=-g_{d(j)i}(X)$, $\quad 1\le j<i\le n$, which holds if and only if $x_{d(j)i}=-g_{d(i)j}(X)$, $\quad 1\le j<i\le n$.

(iii) If both (L1) and (L2) hold, then $x_{d(i)i} = -g_{d(i)i}(X)$ for $i = 1, \ldots, n$. In particular, $x_{n+1,n} = 0$.

From the above lemmas, one derives the following:

(V.3) Proposition: Let $X \in L^{+}(2n)$ be such that $\bar{X} \in L$. Then

$$f_{ij}^{(\mu)}(X) = -f_{d(j)d(i)}^{(\mu)}(X), \quad 1 \leq j < i \leq 2n.$$

In particular, for $i = 1, \ldots, n-1$, $\quad f_{d(i)i}^{(\mu)}(X) = 0$.

Put $\hat{g}_{\alpha\beta}(X) \doteq (1/2) \sum_{t=1}^{\alpha-\beta-1} \sum_{\alpha > \Gamma_t > \beta} (-1)^t x_{\alpha\gamma_1} \cdots x_{\gamma_t \beta}$. Then the relation $x_{d(i)i} = -g_{d(i)i}(X)$ can be written as $x_{d(i)i} = -\hat{g}_{d(i)i}(X)$ where now, on the righthand side, there is no dependence on $x_{d(i)i}$. Thus, a matrix $X \in L^{+}(2n)$ such that $\bar{X} \in L$ can be thought of as having free variables with indices

(5.16) $$(\alpha, \beta) \in I_n^L \doteq \left\{ (i,j) \,\middle|\, 1 \leq j < i \leq 2n, \quad i + j < 2n + 1 \right\}.$$

Let

(5.17) $$I_{n,p}^L \doteq \left\{ (i,j) \in I_n^L \,\middle|\, i - j > P(i,j) \right\}.$$

(V.4) Corollary: Let $X \in L^{+}(2n)$ be such that $\bar{X} \in L$, and let $\pi \in W_G$. Then $\langle \pi \bar{X} \rangle_B \in$ LHess(p, A) if and only if $f_{\alpha\beta}^{(\mu)}(X) = 0, \ \forall \, (\alpha, \beta) \in I_{n,p}^L$.

Remark: The set $I_n^L - I_{n,p}^L$ has cardinality $\min(p, n-1) + p(n-1) - [p^2/4]$.

(V.5) Theorem: Let $A \in \text{Reg}(\mathbf{g})$. Then for $p \geq 1$, LHess(p, A) is a smooth submanifold of LFlag$_0(n)$ of dimension $\min(p, n-1) + p(n-1) - [p^2/4]$.

The proofs of (V.1)–(V.5) are easy modifications of the arguments used in Section IV to prove the corresponding results; hence we omit them. For the same reason, we only outline the proof of the following

(V.6) Theorem: If $A \in \text{Reg}(\mathbf{g})$, then for $p \geq 1$, LHess(p, A) is connected.

Sketch of Proof: As for the C_n case, one shows that for $\pi \in W_G$, $ch(\pi) \cap$ LHess(p, A) is contractible and hence connected. To prove the theorem, it is enough to show that

(5.18) $$ch(\pi) \cap ch(\pi\omega) \cap \text{LHess}(1, A) \neq \emptyset$$

(5.19) $$ch(\pi) \cap ch(\pi\tau_i^{\natural}) \cap \text{LHess}(1, A) \neq \emptyset, \qquad i = 1, \ldots, n-1,$$

where $\omega \doteq \sigma^{+}_{(1,\dots,1,-1,-1)}\tau^{\natural}_{n-1}$, and as before, $\tau_i \in \Sigma(n)$ is the adjacent transposition which interchanges i and $i+1$.

The proof of (5.19) is essentially identical to that of (4.18). To prove (5.18), let $X = (x_{ij}) \in L^{+}(2n)$ be defined by

$$x_{ij} \doteq \begin{cases} 1 & \text{if } (i,j) = (n+1, n-1) \\ -1 & \text{if } (i,j) = (n+2, n) = (d(n-1), d(n+1)) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\langle \pi \bar{X} \rangle_B \in \text{LHess}(1, A)$ for all $\pi \in W_G$, and the flag $\langle \bar{X} \rangle_B$ is fixed by ω–i.e., $\langle \omega \bar{X} \rangle_B = \langle \bar{X} \rangle_B$. Therefore, $\langle \pi \bar{X} \rangle_B = \langle \pi \omega \bar{X} \rangle_B$, proving (5.18). ∎

Arguing as for the C_n case, it is easy to see that a Bruhat cell $B_\pi \doteq \langle B\pi \rangle_B$, $\pi \in W_G$, can be written as $\langle \pi(\overline{L^{+}_{\pi}(2n) \cap G}) \rangle_B$, and it is therefore the slice of $ch(\pi)$ obtained by setting equal to zero all the entries x_{ij} for which $\bar{\pi}(i) > \bar{\pi}(j)$.

(V.8) Proposition: Let B_π be a Bruhat cell of G/B. Then $B_\pi \cap \text{LHess}(p, A)$ is a (quasiprojective) subvariety of $\text{LHess}(p, A)$ which is analytically isomorphic to a $\mathbb{C}$-affine space of dimension

$$E^{L}_{p}(\pi) \doteq card \left\{ (i,j) \in I^{L}_{n} - I^{L}_{n,p} \;\middle|\; \bar{\pi}(i) < \bar{\pi}(j) \right\}. \tag{5.20}$$

(V.9) Definition: For $p < 2n$ and $k = 1, \dots, \min(n-1, p) + p(n-1) - [p^2/4] + 1$, we define the ***generalized Eulerian numbers of type D_n and height p*** by

$$D_n(p, k) \doteq card \left\{ \pi \in W_G \;\middle|\; E^{L}_{p}(\pi) = k - 1 \right\}. \tag{5.21}$$

(V.10) Theorem: Let $A \in \text{Reg}(g)$. Then for $p \geq 1$,

$$b_{2k+1}(\text{LHess}(p, A)) = 0$$

$$b_{2k}(\text{LHess}(p, A)) = D_n(p, k+1).$$

VI. B_n-type Hessenberg Varieties

Let G denote the identity component of the group of linear transformations preserving the symmetric bilinear form on $\mathbb{C}^{2n+1}$ defined by the matrix

$$K = \begin{pmatrix} 0 & 0 & \theta \\ 0 & 1 & 0 \\ \theta & 0 & 0 \end{pmatrix} \in GL(2n+1, \mathbb{C}). \tag{6.1}$$

Thus,

$$G = \left\{ X \in GL(2n+1, \mathbb{C}) \,\middle|\, {}^tXKX = K, \quad \det X = 1 \right\}, \tag{6.2}$$

and the Lie algebra $\mathbf{g}$ of G is

$$\begin{aligned} \mathbf{g} &= \left\{ X \in gl(2n+1, \mathbb{C}) \,\middle|\, {}^tXK + KX = 0 \right\} \\ &= \left\{ X \in gl(2n+1, \mathbb{C}) \,\middle|\, x_{\delta(j)\delta(i)} = -x_{ij}, \quad 1 \le i,j \le 2n+1 \right\}, \end{aligned} \tag{6.3}$$

where $\delta(\alpha) \doteq 2n+2-\alpha$. $\mathbf{g}$ is a semisimple Lie algebra of the classical type B_n. An explicit isomorphism $\omega: so(2n+1, \mathbb{C}) \to \mathbf{g}$ is given by $\omega(X) = T^{-1}XT$, where

$$T = \begin{pmatrix} I & 0 & \theta \\ 0 & \sqrt{2} & 0 \\ iI & 0 & -i\theta \end{pmatrix}.$$

Remark: Alternatively, the bilinear form could be defined by the matrix

$$\tilde{K} = \begin{pmatrix} 0 & 0 & I \\ 0 & 1 & 0 \\ I & 0 & 0 \end{pmatrix},$$

which more closely resembles the matrices J and L in Sections IV and V. However, the use of K conveniently simplifies the notation in the case B_n.

We make the standard choice of Borel subalgebra, namely

$$\mathbf{b} \doteq V(2n+1) \cap \mathbf{g}. \tag{6.4}$$

I.e., **b** consists of the upper-triangular elements of **g**. A Cartan subalgebra contained in **b** is

(6.5)
$$\mathbf{s} \doteq \sum_{i=1}^{n} \mathbb{C}(E_{ii} - E_{\delta(i)\delta(i)}).$$

The corresponding root-system is

(6.6)
$$B_n: \quad \{e_j - e_i \mid 1 \le i, j \le n, \quad i \ne j\} \cup \{\pm(e_j + e_i) \mid 1 \le j < i \le n\} \\ \cup \{\pm e_i \mid 1 \le i \le n\}.$$

The root-spaces and heights of the roots (with respect to the base determined by **b**) are as follows: The root $e_j - e_i$ has root-space $\mathbb{C}(E_{ji} - E_{\delta(i)\delta(j)})$ and height $i - j$; the root $e_j + e_i$ has root-space $\mathbb{C}(E_{j\delta(i)} - E_{i\delta(j)})$ and height $\delta(i+j)$; the root $-e_j - e_i$ has root-space $\mathbb{C}(E_{\delta(i)j} - E_{\delta(j)i})$ and height $-\delta(i+j)$; the root e_i has root-space $\mathbb{C}(E_{i,n+1} - E_{n+1,\delta(i)})$ and height $n+1-i$; the root $-e_i$ has root-space $\mathbb{C}(E_{n+1,i} - E_{\delta(i),n+1})$ and height $i - (n+1)$.

It follows immediately from Definition (III.1) that the p^{th} Hessenberg subspace of **g** relative to **b** is given by

(6.7)
$$h_p(\mathbf{b}, \mathbf{g}) = \left\{ X \in \mathbf{g} \,\middle|\, x_{ij} = 0, \quad \forall\ i - j > p \right\}.$$

I.e., $h_p(\mathbf{b}, \mathbf{g}) = \mathbf{g} \cap \mathbf{H}_p(2n+1)$.

Let B be the connected Lie subgroup of G with Lie algebra **b**. I.e., $B = G \cap V(2n+1)$. Let KFlag(n) denote the subvariety of Flag($2n+1$) consisting of those flags which are isotropic with respect to K–i.e., for which $S_i^\perp = S_{2n+1-i}, \quad i = 1, \ldots, n$, where orthogonals are taken with respect to K. G acts transitively on KFlag(n), and B is the stabilizer of the flag $sp\{e_1\} \subset \cdots \subset sp\{e_1, \ldots, e_{2n}\}$. Thus, G/B is identified with KFlag(n). In this identification, $\langle g \rangle_B$ is identified with the flag $T_1 \subset \cdots \subset T_{2n}$, where T_i is the subspace spanned by the first i columns of the matrix g. Notice that there is no reversal of the order of the last n columns of g, as was the case for JFlag(n) and LFlag(n). (If $\tilde{K}$ had been used to define the bilinear form in place of K, then the identification of G/B with

KFlag(n) would reverse the order of the last n columns of g.) From Definition (III.2) and (6.7), it follows that the *B_n-type Hessenberg variety of height p* is given by

$$\text{KHess}(p, A) \doteq \left\{(T_1, \ldots, T_{2n}) \in \text{KFlag}(n) \,\middle|\, AT_i \subset T_{i+p}, \quad \forall i\right\}. \tag{6.8}$$

The Weyl group of G is isomorphic to the group of conjugations of $\mathbf{s}$ by elements of the form

$$\sigma_\epsilon^\theta \tau^\theta = (-1)^\epsilon \begin{pmatrix} S_+(\epsilon) & 0 & S_-(\epsilon)\theta \\ 0 & 1 & 0 \\ \theta S_-(\epsilon) & 0 & \theta S_+(\epsilon)\theta \end{pmatrix} \begin{pmatrix} \tau & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \theta\tau\theta \end{pmatrix}, \tag{6.9}$$

where $\epsilon \in Z_n$, $\tau \in \Sigma(n)$, $S_+(\epsilon)$, $S_-(\epsilon)$ are defined as in (4.2), and $(-1)^\epsilon$ is minus one if ϵ has an odd number of -1 entries and plus one otherwise. We put

$$W_G \doteq \left\{\sigma_\epsilon^\theta \tau^\theta \,\middle|\, \epsilon \in Z_n, \ \tau \in \Sigma(n)\right\} \subset G \cap \Sigma(2n+1). \tag{6.10}$$

Let now

$$L \doteq L^+(2n+1) \cap G. \tag{6.11}$$

Then the sets

$$ch(\pi) \doteq \left\{\langle \pi Y\rangle_B \,\middle|\, Y \in L\right\}, \qquad \pi \in W_G \tag{6.12}$$

give rise to a system of analytic charts for G/B.

Let $X \in L$. If $\pi \in W_G$, then $\langle \pi X\rangle_B \in \text{KHess}(p, A)$ if and only if

$$(\pi^{-1} A \pi) X = X H_p, \qquad H_p \in h_p(\mathbf{g}). \tag{6.13}$$

It is easy to see that if $\pi = \sigma_\epsilon^\theta \tau^\theta$ and A is the regular element

$$A \doteq diag(\lambda_1, \ldots, \lambda_n, 0, -\lambda_n, \ldots, -\lambda_1), \quad \lambda_i \pm \lambda_j \neq 0, \ \forall i \neq j,$$

then

$$\pi^{-1} A \pi = diag(\nu_1, \ldots, \nu_{2n+1}) \doteq \nu, \tag{6.14}$$

where $\nu_i = \epsilon_{\tau(i)}\lambda_{\tau(i)}$ for $i \leq n$, $\nu_{n+1} = 0$, and $\nu_i = -\epsilon_{\tau(\delta(i))}\lambda_{\tau(\delta(i))}$ for $i > n+1$. Hence we obtain

(VI.1) Proposition: Let $X \in L$, $\pi \in W_G$. Then $\langle \pi X\rangle_B \in \mathrm{KHess}(p, A)$ if and only if $f^{(\nu)}_{\alpha\beta}(X) = 0$, $\forall\ \alpha - \beta > p$.

Let now

$$X \doteq \begin{pmatrix} X_1 & 0 & 0 \\ {}^t v & 1 & 0 \\ Z & u & X_2 \end{pmatrix} \in L^+(2n+1).$$

Then $X \in L$ if and only if

$$(K1) \qquad {}^tX_1\theta X_2 = \theta$$

$$(K2) \qquad v + {}^tX_1\theta u = 0$$

$$(K3) \qquad {}^tZ\theta X_1 + v\,{}^tv + {}^tX_1\theta Z = 0.$$

These conditions are equivalent to the set of equations

$$(K1)(i,j) \qquad x_{ij} + x_{\delta(j)\delta(i)} + \sum_{t=1}^{i-j-1} x_{i-t,j}x_{\delta(i-t)\delta(i)} = 0, \qquad 1 \leq j < i \leq n$$

$$(K2)(i) \qquad x_{n+1,i} + x_{\delta(i),n+1} + \sum_{t=1}^{n-i} x_{i+t,i}x_{\delta(i+t),n+1} = 0, \qquad 1 \leq i \leq n$$

$$(K3)(i,j) \qquad \left(x_{\delta(i)j} + \sum_{t=1}^{n-i} x_{i+t,i}x_{\delta(i+t)j}\right) + \left(x_{\delta(j)i} + \sum_{t=1}^{n-j} x_{j+t,j}x_{\delta(j+t)i}\right)$$
$$+ x_{n+1,i}x_{n+1,j} = 0, \quad 1 \leq j \leq i \leq n.$$

Similarly to the cases C_n and D_n, one obtains the following equivalent formulations:

(VI.2) Lemma: Let $X = (x_{ij}) \in L^+(2n+1)$. Then

(i) (K1) holds if and only if $x_{ij} = -g_{\delta(j)\delta(i)}(X)$, $\quad 1 \leq j < i \leq n$, which holds if and only if $x_{\delta(j)\delta(i)} = -g_{ij}(X)$, $\quad 1 \leq j < i \leq n$.

(ii) If (K1) holds, then (K2) holds if and only if $x_{n+1,i} = -g_{\delta(i),n+1}(X)$, $\quad 1 \leq i \leq n$, which holds if and only if $x_{\delta(i),n+1} = -g_{n+1,i}(X)$, $\quad 1 \leq i \leq n$.

(iii) If (K1) holds, then (K3) holds if and only if $x_{\delta(i)j} = -g_{\delta(j)i}(X), \quad 1 \le j \le i \le n$, which holds if and only if $x_{\delta(j)i} = -g_{\delta(i)j}(X), \quad 1 \le j \le i \le n$.

(iv) (K1), (K2) and (K3) hold if and only if $x_{ij} = -g_{\delta(j)\delta(i)}(X), \quad 1 \le j < i \le 2n+1$.

From the above lemmas, one derives the following

(VI.3) Proposition: If $X \in L$, then for all pairs (i,j) with $1 \le j < i \le 2n+1$, $f^{\nu}_{ij}(X) = -f^{(\nu)}_{\delta(j)\delta(i)}(X)$. In particular, for $i = 1, \ldots, n$, $\quad f^{(\nu)}_{\delta(i)i}(X) = 0$.

Put now

$$
\begin{aligned}
I^K_n &\doteq \left\{(i,j) \,\middle|\, 1 \le j < i \le 2n+1, \quad i+j \le 2n+1\right\} \\
I^K_{n,p} &\doteq \left\{(i,j) \in I^K_n \,\middle|\, i-j > p\right\}.
\end{aligned}
\tag{6.15}
$$

From (VI.1) and (VI.3), we obtain the following

(VI.4) Corollary: Let $X \in L$, $\pi \in W_G$. Then $\langle \pi X \rangle_B \in \mathrm{KHess}(p, A)$ if and only if $f^{(\nu)}_{\alpha\beta}(X) = 0, \, \forall \, (\alpha, \beta) \in I^K_{n,p}$.

Remark: $card(I^K_n - I^K_{n,p}) = np - [p^2/4]$.

(VI.5) Theorem: Let $A \in \mathrm{Reg}(\mathbf{g})$. Then for $p \ge 1$, $\mathrm{KHess}(p, A)$ is a smooth submanifold of $\mathrm{KFlag}(n)$ of dimension $np - [p^2/4]$.

(VI.6) Theorem: If $A \in \mathrm{Reg}(\mathbf{g})$, then for $p \ge 1$, $\mathrm{KHess}(p, A)$ is connected.

Sketch of Proof: As for the C_n and D_n cases, one shows that for $\pi \in W_G$, $ch(\pi) \cap \mathrm{KHess}(p, A)$ is contractible and hence connected. Thus, it is enough to show that for all $\pi \in W_G$,

$$ch(\pi) \cap ch(\pi \sigma^{\theta}_{\epsilon(n)}) \cap \mathrm{KHess}(1, A) \neq \emptyset \tag{6.16}$$

$$ch(\pi) \cap ch(\pi \tau^{\theta}_i) \cap \mathrm{KHess}(1, A) \neq \emptyset, \qquad i = 1, \ldots, n-1, \tag{6.17}$$

where $\epsilon(n) = (1, \ldots, 1, -1)$ and $\tau_i \in \Sigma(n)$ is the adjacent transposition which interchanges i and $i+1$.

The proof of (6.17) is essentially identical to that of (4.18). To prove (6.16), let $X = (x_{ij}) \in L^+(2n+1)$ be defined by

$$x_{ij} \doteq \begin{cases} \sqrt{2} & \text{if } (i,j) = (n+1, n) \\ -\sqrt{2} & \text{if } (i,j) = (n+2, n+1) \\ -1 & \text{if } (i,j) = (n+2, n) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\langle \pi X \rangle_B \in \mathrm{KHess}(1, A)$ for all $\pi \in W_G$. Next, let b be the element of B given by

$$b \doteq \begin{pmatrix} u & w & -w\,{}^t\!w\,{}^t\!u^{-1}\theta/2 \\ 0 & 1 & -{}^t\!w\,{}^t\!u^{-1}\theta \\ 0 & 0 & \theta\,{}^t\!u^{-1}\theta \end{pmatrix},$$

where $u \doteq diag(-1, \ldots, -1, 1)$ and $w \doteq {}^t[0, \ldots, 0, -\sqrt{2}]$. Then it is straightforward to check that $\sigma^{\theta}_{\epsilon(n)} X b \in L$. ■

Arguing in the usual way, one sees that a Bruhat cell $B_\pi \doteq \langle B\pi \rangle_B$, $\pi \in W_G$, of G/B can be written as $\langle \pi(L^+_{\pi^+} \cap G) \rangle_B$, where if $\pi = \sigma^{\theta}_{\epsilon}\tau^{\theta}$, $\pi^+ \doteq (-1)^{\epsilon}\pi$. We put

$$W^+_G \doteq \left\{ \pi^+ \;\middle|\; \pi \in W_G \right\}. \tag{6.18}$$

Thus, B_π is the slice of $ch(\pi)$ obtained by setting equal to zero all the entries x_{ij} for which $\pi^+(i) > \pi^+(j)$.

(VI.7) Proposition: Let B_π be a Bruhat cell of G/B. Then $B_\pi \cap \mathrm{KHess}(p, A)$ is a (quasiprojective) subvariety of $\mathrm{KHess}(p, A)$ which is analytically isomorphic to a $\mathbb{C}$-affine space of dimension

$$E^K_p(\pi) \doteq card\left\{ (i,j) \in I^K_n - I^K_{n,p} \;\middle|\; \pi^+(i) < \pi^+(j) \right\}. \tag{6.19}$$

(VI.8) Definition: For $p \leq 2n$ and $k = 1, \ldots, np - [p^2/4] + 1$, we define the *generalized Eulerian numbers of type B_n and height p* by

$$B_n(p,k) \doteq card\left\{ \pi \in W_G \;\middle|\; E^K_p(\pi) = k - 1 \right\}. \tag{6.20}$$

(VI.9) Theorem: Let $A \in \mathrm{Reg}(\mathfrak{g})$. Then for $p \geq 1$,

$$b_{2k+1}(\mathrm{KHess}(p, A)) = 0$$
$$b_{2k}(\mathrm{KHess}(p, A)) = B_n(p, k+1).$$

VII. Generalized Eulerian Numbers

Let $\sigma \in \Sigma(n)$. Then, σ is said to have a fall (or descent) at i, $1 \leq i \leq n-1$, if $\sigma(i) > \sigma(i+1)$. The total number of falls in σ is evidently $E_1(\sigma)$, the first Eulerian dimension of σ. (See Theorem (II.3).) The numbers

$$A(n, k) \doteq card\left\{\sigma \in \Sigma(n) \,\middle|\, E_1(\sigma) = k-1\right\}$$

are the well-known Eulerian numbers, which occur in a variety of combinatorial problems. (See e.g., [29,30].)

The fact that the numbers $\{A_{n-1}(p, k)\}$ as defined in (2.7) coincide with $\{A(n, k)\}$ for $p = 1$ is our primary motivation for referring to $\{A_{n-1}(p, k)\}$ as *generalized Eulerian numbers*. We observe that the numbers $A_{n-1}(p, k)$ are also well-known in the case $p = n-1$. Indeed, for $p = n-1$, $E_{n-1}(\sigma)$ is the number of inversions in σ, an inversion being a pair (i, j) with $1 \leq j < i \leq n$ and $\sigma(i) < \sigma(j)$. Thus, $\{A_{n-1}(n-1, k)\}$ coincide with the so-called Mahonian numbers (see e.g., [31]), which are best-known as the coefficients in the expansion of $(1+t)(1+t+t^2)\cdots(1+t+\ldots+t^{n-1})$. However, to the best of the authors' knowledge, the numbers $\{A_{n-1}(p, k)\}$, $\quad 1 < p < n-1$, have not been studied in the literature.

In this section, we show that the numbers $A_{n-1}(p, k)$, $B_n(p, k)$, $C_n(p, k)$ and $D_n(p, k)$ as defined in the previous sections admit a unified interpretation as statistics on (reduced) root-systems, thereby justifying Definition (VII.1) below. Next, we give some explicit formulas for $C_n(1, k) = B_n(1, k)$. The similarity of the obtained formulas with the corresponding ones for the classical Eulerian numbers is striking. In particular, we discover that these numbers are f-Eulerian in the sense of Stanley [20], with $f(s) = (2s+1)^n$.

(VII.1) Definition: Let Φ be a reduced root system with Weyl group W, and let Φ^+ (respectively, Φ^-) be the set of positive (respectively, negative) roots with respect to some fixed basis Δ. Let $h(\cdot)$ be the height function on Φ with respect to Δ. Let $w \in W$. We define the p^{th} *Eulerian dimension of* w by

$$(7.1) \qquad E_p^{\Phi}(w) \doteq card\left\{\alpha \in \Phi^+ \,\middle|\, h(\alpha) \le p, \quad w(\alpha) \in \Phi^-\right\},$$

and the *generalized Eulerian numbers of height p on* Φ by

$$(7.2) \qquad \Phi(p,k) \doteq card\left\{w \in W \,\middle|\, E_p^{\Phi}(w) = k-1\right\}.$$

Remark: A comparison of the data for B_n and C_n in the Appendix reveals immediately that, in general, two different root systems with isomorphic Weyl groups have different generalized Eulerian numbers.

(VII.2) Theorem: For Φ of type A_{n-1}, B_n, C_n and D_n, the generalized Eulerian numbers as defined in (2.7), (6.20), (4.22) and (5.21) respectively coincide with those defined in (7.2).

Proof: For A_{n-1}, this is Proposition (III.3).

Case B_n: Let Φ be as in (6.6). Then W is the group of $n \times n$ signed permutation matrices. Let W_G^+ be as in (6.18) and define the bijection $\psi: \ W_G^+ \to W$, where for $i = 1, \ldots, n$,

$$\psi(\pi^+)(i) \doteq \begin{cases} \pi^+(i) & \text{if } 1 \le \pi^+(i) \le n \\ -\delta(\pi^+(i)) & \text{otherwise.} \end{cases}$$

(Note that $\pi^+(\delta(i)) = \delta(\pi^+(i))$. In particular, $\pi^+(n+1) = n+1$.)

Next, if I_n^K and $I_{n,p}^K$ are as in (6.15), for $(i,j) \in I_n^K$ put

$$\alpha_{ij} \doteq \begin{cases} e_j - e_i & \text{if } i \le n \\ e_j & \text{if } i = n+1 \\ e_j + e_{\delta(i)} & \text{if } i > n+1. \end{cases}$$

Then $(i,j) \to \alpha_{ij}$ is a bijection of I_n^K onto Φ^+, and it is tedious but trivial to verify that $h(\alpha_{ij}) \le p$ if and only if $(i,j) \in I_n^K - I_{n,p}^K$, and $\psi(\pi^+)(\alpha_{ij}) \in \Phi^-$ if and only if $\pi^+(i) < \pi^+(j)$.

Case C_n: Let Φ be as in (3.4). Then W is again the group of $n \times n$ signed permutation matrices. Let W_G^+ be as in (4.5). As usual, $\bar{W}_G^+$ denotes $\Omega W_G^+ \Omega$. Define the bijection

$\psi: \bar{W}_G^+ \to W$, where for $i = 1, \ldots, n$,

$$\psi(\bar{\pi}^+)(i) \doteq \begin{cases} \bar{\pi}^+(i) & \text{if } 1 \le \bar{\pi}^+(i) \le n \\ -d(\bar{\pi}^+(i)) & \text{otherwise.} \end{cases}$$

(Note that $\bar{\pi}^+(d(i)) = d(\bar{\pi}^+(i))$.)

Next, if I_n^J and $I_{n,p}^J$ are as in (4.14), (4.15), for $(i,j) \in I_n^J$, put

$$\alpha_{ij} \doteq \begin{cases} e_j - e_i & \text{if } 1 \le j < i \le n \\ e_j + e_{d(i)} & \text{if } 1 \le j \le n < i \le d(j). \end{cases}$$

Then $(i,j) \to \alpha_{ij}$ is a bijection of I_n^J onto Φ^+. Furthermore, $h(\alpha_{ij}) \le p$ if and only if $(i,j) \in I_n^J - I_{n,p}^J$, and $\psi(\bar{\pi}^+)(\alpha_{ij}) \in \Phi^-$ if and only if $\bar{\pi}^+(i) < \bar{\pi}^+(j)$.

Case D_n: Let Φ be as in (5.5) Then W is the group of $n \times n$ even-signed permutation matrices. Let W_G be as in (5.9). As usual, $\bar{W}_G$ denotes $\Omega W_G \Omega$. Define the bijection $\psi: \bar{W}_G \to W$, where for $i = 1, \ldots, n$,

$$\psi(\bar{\pi})(i) \doteq \begin{cases} \bar{\pi}(i) & \text{if } 1 \le \bar{\pi}(i) \le n \\ -d(\bar{\pi}(i)) & \text{otherwise.} \end{cases}$$

Next, if I_n^L and $I_{n,p}^L$ are as in (5.16), (5.17), for $(i,j) \in I_n^L$, put

$$\alpha_{ij} \doteq \begin{cases} e_j - e_i & \text{if } 1 \le j < i \le n \\ e_j + e_{d(i)} & \text{if } 1 \le j < n < i < d(j). \end{cases}$$

Then $(i,j) \to \alpha_{ij}$ is a bijection of I_n^L onto Φ^+. Furthermore, $h(\alpha_{ij}) \le p$ if and only if $(i,j) \in I_n^L - I_{n,p}^L$, and $\psi(\bar{\pi})(\alpha_{ij}) \in \Phi^-$ if and only if $\bar{\pi}(i) < \bar{\pi}(j)$. ■

Remark: In the very special case where the root system is of type A_{n-1} and $p = 1$, Theorem VII.2 gives the following characterization of the classical Eulerian numbers $\{A(n,k)\}$: $A(n,k)$ is equal to the number of permutations on n letters which map exactly $k-1$ *simple* roots to negative roots. Although this fact is rather obvious, we were not able to locate it in the literature.

Remark: It follows from Theorem VII.2 that $C_n(1,k) = B_n(1,k)$. Indeed, the root systems of types B_n and C_n have the same Weyl group, and the sets of simple roots are, respectively, $\{e_i - e_{i+1} \mid 1 \le i \le n-1\} \cup \{e_n\}$ and $\{e_i - e_{i+1} \mid 1 \le i \le n-1\} \cup \{2e_n\}$.

Remark: For the classical linear Lie algebras, we now have three characterizations of the p^{th} Eulerian dimension of an element of the Weyl group, and hence of the corresponding generalized Eulerian numbers as well–one which is purely combinatorial, a second which is Lie algebraic, and a third which is topological. For example, in the case of $sp(n, \mathbb{C})$, the p^{th} symplectic Eulerian dimension of $\pi \in W_G$ is defined combinatorially (see (4.21)) as

$$E_p^J(\pi) \doteq card\{(i,j) \mid 1 \leq j < i \leq 2n,\ \ i+j \leq 2n+1,\ \ i-j \leq p,\ \ \bar{\pi}^+(i) < \bar{\pi}^+(j)\}.$$

Equivalently, $E_p^J(\pi)$ can be characterized Lie algebraically (see the proof of Theorem VII.2) as the number of positive roots of height at most p which are taken to negative roots by the corresponding element $\psi(\bar{\pi}^+)$ in the Weyl group W. Topologically, $E_p^J(\pi)$ is characterized as the dimension of the intersection of the Bruhat cell B_π with the variety JHess(p, A), for any regular $A \in \mathbf{s}$. These three characterizations of $E_p^J(\pi)$ yield three corresponding characterizations of the generalized Eulerian numbers $\{C_n(p,k)\}$. In particular, the topological characterization of these numbers is as the (even) Betti numbers of JHess(p, A) (Theorem IV.12)). Similar statements apply to the Eulerian dimensions and generalized Eulerian numbers for the other three classical cases–A_{n-1}, B_n and D_n.

The topological characterization of the generalized Eulerian numbers in the classical cases enables us to obtain a result concerning unimodality. A sequence of numbers $a_0, \ldots, a_n$ is said to be *unimodal* if $a_0 \leq \ldots \leq a_{[n/2]}$ and $a_{[n/2]+1} \geq \ldots \geq a_n$. It is *symmetric* if $a_k = a_{n-k}$.

(VII.3) Theorem: If Φ is a reduced root system of classical type, then the generalized Eulerian numbers of height p on Φ are unimodal and symmetric.

Proof: From the results in Sections II, IV, V and VI, we know that the generalized Eulerian numbers $\{\Phi(p,k)\}$ are the even Betti numbers of an appropriate p^{th} Hessenberg variety, a smooth, connected, compact complex projective variety. By Poincaré duality, the sequence is symmetric. It follows from a result of Stanley [19] (applying the hard Lefschetz theorem for nonsingular irreducible projective varieties) that the even Betti numbers form a unimodal sequence. ■

We now examine in more detail the properties of the generalized symplectic Eulerian numbers of height one. There is a rather striking similarity between these properties and those of the classical Eulerian numbers.

(VII.4) Proposition: The numbers $C_n(1,k)$ satisfy the recurrence relation

$$C_n(1,k) = (2k-1)C_{n-1}(1,k) + (2n-2k+3)C_{n-1}(1,k-1). \tag{7.3}$$

Proof: Let $\bar{W}_G^+(n)$ be as in (4.5), where we now make explicit the size n. If $\sigma = \Omega\sigma_\epsilon^+\tau^\natural\Omega \in \bar{W}_G^+(n)$, then for $1 \le i \le n$,

$$\sigma(i) = \begin{cases} \tau(i) & \text{if } \epsilon_{\tau(i)} = 1 \\ d(\tau(i)) & \text{if } \epsilon_{\tau(i)} = -1 \end{cases}$$

$$\sigma(d(i)) = d(\sigma(i)).$$

For $j = 1, \ldots, 2n$, we define maps $\eta_j : \bar{W}_G^+(n-1) \to \bar{W}_G^+(n)$ as follows: If $\sigma \in \bar{W}_G^+(n-1)$, then $\eta_j(\sigma)$ is obtained from σ by inserting the values 0 and $2n-1$ in σ at positions j and $d(j) = 2n+1-j$ respectively, and then adding 1 to each of the $2n$ values. (For example, if $\sigma = (5,1,4|3,6,2) \in \bar{W}_G^+(3)$, then $\eta_2(\sigma) = (6,1,2,5|4,7,8,3)$.) Such maps are clearly injective, and $\cup_{j=1}^{2n} \eta_j(\bar{W}_G^+(n-1)) = \bar{W}_G^+(n)$ or, equivalently, $\eta_i(\bar{W}_G^+(n-1)) \cap \eta_j(\bar{W}_G^+(n-1)) = \emptyset$ whenever $i \ne j$.

Let now $\sigma \in \bar{W}_G^+(n-1)$ and $1 \le j \le n$. Then

$$E_1^J(\eta_j(\sigma)) = \begin{cases} E_1^J(\sigma) & \text{if } j = 1 \\ E_1^J(\sigma) & \text{if } j > 1 \text{ and } \sigma(j-1) > \sigma(j) \\ E_1^J(\sigma) + 1 & \text{if } j > 1 \text{ and } \sigma(j-1) < \sigma(j) \end{cases}$$

$$E_1^J(\eta_{d(j)}(\sigma)) = \begin{cases} E_1^J(\sigma) + 1 & \text{if } j = 1 \\ E_1^J(\sigma) & \text{if } j > 1 \text{ and } \sigma(j-1) > \sigma(j) \\ E_1^J(\sigma) + 1 & \text{if } j > 1 \text{ and } \sigma(j-1) < \sigma(j). \end{cases}$$

Hence,

$$card\{j \mid 1 \le j \le 2n, \quad E_1^J(\eta_j(\sigma)) = E_1^J(\sigma)\} = 2E_1^J(\sigma) + 1$$

$$card\{j \mid 1 \le j \le 2n, \quad E_1^J(\eta_j(\sigma)) = E_1^J(\sigma) + 1\} = 2(n - E_1^J(\sigma)) - 1.$$

Fix k. All the $\sigma \in \bar{W}_G^+(n-1)$ for which $E_1^J(\sigma) = k-2$ will be mapped, under the various η_j's, into $[2(n-(k-2))-1]C_{n-1}(1,k-1) = (2n-2k+3)C_{n-1}(1,k-1)$ elements in $\bar{W}_G^+(n)$

with E_1^J equal to $k-1$, and all the elements $\sigma \in \bar{W}_G^+(n-1)$ for which $E_1^J(\sigma) = k-1$ will be mapped into $[2(k-1)+1]C_{n-1}(1,k) = (2k-1)C_{n-1}(1,k)$ elements in $\bar{W}_G^+(n)$ with E_1^J equal to $k-1$. This implies (7.3). ∎

Remark: For the classical Eulerian numbers $A_{n-1}(1,k)$, the corresponding recurrence relation is

$$A_{n-1}(1,k) = kA_{n-2}(1,k) + (n-k+1)A_{n-2}(1,k-1).$$

(VII.5) Proposition: The exact value of the numbers $C_n(1,k)$ is

$$C_n(1,k) = \sum_{j=0}^{k-1}(-1)^j \binom{n+1}{j}(2k-2j-1)^n. \tag{7.4}$$

Proof: The recurrence relation (7.3) together with $C_1(1,1) = C_1(1,2) = 1$ determines all the numbers $\{C_n(1,k)\}$. Now, for $k=1$, the righthand side of (7.4) is equal to 1 for any n. It is also equal to 1 if $k=2$ and $n=1$. Thus, it suffices to show that the righthand side of (7.4) satisfies the recurrence (7.3). This is an elementary exercise. ∎

Remark: The exact values of the classical Eulerian numbers are given by

$$A_{n-1}(1,k) = \sum_{j=0}^{k-1}(-1)^j \binom{n+1}{j}(k-j)^n.$$

(VII.6) Proposition: If $C_{n,1}(x) \doteq \sum_{k=1}^{n+1} C_n(1,k)x^k$, then

$$C_{n,1}(x) = (1-x)^{n+1}\sum_{s=0}^{\infty}(2s+1)^n x^{s+1}. \tag{7.6}$$

Proof: A classical result in the calculus of finite differences (see e.g., [20]) is that if $f: \mathbf{N} \to \mathbb{C}$ and $n \in \mathbf{N}$ (where $\mathbf{N}$ denotes the nonnegative integers), then $f(s)$ is a polynomial in s of degree at most n if and only if $(1-x)^{n+1}\sum_{s=0}^{\infty} f(s)x^s$ is a polynomial in x of degree at most n.

Consider now the degree n polynomial function $f(s) \doteq (2s+1)^n$. Then

$$(1-x)^{n+1}\sum_{s=0}^{\infty} f(s)x^s$$

is a polynomial in x of degree at most n, and we have

$$(1-x)^{n+1}\sum_{s=0}^{\infty} f(s)x^s = \left(\sum_{j=0}^{n+1}\binom{n+1}{j}(-1)^j x^j\right)\left(\sum_{t=1}^{\infty}(2t-1)^n x^{t-1}\right)$$

$$= \sum_{k=1}^{n+1}\left\{\sum_{j=0}^{k-1}(-1)^j\binom{n+1}{j}(2k-2j-1)^n\right\}x^{k-1}$$

$$= \sum_{k=1}^{n+1} C_n(1,k)x^{k-1}.$$

Thus, $C_{n,1}(x) = x(1-x)^{n+1}\sum_{s=0}^{\infty} f(s)x^s$. ■

Remark: For the classical Eulerian polynomial $A_{n-1,1}(x) \doteq \sum_{k=1}^{n} A_{n-1}(1,k)x^k$, we have

$$A_{n-1,1}(x) = (1-x)^{n+1}\sum_{s=0}^{\infty} s^n x^s.$$

Remark: In [20], Stanley has introduced the following terminology: If $f:\ \mathbf{N} \to \mathbb{C}$, $n \in \mathbf{N}$ and $\sum_{s=0}^{\infty} f(s)x^s = P(x)/(1-x)^{n+1}$, where the polynomial $P(x)$ has degree at most n, then the coefficients of $P(x)$ are called *f-Eulerian numbers*. Thus, the numbers $C_n(1,k) = B_n(1,k)$ are $(2s+1)^n$-Eulerian numbers.

Tables of the first values of the numbers $A_{n-1}(p,k)$, $B_n(p,k)$, $C_n(p,k)$ and $D_n(p,k)$ are included in the Appendix.

Acknowledgement: We would like to thank Philip Hanlon for computing the tables of generalized Eulerian numbers in the Appendix. Also, we would like to thank both he and Robert Proctor for useful suggestions.

References

[1] G.H. Golub and C.F. Van Loan, *Matrix Computations*, John Hopkins Univ. Press, Baltimore, MD, 1984.

[2] R. Hermann, *Cartanian Geometry, Nonlinear Waves, and Control Theory, Part A*, Math Sci Press, Brookline, 1979.

[3] G.S. Ammar and C.F. Martin, "The geometry of matrix eigenvalue methods," *Acta Aplicandae Mathematicae*, 5 (1986), 239-278.

[4] M. Shub and A.T. Vasquez, "Some linearly induced Morse-Smale systems, the QR-algorithm and the Toda lattice," *Contemp. Math*, 64 (1986), 181-194.

[5] L.E. Faibusovich, "Generalized Toda flows, Riccati equations on Grassmann manifolds and QR-algorithm," *Functional Analysis and Its Applications*, 21 (1987), 88-89. (In Russian)

[6] G.S. Ammar, "Geometric aspects of Hessenberg matrices," *Contemporary Math.*, 68 (1987), 1-21.

[7] R. Steinberg, "Desingularization of the unipotent variety," *Invent. Math.*, 36 (1976), 209-224.

[8] T.A. Springer, "A construction of representations of Weyl groups," *Invent. Math.*, 44 (1978), 279-293.

[9] N. Spaltenstein, "The fixed point set of a unipotent transformation on the flag manifold," *Proc. Kon. Akad. Wetensch. Amsterdam*, 79 (1978), 452-456.

[10] R. Hotta and N. Shimomura, "The fixed point subvarieties of unipotent transformations on generalized flag varieties and the Green functions–combinatorial and cohomological treatments centering GL(n)," *Math. Ann.*, 241 (1979), 193-208.

[11] N. Shimomura, "A theorem on the fixed point set of a unipotent transformation on the flag manifold," *J. Math. Soc. Japan*, 32 (1980), 55-64.

[12] N. Spaltenstein, *Sous-groupes de Borel Contenant un Unipotent Donné*, Lecture Notes in Math. 948, Springer, New York, 1982.

[13] N. Shimomura, "The fixed point subvarieties of unipotent transformations on the flag varieties," *J. Math. Soc. Japan*, 37 (1985), 537-556.

[14] U. Helmke and M.A. Shayman, "The biflag manifold and the fixed points of a unipotent transformation on the flag manifold," *Linear Alg. Appl.*, 92 (1987), 125-159.

[15] C. De Concini, G. Lusztig and C. Procesi, "Homology of the zero-set of a nilpotent vector field on a flag manifold," *J. Amer. Math. Soc.*, 1 (1988).

[16] F. De Mari, *On the Topology of the Hessenberg Varieties of a Matrix*, Ph.D. thesis, Washington University, St. Louis, Missouri, 1987.

[17] F. De Mari and M.A. Shayman, "Generalized Eulerian numbers and the topology of the Hessenberg variety of a matrix," *Acta Aplicandae Mathematicae*, to appear.

[18] R. Byers, "A Hamiltonian QR-algorithm," *SIAM J. Sci. Stat. Comput.*, 7 (1986), 212-229.

[19] R.P. Stanley, "Weyl groups, the hard Lefschetz theorem, and the Sperner property," *SIAM J. Alg. Disc. Meth.*, 1 (1980), 168-184.

[20] R.P. Stanley, *Enumerative Combinatorics*, Vol I, Wadsworth and Brooks/Cole, Monterey, 1986.

[21] L. Carlitz, "Eulerian numbers and polynomials of higher order," *Duke Math. J.*, 27 (1960), 401-423.

[22] L. Carlitz and R. Scoville, "Generalized Eulerian numbers: combinatorial applications," *Jour. für die Reine und Angew. Math.*, 265 (1974), 110-137.

[23] J. Dillon and D. Roselle, "Eulerian numbers of higher order," *Duke Math. J.*, 35 (1968), 247-256.

[24] D. Lehmer, "Generalized Eulerian numbers," *Jour. of Combin. Theory*, Series A, 32 (1982), 195-215.

[25] D. Rawlings, "The r-major index," *Jour. of Combin. Theory*, Series A, 31 (1981), 175-183.

[26] H. Hiller, *The Geometry of Coxeter Groups*, Pitman, London, 1982.

[27] A.H. Durfee, "Algebraic varieties which are a disjoint union of subvarieties," in *Geometry and Topology, Manifolds, Varieties, and Knots*, (ed. C. McCrory and T. Shirfin), Dekker, New York, 1987.

[28] A. Bialynicki-Birula, "Some theorems on actions of algebraic groups," *Annals Math.*, 98 (1973), 480-497.

[29] L. Comtet, *Analyse Combinatoire*, Vol. I& II, Presses Univ. de France, Paris, 1970.

[30] J. Riordan, *An Introduction to Combinatorial Analysis*, Wiley, New York, 1958.

[31] D. Foata, "Distributions Eulériennes et Mahoniennes sur le groupe des permutations," in *Higher Combinatorics* (ed. M. Aigner), Reidel, Dordrecht, 1977.

Appendix: Tables of Generalized Eulerian Numbers

$A_2(p,k)$	1	2	3	4
1	1	4	1	
2	1	2	2	1

$A_3(p,k)$	1	2	3	4	5	6	7
1	1	11	11	1			
2	1	3	8	8	3	1	
3	1	3	5	6	5	3	1

$A_4(p,k)$	1	2	3	4	5	6	7	8	9	10	11
1	1	26	66	26	1						
2	1	4	17	38	38	17	4	1			
3	1	4	9	19	27	27	19	9	4	1	
4	1	4	9	15	20	22	20	15	9	4	1

$B_2(p,k)$	1	2	3	4	5
1	1	6	1		
2	1	3	3	1	
3	1	2	2	2	1

$B_3(p,k)$	1	2	3	4	5	6	7	8	9	10
1	1	23	23	1						
2	1	5	18	18	5	1				
3	1	3	7	13	13	7	3	1		
4	1	3	5	9	12	9	5	3	1	
5	1	3	5	7	8	8	7	5	3	1

$B_4(p,k)$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	1	76	230	76	1												
2	1	7	45	139	139	45	7	1									
3	1	4	12	38	83	108	83	38	12	4	1						
4	1	4	9	22	47	70	78	70	47	22	9	4	1				
5	1	4	9	16	27	44	59	64	59	44	27	16	9	4	1		
6	1	4	9	16	24	35	48	55	55	48	35	24	16	9	4	1	
7	1	4	9	16	24	32	39	44	46	44	39	32	24	16	9	4	1

$C_3(p,k)$	1	2	3	4	5	6	7	8	9	10
1	1	23	23	1						
2	1	4	19	19	4	1				
3	1	3	6	14	14	6	3	1		
4	1	3	6	9	10	9	6	3	1	
5	1	3	5	7	8	8	7	5	3	1

$C_4(p,k)$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	1	76	230	76	1												
2	1	5	51	135	135	51	5	1									
3	1	4	10	34	85	116	85	34	10	4	1						
4	1	4	10	20	43	72	84	72	43	20	10	4	1				
5	1	4	9	17	27	44	59	62	59	44	27	17	9	4	1		
6	1	4	9	17	27	37	46	51	51	46	37	27	17	9	4	1	
7	1	4	9	16	24	32	39	44	46	44	39	32	24	16	9	4	1

$D_4(p,k)$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	44	102	44	1								
2	1	4	30	61	61	30	4	1					
3	1	4	9	20	38	48	38	20	9	4	1		
4	1	4	9	16	27	39	39	27	16	9	4	1	
5	1	4	9	16	23	28	30	28	23	16	9	4	1