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Finding shortest gentle paths on polyhedral terrains by the method of multiple shooting

Phan Thanh An [a,b,*], Nguyen Thi Le [c], Le Hong Trang [a,b], Raymond Chi-Wing Wong [d]

[a] Institute of Mathematical and Computational Sciences - IMACS and Faculty of Applied Science, Ho Chi Minh City University of Technology (HCMUT), 268 Ly Thuong Kiet Street, District 10, Ho Chi Minh City, Viet Nam
[b] Vietnam National University Ho Chi Minh City, Linh Trung Ward, Thu Duc City, Ho Chi Minh City, Viet Nam
[c] Institute of Mathematics, Vietnam Academy of Science and Technology (VAST), 18 Hoang Quoc Viet Road, Hanoi, Viet Nam
[d] Department of Computer Science and Engineering, Hong Kong University of Science and Technology, Hong Kong, China

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ABSTRACT

The problem of finding shortest θ-gentle paths can be stated as follows: given two points p, q on a polyhedral terrain and a slope parameter $\theta \in (0, \pi/2)$, the objective is to find a path joining p and q on the terrain which is shortest such that the slope of the path does not exceed θ. In this paper, we introduce some geometric and analysis properties of such paths and answer the question of whether known results of classical shortest paths hold for shortest θ-gentle paths. An algorithm for approximately computing such shortest θ-gentle paths on terrains is presented, where an approximate shortest θ-gentle path joining two points is a θ-gentle path whose length is the infimum of a sequence of that of θ-gentle paths in which they are decreasing. We also show that the sequence of lengths of paths obtained by the proposed algorithm is convergent. The algorithm is implemented in C++ using CGAL and Open GL in some specific circumstances.

1. Introduction

A variant of the shortest path problem is the shortest θ-gentle path problem (SGP problem for briefly): given a polyhedral terrain $\mathcal{T}$, a source point p, a destination point q on $\mathcal{T}$, and a slope value $\theta \in (0, \pi/2)$, find a shortest θ-gentle path joining p and q on $\mathcal{T}$, in the sense that it is shortest such that its slope does not exceed θ. This problem can be seen as a generalization of the single source shortest path whose goal is to find classical shortest paths joining two points on a polyhedral terrain ($\theta = \pi/2$), which is a well-studied problem in computational geometry and robotics. Applications of the slope-constrained shortest path problem appear in some fields. For example, when we design mobile robots for traversing on non-planar surfaces or when we ski down a mountain and avoid a too steep path, paths that are too steep should be replaced by "zig-zag lines" satisfying the slope constraint. In train transport, the railroads in Linz, Austria with a slope of 11.6%, and the one in Tram 28 in Lisbon, Portugal which has a 14.5% grade are considered to be the steepest railroads in the world.

When $\theta = \pi/2$, the SGP problem becomes the single source shortest path problem. Several works [1–3] presented algorithms for solving the problem in this case. Unfortunately, when $0 < \theta < \pi/2$, known methods for solving the single source shortest path problem such as using the star unfolding technique of Agarwal et al. [1], the modified Chen and Han's sequence tree [2,4], and the sequence of edges that the shortest path goes through [3] does not work. Some special cases of the SGP problem have been investigated by Ahmed, Lubiw, and Maheshwari in [5,6]. Amed and Lubiw also showed that the problem of minimizing the total number of bends (of shortest θ-gentle paths) is NP-hard, and no polynomial time solution is known [5]. Thus, in the general case, the SGP problem is NP-hard.

To date, there are several approaches for finding such shortest θ-gentle paths. Nöllenburg and Sautter [7] presented an approximation approach based on determining the norm for finding shortest θ-gentle paths on a sequence of adjacent triangles. An algorithm proposed by Ahmed, Lubiw, Maheshwari [6] models the problem as a graph whose nodes are Steiner points added along the edges of the terrain. After discretizing, the required path is found on the whole terrain and therefore they take much memory of computers when the size of the problem

* Corresponding author at: Institute of Mathematical and Computational Sciences - IMACS and Faculty of Applied Science, Ho Chi Minh City University of Technology (HCMUT), 268 Ly Thuong Kiet Street, District 10, Ho Chi Minh City, Viet Nam.
E-mail address: thanhan@hcmut.edu.vn (P.T. An).

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is large. These two algorithms have not been implemented, and thus it is not clear how practical they are. Liu and Wong [8] proposed an algorithm to solve approximately the SGP problem using a technique of simplifying terrains and some graph tools. They concentrated on reducing the size of terrains rather than solving the SGP problem. To overcome this issue, we will present a new approach, namely the method of multiple shooting (MMS for short), for computing shortest θ-gentle paths on sequences of adjacent triangles of sub-terrains with a fewer number of triangles and implement it on computers therefore until now, there are only two methods (i.e., Liu and Wong's method and our method of multiple shooting) having their implementations on computers. The method was proposed for solving the geometric shortest path problems in 2D and 3D [9–11]. We then use successfully MMS to deal with approximately the SGP problem but under the assumption of the connection given in Section 4.1 for terrains.

According to [6–8], solutions to the SGP problem are computed based on deducing the subproblem of finding a shortest θ-gentle path joining two given points along a sequence of adjacent triangles. Not many properties of such a path have been shown. A natural question is "whether the properties of classical shortest paths hold for shortest θ-gentle paths?". Although the authors in [6,8] proved that a shortest θ-gentle path is a polyline and not unique if it exists, the existence of such a path has been not stated.

In the paper, we answer the question of the existence of shortest θ-gentle paths joining two points along a sequence of adjacent triangles (Proposition 2). We show that shortest θ-gentle paths can go through convex vertices of polyhedral terrains (Example 1). The characterization of unreachable vertices by a θ-cone is presented (Proposition 1). An iterative algorithm based on MMS is given, and we prove that the sequence of lengths of paths obtained by the proposed algorithm is convergent (Proposition 5). Furthermore, the path obtained after some iterative steps of the algorithm is an approximate shortest θ-gentle path if the number of iterative steps is large enough (Theorem 1), where the notion of an approximate shortest θ-gentle path joining two points is presented in Definition 5.

The rest of the paper is organized as follows. Section 2 recalls preliminary notions. Section 3 presents some properties of shortest θ-gentle paths joining two points. Section 4 introduces an iterative algorithm using MMS for the SGP problem. The algorithm is implemented in C++ using CGAL and numerical results are given and visualized to describe how our method works in Section 5. We used MMS to compare with Liu and Wong's algorithm [8] for solving the SGP problem. The lengths of final θ-gentle paths obtained by our algorithm are similar to that get by Liu and Wong's one, while the running time of Liu and Wong's algorithm is thousands of times higher than the proposed algorithm. Proofs of the correctness of the proposed algorithm are arranged in Appendix.

2. Preliminaries

We recall some definitions and properties. For any points p, q in $\mathbb{R}^3$, we denote $[p,q] := \{(1-\lambda)p + \lambda q : 0 \le \lambda \le 1\}$, $(p,q) := \{(1-\lambda)p + \lambda q : 0 < \lambda < 1\}$.

A *polyhedral terrain* or simply a *terrain*, denoted by $\mathcal{T}$, is a polyhedral surface in $\mathbb{R}^3$ in which every vertical line intersects the surface at most once. This means that the projections of all faces of $\mathcal{T}$ on the xy-plane are pairwise non-overlap. For a point $p \in \mathcal{T}$, let denote the triple $x(p), y(p), z(p)$ its coordinates in $\mathbb{R}^3$. Then $z(p)$ is also called the *height* of p. $\mathcal{T}$ is said to be *convex* if it is a convex polyhedral surface in $\mathbb{R}^3$. W.l.o.g, we assume that all the faces of the terrain are triangles, for if not, we can triangulate these faces to obtain triangles.

Definition 1. A *sequence of adjacent triangles* on $\mathcal{T}$, denoted by $\mathcal{F}$, is defined by a list of distinct triangles $(f_1, f_2, \dots, f_m)$ of $\mathcal{T}$, where f_i and f_{i+1} share a common edge e_i, for $i = 1, 2, \dots, m-1$.

By abuse of notation, we use the same letter $\mathcal{F}$ for the union of all triangles of the sequence $\mathcal{F}$ of adjacent triangles. Let p and q be two points on $\mathcal{F}$. If p is in the first triangle and q is in the final triangle, then $\mathcal{F}$ is called a sequence of adjacent triangles *joining p and q*.

Definition 2 (*[12]*)**.** Let $[t_0, t_1] \subset \mathbb{R}$. A *path* on $\mathcal{T}$ is a continuous map $\gamma : [t_0, t_1] \to \mathcal{T}$.

If $\gamma(t_0) = p, \gamma(t_1) = q$, where $p, q \in \mathcal{T}$, then we say that γ *joins p and q on $\mathcal{T}$*. If $\gamma : [t_0, t_1] \to \mathcal{F}$ and $\gamma(t_0) = p, \gamma(t_1) = q$, where $p, q \in \mathcal{F}$, then we say that γ is *a path joining p and q along the sequence $\mathcal{F}$ of adjacent triangles.*

We also call the image $\gamma([t_0, t_1])$ to be the path γ. All paths considered in the paper are piece-wise linear. The *slope of a segment* $[a,b]$ is the angle (in radian) between the line joining a, b and the horizontal plane, denoted by $\mathrm{sl}([a,b])$. Here the angle between a line and a plane is defined by the angle between the line and its projection onto the plane. The *slope of a path* is the maximum slope among its pieces. Consequently, the slope is in $[0, \pi/2]$.

We assume throughout the paper that the slope parameter $\theta \in (0, \pi/2)$ and p, q are two points on $\mathcal{T}$.

Definition 3 (*[8]*)**.** A path is *θ-gentle* if its slope does not exceed θ. A θ-gentle path γ joining p and q on $\mathcal{T}$ is called a *shortest θ-gentle path*, denoted by $\mathrm{SGP}_{\mathcal{T}}(p,q|\theta)$ or simply $\mathrm{SGP}_{\mathcal{T}}(p,q)$, if there is no other θ-gentle path γ' joining p and q on $\mathcal{T}$ such that $l(\gamma') < l(\gamma)$, where $l(.)$ denotes an arclength function.

Reachability

Definition 4 (*[8]*)**.** Given two distinct points a and a' on $\mathcal{T}$, a is said to be *θ-reachable from a'* if there exists a θ-gentle path joining a and a' on $\mathcal{T}$.

A point a is said to be *θ-reachable* if there exists another point a' on $\mathcal{T}$ such that a is reachable from a'. Otherwise, a is said to be *θ-unreachable*.

By Lemma 5 [8], if a point of $\mathcal{T}$ is θ-unreachable, then it is a vertex of $\mathcal{T}$. To deal with the SGP problem, we thus only have to check whether each vertex (instead of all possible points on $\mathcal{T}$) is θ-unreachable or not. If the source point p and the destination point q are θ-reachable, then there is a θ-gentle path joining p and q on $\mathcal{T}$ by Lemma 6 [8]. When one of them, say p, is θ-unreachable, we deduce that p is a vertex. In this case, an approximate solution is obtained by choosing a non-vertex point p' such that the distance between p' and p is at most ε, where ε is a given positive number. Then the solution to the problem is approximately replaced by a θ-gentle path joining p' and q on $\mathcal{T}$.

A θ-unreachable vertex is also known as a *sharp* vertex defined by Ahmed et al. [6]. Additionally, there may exist some vertices in a triangle, say f, which are not θ-reachable from any points in f, however, these vertices may be θ-reachable from some points on a triangle that is adjacent to f. Such vertices are said to be *locally sharp in f* (see [6]).

The uniqueness of shortest θ-gentle paths

Let f be a triangle of $\mathcal{T}$ and $a, b \in f$. Assume that a and b are θ-reachable points from some points of f. If $\mathrm{sl}([a,b]) \le \theta$ (i.e., $[a,b]$ is not too steep), then $\mathrm{SGP}_f(a,b) = [a,b]$. Otherwise, $\mathrm{sl}([a,b]) > \theta$ (i.e., $[a,b]$ is too steep), there exists a polyline in f from b to a such that the slopes of all segments of the polyline are equal to θ (see Fig. 1). Such a zig-zag line is called an *adjusted path* of $[a,b]$, denoted by $\mathrm{adj}[a,b]$. It is shown that $\mathrm{adj}[a,b]$ indeed is $\mathrm{SGP}_f(a,b)$ and its length is $\dfrac{|z(a)-z(b)|}{\sin\theta}$. Hence

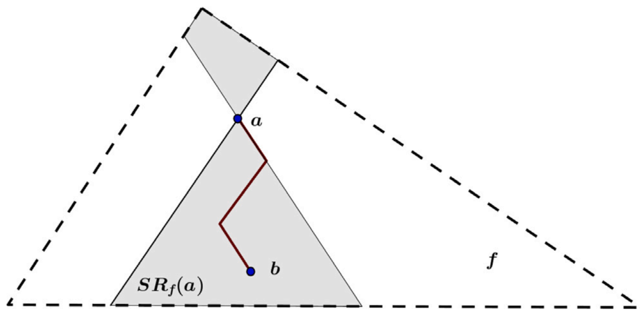


Fig. 1. An adjusted path of $[a, b]$ when $[a, b]$ is too steep.

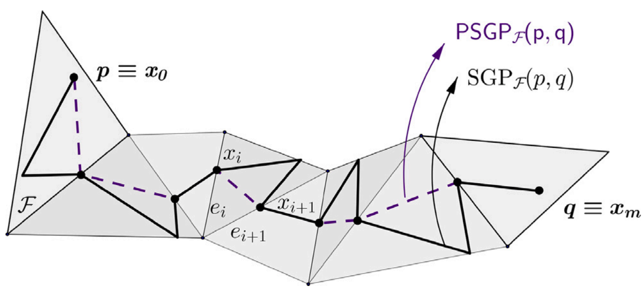


Fig. 2. The solid curve shows $\mathrm{SGP}_{\mathcal{F}}(p,q)$ and the dashed curve shows a pseudo path $\mathrm{PSGP}_{\mathcal{F}}(p,q)$ of $\mathrm{SGP}_{\mathcal{F}}(p,q)$.

in both cases ($[a,b]$ is too steep or not), Liu and Wong [8] prove that the length of $\mathrm{SGP}_f(a,b)$ is given by:

$$l\left(\mathrm{SGP}_f(a,b)\right) = \max\left\{ l([a,b]), \frac{|z(a)-z(b)|}{\sin\theta} \right\}. \tag{1}$$

Moreover, Lemma 3 [7] shows that the following expression defines a norm on $\mathbb{R}^3$:

$$\|a\|_s = \max\left\{ \|a\|_2, \frac{|z(a)|}{\sin\theta} \right\}, \tag{2}$$

where $a \in \mathbb{R}^3$, $\|.\|_2$ is the Euclidean norm.

Combining (1) and (2) yields

$$\|a-b\|_s = \max\left\{ \|a-b\|_2, \frac{|z(a)-z(b)|}{\sin\theta} \right\} = \max\left\{ l([a,b]), l(\mathrm{adj}[a,b]) \right\}. \tag{3}$$

To conclude, if $b \in f$ and $[a,b]$ is too steep, the θ-gentle adjusted paths of $[a,b]$ are not unique. It follows that shortest θ-gentle paths are not unique in general. Furthermore, shortest θ-gentle paths are polylines (see [6,8]).

Pseudo paths of shortest θ-gentle paths

Assume that $\mathcal{F}$ is a sequence of m adjacent triangles joining two θ-reachable points p and q on $\mathcal{T}$ and common edges are $e_1, e_2, \ldots, e_{m-1}$. Without loss of generality, we suppose that $p \notin e_1, q \notin e_{m-1}$. Lemma 4 [6] states that there exists at least one shortest θ-gentle path joining p and q that crosses each triangle of a sequence of adjacent triangles at most once. We denote by x_i the intersection point of $\mathrm{SGP}_{\mathcal{F}}(p,q)$ and e_i, for $i = 1, 2, \ldots, m-1$. Then $\mathrm{SGP}_{\mathcal{F}}(p,q)$ is the union of sub-paths $\mathrm{SGP}_{f_i}(x_i, x_{i+1})$ in each triangle. Moreover, these sub-paths, which are $[x_i, x_{i+1}]$ or adjusted paths of $[x_i, x_{i+1}]$ depend on the slope of $[x_i, x_{i+1}], i = 0, 1 \ldots, m-1$, where $x_0 = p, x_m = q$ (see Fig. 2).

Remark 1. Combining (1), (2), and (3) yields

$$l\left(\mathrm{SGP}_{\mathcal{F}}(p,q)\right) = \|x_0 - x_1\|_s + \|x_1 - x_2\|_s + \cdots + \|x_{m-1} - x_m\|_s. \tag{4}$$

The polyline formed by consecutively connecting $x_0, x_1, \ldots, x_m$ is called a *pseudo path* of $\mathrm{SGP}_{\mathcal{F}}(p,q)$, denoted by $\mathrm{PSGP}_{\mathcal{F}}(p,q)$ (see Fig. 2), i.e.,

$$\mathrm{PSGP}_{\mathcal{F}}(p,q) = [x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{m-1}, x_m].$$

Then a θ-gentle path whose pseudo path is $\mathrm{PSGP}_{\mathcal{F}}(p,q)$ is also called an *adjusted path* of $\mathrm{PSGP}_{\mathcal{F}}(p,q)$. Let us denote by $lp(\mathrm{PSGP}_{\mathcal{F}}(p,q))$ the right-hand side of Eq. (4), then we can rewrite

$$lp\left(\mathrm{PSGP}_{\mathcal{F}}(p,q)\right) = l\left(\mathrm{SGP}_{\mathcal{F}}(p,q)\right).$$

3. Some properties of shortest θ-gentle paths

In this section, we give some geometric and analysis properties of shortest gentle paths: characterization of a θ-unreachable vertex, the existence of shortest θ gentle paths along a sequence of adjacent triangles, and an example of shortest θ-gentle paths which can visit a convex vertex of a polyhedral terrain.

To characterize a θ-unreachable vertex, we introduce the concept of a θ-cone. Take $a \in \mathbb{R}^3$, a *θ-cone* whose vertex is a is constructed as follows: let Δ be the line passing through a and being perpendicular to the horizontal plane. A θ-cone is formed by a set of all lines passing through a and creating with Δ an angle that does not exceed $\pi/2 - \theta$. Lines that create with Δ an angle equal to $\pi/2-\theta$ are called generating lines of θ-cone. A θ-cone consists of two *θ-convex cones*, upper and lower ones, incident to a (see Fig. 3(i)).

Given a triangle f on $\mathcal{T}$, suppose that a is a point of f. The intersection between the interior of the θ-cone whose vertex is a and f is a so-called *steep region* of a on f, denoted by $\mathrm{SR}_f(a)$. The steep region $\mathrm{SR}_f(a)$ can be an empty set, a triangle, polygons, or an entire triangle f in which they do not include edges sharing the vertex a (see Fig. 4).

Proposition 1. *A vertex a of a polyhedral terrain is θ-unreachable if and only if all its adjacent triangles lie completely in (i.e., in and not on the generating lines) one of the two θ-convex cones of the θ-cone whose vertex is a.*

Proof (⇒). Assume that a is θ-unreachable. Let us prove that all triangles which are adjacent to a lie completely in one of two θ-convex cones (see Fig. 3(i)). On the contrary, suppose that there exists a triangle, say f, which is adjacent to a such that f does not lie completely in one of the two θ-convex cones. Then there is an edge e of f which is adjacent to a such that e does not lie completely in one of the two θ-convex cones. It follows that the angle between e and the horizontal plane does not exceed θ. Hence e is not too steep. Therefore a is θ-reachable from some point in e. This contradicts our assumption that a is θ-unreachable.

(⇐) We suppose that all triangles adjacent to a lie completely in one of two θ-convex cones. If a is not θ-unreachable, then there is a θ-gentle path γ joining a and a point of a triangle, say f, which is adjacent to a. Let m be the line segment of γ containing a. Thus the slope of m does not exceed θ. As $m \in f$, m lies completely in one of two θ-convex cones. Then $\mathrm{sl}(m) > \theta$. Thus $\mathrm{sl}_\gamma(a) > \theta$, a contradiction. The proof is complete. □

Proposition 1 indicates that to check the θ-unreachability of a vertex, we can construct a θ-cone whose vertex is a and then check whether all triangles that are adjacent to a lie completely in one of the two θ-convex cones of the θ-cone or not (see Figs. 3(ii) and (iii)).

Due to the definition of the slope of θ-gentle paths, the union of two θ-gentle paths is θ-gentle. It is also easy to obtain the triangle inequality for shortest θ-gentle paths: $l\left(\mathrm{SGP}_{\mathcal{T}}(a,b)\right) \le l\left(\mathrm{SGP}_{\mathcal{T}}(a,c)\right) + l\left(\mathrm{SGP}_{\mathcal{T}}(c,b)\right)$, where $a,b,c \in \mathcal{T}$ assuming that $\mathrm{SGP}_{\mathcal{T}}(a,b)$, $\mathrm{SGP}_{\mathcal{T}}(b,c)$, $\mathrm{SGP}_{\mathcal{T}}(c,a)$ exist. In triangle inequality, $\mathcal{T}$ can be replaced by a triangle or a sequence of adjacent triangles.

We assume $\mathcal{G}_{\mathcal{F}}$ to be the set of all θ-gentle paths joining p and q on a sequence $\mathcal{F}$ of adjacent triangles and $\mathcal{G}_{\mathcal{F}} \neq \emptyset$. Let $m = \inf\{l(\gamma) : \gamma \in \mathcal{G}_{\mathcal{F}}\}$.

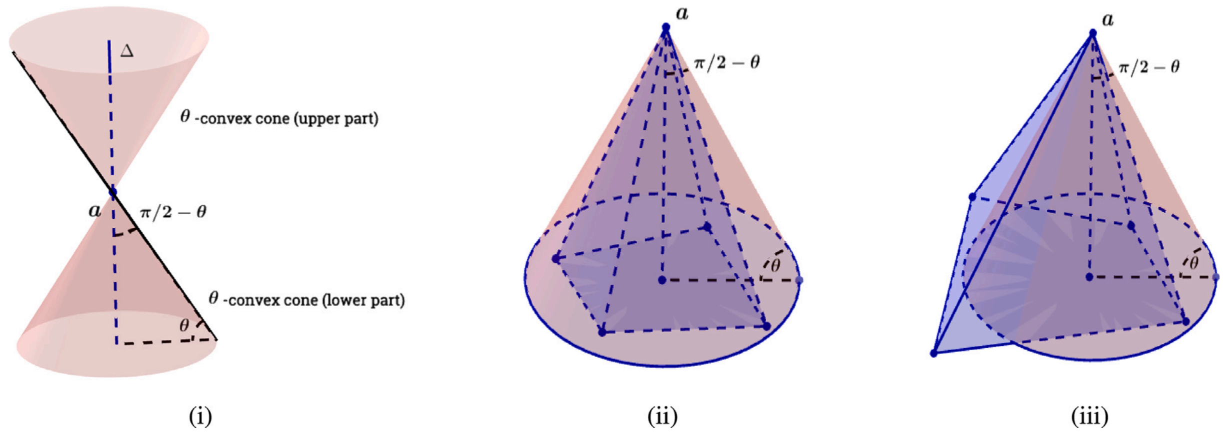


Fig. 3. (i) a θ-cone consists of two θ-convex cones; (ii) a is θ-unreachable; (iii) a is θ-reachable.

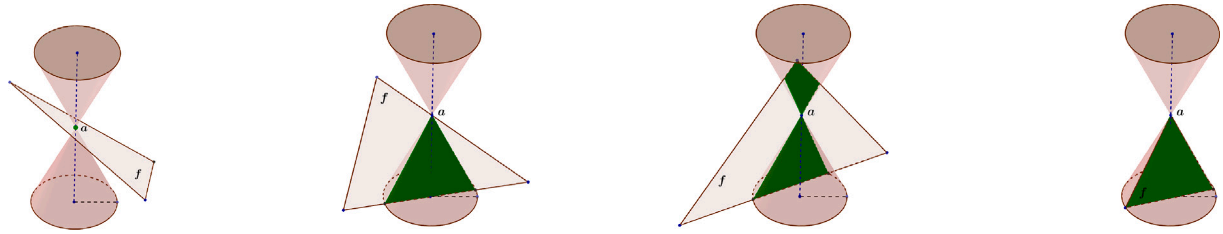


Fig. 4. Steep regions in the triangle f.

A question is "Is there a θ-gentle path γ_0 satisfying $l(\gamma_0) = m$?" It means that " Does there exist a shortest θ-gentle path joining p and q on $\mathcal{F}$?".

This is answered by our following proposition.

Proposition 2. *Given $p, q \in \mathcal{T}$, let $\mathcal{F}$ be a sequence of adjacent triangles joining p and q on $\mathcal{T}$, such that all vertices in $\mathcal{F}$ are θ-reachable from some points in $\mathcal{F}$. Then, there exists γ_0 in $\mathcal{G}_{\mathcal{F}}$ such that $l(\gamma_0) \le l(\gamma)$, for all $\gamma \in \mathcal{G}_{\mathcal{F}}$. The path γ_0 is then a shortest θ-gentle path joining p and q along $\mathcal{F}$, where $\mathcal{G}_{\mathcal{F}}$ denotes the set of all θ-gentle paths joining p and q along $\mathcal{F}$.*

Proof. Because all vertices of $\mathcal{T}$ are θ-reachable from some points in $\mathcal{F}$, there exists at least one θ-gentle path in $\mathcal{F}$ joining two arbitrary vertices of $\mathcal{T}$. Since all θ-unreachable points are vertices of the terrain, all points in $\mathcal{F}$ are θ-reachable from some point in $\mathcal{F}$. Therefore there always exists at least one θ-gentle path in $\mathcal{F}$ joining p and q, hence $\mathcal{G}_{\mathcal{F}} \neq \emptyset$. Suppose that $\mathcal{F}$ consists of $m+1$ adjacent triangles $f_1, f_2, \dots, f_{m+1}$ and $p \in f_1, q \in f_{m+1}$. We denote $\mathbb{E} = \mathbb{R}^3$, $\mathcal{E} := e_1 \times e_2 \times \cdots \times e_m \in \mathbb{E}^m$, where e_i is the common edge of f_i and f_{i+1}, for $i = 1, 2, \dots, m$. The set $\mathcal{E}$ is closed and bounded in $\mathbb{E}^m$ with product topology. Then $\mathcal{E}$ is compact. Without loss of generality, we assume that $p \notin e_1, q \notin e_m$. Let an arbitrary point $x_i \in e_i$, for $i = 1, 2, \dots, m$ and $x_0 = p, x_{m+1} = q$. Consider the function $\Phi : \mathbb{E}^m \longrightarrow \mathbb{R}, \Phi(x_1, x_2, \dots, x_m) = \sum_{i=0}^m \|x_i - x_{i+1}\|_s$, where $\|.\|_s$ is determined by (2):

$$\|a\|_s = \max\left\{ \|a\|_2, \frac{|z(a)|}{\sin\theta} \right\}$$

Our next claim is that Φ is continuous. For all sequences $(x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)})$ converging to $(x_1, x_2, \dots, x_m)$ in $\mathbb{E}^m$, we have to prove that $\Phi(x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)}) \to \Phi(x_1, x_2, \dots, x_m)$ in $\mathbb{R}$ as $n \to \infty$.

$$\begin{aligned} \Phi(x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)}) &= \sum_{i=0}^m \|x_i^{(n)} - x_{i+1}^{(n)}\|_s \\ &\le \sum_{i=0}^m (\|x_i^{(n)} - x_i\|_s + \|x_i - x_{i+1}\|_s + \|x_{i+1} - x_{i+1}^{(n)}\|_s) \end{aligned} \tag{*}$$

where $x_0^{(n)} = p, x_{m+1}^{(n)} = q$. As $\Phi(x_1, x_2, \dots, x_m) = \sum_{i=0}^m \|x_i - x_{i+1}\|_s$, now (*) becomes

$$\Phi(x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)}) - \Phi(x_1, x_2, \dots, x_m) \le \sum_{i=0}^m (\|x_i^{(n)} - x_i\|_s + \|x_{i+1} - x_{i+1}^{(n)}\|_s)$$

Similarly, we have

$$\Phi(x_1, x_2, \dots, x_m) - \Phi(x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)}) \le \sum_{i=0}^m (\|x_i - x_i^{(n)}\|_s + \|x_{i+1}^{(n)} - x_{i+1}\|_s)$$

This gives

$$|\Phi(x_1, x_2, \dots, x_m) - \Phi(x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)})| \le \sum_{i=0}^m (\|x_i - x_i^{(n)}\|_s + \|x_{i+1}^{(n)} - x_{i+1}\|_s) \tag{**}$$

Since $(x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)})$ converges to $(x_1, x_2, \dots, x_m)$ in $\mathbb{E}^m$, we have $x_i^{(n)} \to x_i$ in $\mathbb{R}^3$ as $n \to \infty$, for $i = 1, 2, \dots, m$. Hence for $i = 1, 2, \dots, m$, $\sum_{i=0}^m \|x_i^{(n)} - x_i\|_2 \to 0$ and $\sum_{i=0}^m \|x_{i+1} - x_{i+1}^{(n)}\|_2 \to 0$ as $n \to \infty$, where $\|.\|_2$ is the Euclidean norm in $\mathbb{R}^3$. As two norms in a finite-dimensional space are equivalent, $\|.\|_2$ and $\|.\|_s$ are equivalent in $\mathbb{R}^3$. Accordingly, the sequence $\{x_i^{(n)}\}$ converges with the norm $\|.\|_2$ if and only if $\{x_i^{(n)}\}$ converges with the norm $\|.\|_s$ in $\mathbb{R}^3$. Consequently, $\sum_{i=0}^m \|x_i^{(n)} - x_i\|_s \to 0$ and $\sum_{i=0}^m \|x_{i+1} - x_{i+1}^{(n)}\|_s \to 0$ as $n \to \infty$. Applying (**) gives $|\Phi(x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)}) - \Phi(x_1, x_2, \dots, x_m)| \to 0$ as $n \to \infty$. Then $\Phi(x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)}) \to \Phi(x_1, x_2, \dots, x_m)$ in $\mathbb{R}$ as $n \to \infty$. This implies that Φ is continuous. Because Φ is continuous on the compact set $\mathcal{E}$, there exists $(x_1^*, x_2^*, \dots, x_m^*) \in \mathcal{E}$ such that

$$\Phi(x_1^*, x_2^*, \dots, x_m^*) = \min\{\Phi(x_1, x_2, \dots, x_m) : (x_1, x_2, \dots, x_m) \in \mathcal{E}\}.$$

We connect $x_0, x_1^*, x_2^*, \dots, x_m^*, x_{m+1}$ by the polyline including segments $[x_0, x_1^*]$, $[x_1^*, x_2^*]$, …, $[x_m^*, x_{m+1}]$. Note that all points in $\mathcal{F}$ are θ-reachable from some point in $\mathcal{F}$, by constructing an adjusted path of the polyline if necessary, we can get a θ-gentle path, say σ, with its length $l(\sigma) = \|x_0 - x_1^*\|_s + \|x_1^* - x_2^*\|_s + \dots + \|x_m^* - x_{m+1}\|_s = \Phi(x_1^*, x_2^*, \dots, x_m^*)$. We are now in a position to prove that σ is a shortest θ-gentle path joining p and q along $\mathcal{F}$.

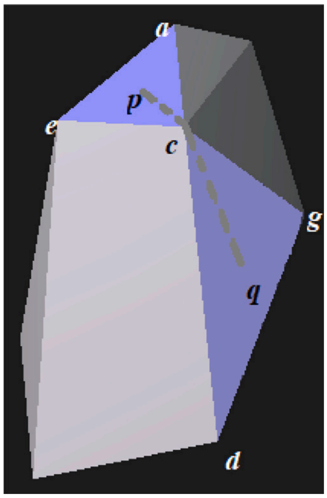


Fig. 5. A pseudo path of shortest θ-gentle paths joining p and q on $\mathcal{T}$ is the polyline joining p, c and q. Therefore there exists a shortest θ-gentle path passing through a convex vertex.

For an arbitrary element $\gamma \in \mathcal{G}_{\mathcal{F}}$, γ is a θ-gentle path joining p and q along $\mathcal{F}$. Let x'_i be the intersection of γ and e_i, for $i = 1, 2, \ldots, m$ and $x'_0 = p, x'_{m+1} = q$. Because x'_i and x'_{i+1} are in the same triangles, the length of θ-gentle sub-path joining x'_i and x'_{i+1} does not exceed that of $\mathrm{SGP}_{f_{i+1}}(x'_i, x'_{i+1})$ that indeed is $\|x'_i - x'_{i+1}\|_s$. Thus $l(\gamma) \geq \sum_{i=0}^{m} \|x'_i - x'_{i+1}\|_s \geq \Phi(x^*_1, x^*_2, \ldots, x^*_m) = l(\sigma)$ for all $\gamma \in \mathcal{G}_{\mathcal{F}}$. Therefore in $\mathcal{G}_{\mathcal{F}}$, σ is a shortest θ-gentle path joining p and q along $\mathcal{F}$, and the proposition is proved. □

Checking whether a vertex of $\mathcal{F}$ is θ-reachable from some points in $\mathcal{F}$ (as in the assumption of Proposition 2) can use the same way of Proposition 1 for triangles of $\mathcal{F}$ (see Fig. 5).

Remark 2. We know that there is no shortest path joining two given points on a polyhedron passing through a convex vertex unless the vertex is the source or the destination of the path, see [3]. Unlike, shortest θ-gentle paths joining two given points on a polyhedral terrain can visit convex vertices as shown below.

Example 1. Consider a terrain $\mathcal{T}$ with nine vertices $a(0.5, 0.5, 1), b(0, 4, 0.5), c(1, 0, 0), d(2, 0, -5), e(0, -1, 0), f(-1, 0.5, -0.5), g(0, -2, -5), h(0, 10, -5), i(-2, 0, -4.5)$. Take $\theta = \pi/3$. Let p and q be points in the relative interior of triangles $\triangle ace$ and $\triangle cdg$ respectively. Then there always exists a shortest θ-gentle path joining p and q on $\mathcal{T}$ passing through the convex vertex $c(1, 0, 0)$. In Example 1, we take $p = (0.541167, -0.182661, 0.276172)$ and $q = (1.11049, -0.182661, -2.53026)$. Then a shortest θ-gentle path joining p and q on $\mathcal{T}$ passes through the convex vertex c and it is an adjusted path of the polyline joining p, c, and q. The length of shortest θ-gentle paths joining p and q on $\mathcal{T}$ is about 3.77925.

4. Finding approximate θ-gentle paths on polyhedral terrains

To deal with the SGP problem, we use the method of multiple shooting applied for shortest path problems in 2D and 3D [9–11]. The iterative algorithm based on MMS gives a sequence of θ-gentle paths whose lengths are descending and convergent. Moreover, when the number of iterative steps is large enough, the path obtained is an approximate shortest θ-gentle path in the sense of Definition 5, which is similar to the definition of an approximate shortest path in [11].

According to Proposition 2, if γ_0 is a shortest θ-gentle path joining p and q on $\mathcal{T}$, then γ_0 is a θ-gentle path, i.e., $\gamma_0 \in \mathcal{G}_{\mathcal{T}}$, and $l(\gamma_0) = \inf_{\gamma \in \mathcal{G}_{\mathcal{T}}} \{l(\gamma)\}$. For the right-hand side of this expression, we replace the infimum that is taken over the whole the set $\mathcal{G}_{\mathcal{T}}$ by the infimum taken over a subset of $\mathcal{G}_{\mathcal{T}}$ such as $\{\gamma^j\}$ in Definition 5. Then we obtain the concept of an approximate shortest θ-gentle path, which is similar to Definition 2.2 in [11] as follows

Definition 5. Let $\{\mathcal{F}^j\}$ be a family of sequences of adjacent triangles of $\mathcal{T}$ joining p and q of $\mathcal{T}$, for $j > 0$. Assume that there exists a sequence of θ-gentle paths joining p to q on $\mathcal{T}$, denoted by $\{\gamma^j\}$, such that: $\gamma^j \subset \mathcal{F}^j$ and $l(\gamma^{j+1}) \leq l(\gamma^j)$, for $j > 0$. Then a θ-gentle path, denoted by γ^*, joining p to q such that

$$l(\gamma^*) = \inf_j \{l(\gamma^j)\},$$

is called an *approximate shortest θ-gentle path* joining p and q on $\mathcal{T}$.

Next, an algorithm based on MMS for finding an approximate θ-gentle path consists of three following factors

4.1. The factor (f1): Partition

Assuming $v \in \mathcal{T}$. A *cutting slice* of $\mathcal{T}$ is the polygon which is the intersection of $\mathcal{T}$ and the plane through v that is parallel to the horizontal plane. Let $p, q \in \mathcal{T}$ such that $z(p) > z(q)$. We assume throughout the paper that there is at least a vertex v of $\mathcal{T}$ between p and q in terms of height, i.e., $z(p) > z(v) > z(q)$. $\mathcal{T}$ is divided into suitable sub-terrains $\mathcal{T}_i$ by a set of cutting slices $\{\xi_0, \xi_1, \ldots, \xi_{k+1}\}$ passing through all vertices of $\mathcal{T}$ between p and q as follows

$$z(\xi_i) > z(\xi_{i+1}), \text{there is no vertex of } \mathcal{T} \text{ between planes } \xi_i \text{ and } \xi_{i+1}. \tag{5}$$
$$\mathcal{T}_i \text{ is a surface bounded by } \mathcal{T}, \xi_i \text{ and } \xi_{i+1}.$$

for $i = 0, 1, \ldots, k$, where ξ_0 and ξ_{k+1} are cutting slices of $\mathcal{T}$ through p and q, respectively, $z(\xi_i)$ denotes the height of ξ_i, i.e., the height of an arbitrary point of ξ_i.

Suppose that $\mathcal{T}$ has the property that ξ_i is connected, for $i = 1, 2, \ldots, k$. This condition is given to ensure that for any two points a and b on the boundary of ξ_i and ξ_{i+1} (denoted by $a \in \mathrm{bd}\xi_i, b \in \mathrm{bd}\xi_{i+1}$), there exists a path on $\mathcal{T}_i$ which joins a and b. Obviously, if $\mathcal{T}$ is convex, then the condition is completely satisfied. Furthermore, the surface of $\mathcal{T}_i$ is assumed to contain no vertices of $\mathcal{T}$ except vertices having the same height as v_i and v_{i+1}. Since the surface of $\mathcal{T}_i$ is a sequence of adjacent polygons, we can certainly assume that adjacent polygons are triangles, for if not, we can triangulate these polygons to obtain triangles.

At the first step of the algorithm, we initialize a set of ordered points by taking $u_i \in \xi_i$. Since the connectedness assumption of ξ_i, there exists a sequence of adjacent triangles of $\mathcal{T}_i$, denoted by $\mathcal{F}_i$, that contains u_i and u_{i+1}, where u_i belongs to the first triangle and u_{i+1} belongs to the final triangle of $\mathcal{F}_i$, for $i = 0, 1, \ldots, k$ (see Fig. 6).

Each triangle of $\mathcal{F}_i$ has an edge that is parallel to the horizontal plane. Indeed, because each vertex v of any triangle of $\mathcal{F}_i$ is a vertex of the terrain $\mathcal{T}$ or an intersection point of a cutting slice and an edge of $\mathcal{T}$. If v is a vertex of $\mathcal{T}$, then $v \in \mathrm{bd}\xi_i$ or $v \in \mathrm{bd}\xi_{i+1}$, due to the construction of cutting slices passing through all vertices of $\mathcal{T}$ between p and q. If v is an intersection point of a cutting slice and an edge of $\mathcal{T}$, then $v \in \mathrm{bd}\xi_i$ or $v \in \mathrm{bd}\xi_{i+1}$. Each triangle has three vertices, then there are two vertices of each triangle of $\mathcal{F}_i$ belonging to the same cutting slice. Consequently, each triangle of $\mathcal{F}_i$ has an edge that is parallel to the horizontal plane, and thus we have the following remark

Remark 3. Every vertex in $\mathcal{F}_i$ is θ-reachable from some points in $\mathcal{F}_i$. Therefore there exists at least one shortest θ-gentle path joining two given points p and q along $\mathcal{F}_i$, where $p, q \in \mathcal{F}_i$.

Two consecutive initial points are connected by a shortest θ-gentle path along $\mathcal{F}_i$. The path received by combining these shortest θ-gentle paths is called the initial path of the algorithm. For each iterative step which is discussed carefully in the next Sections (4.2 and 4.3),

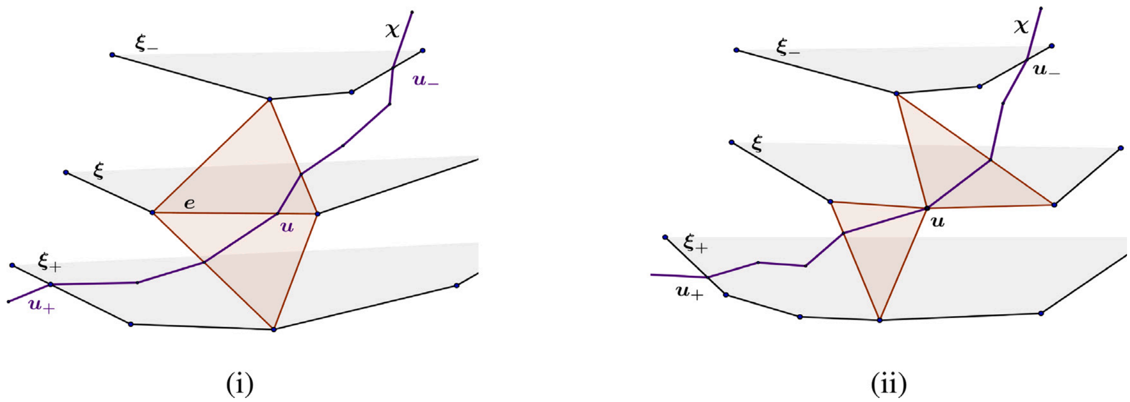


Fig. 6. (i) χ passes through two triangles sharing e which contains u, (ii) χ passes through two triangles sharing only one point that indeed is u.

we obtain a set of points $\{u_i | u_i \in \xi_i, i = 0, 1, \ldots, k+1\}$ and a path $\gamma = \cup_{i=0}^{k} \mathrm{SGP}_{\mathcal{F}_i}(u_i, u_{i+1})$, where $\mathrm{SGP}_{\mathcal{F}_i}(u_i, u_{i+1})$ is the shortest θ-gentle path joining u_i and u_{i+1} along $\mathcal{F}_i$ and $u_0 = p, u_{k+1} = q$. Then u_i is called a *shooting point*.

4.2. The factor (f2): Straightness condition

For simplicity, in Sections 4.2 and 4.3, we use subscripts "+" and "-" to stand for ascending or descending. For instance, if we write u instead of u_i, then u_+ and u_- stand for u_{i+1} and u_{i-1}, respectively. We also use $\cup \mathrm{SGP}_{\mathcal{F}}(u, u_+)$ instead of $\cup_{i=0}^{k} \mathrm{SGP}_{\mathcal{F}_i}(u_i, u_{i+1})$, where $\mathcal{F}$ is understood as the sequence of adjacent triangles joining u and u_+. Thus $\mathcal{F}_-$ is the sequence of adjacent triangles joining u_- and u.

Proposition 3. *Assume that χ is a pseudo path of shortest θ-gentle paths joining p and q along a sequence of adjacent triangles $\hat{\mathcal{F}}$. Let $x \in \chi$. Then there exists at least one shortest θ-gentle path γ joining p and q along $\hat{\mathcal{F}}$ such that χ is the pseudo path of γ and $x \in \gamma$.*

The proof of Proposition 3 is given in Appendix.

We consider $\chi = \cup \mathrm{PSGP}_{\mathcal{F}}(u, u_+)$, where $\mathrm{PSGP}_{\mathcal{F}}(u, u_+)$ is the pseudo path of $\mathrm{SGP}_{\mathcal{F}}(u, u_+)$. According to Proposition 3, we get

Remark 4. Let $x \in \chi$. Then there exists at least one sub-path $\mathrm{SGP}_{\mathcal{F}}(u, u_+)$ containing x.

There are two cases for the shooting point u $(u \neq p, q)$ as follows

Case 1 (Common edge case for u): χ passes through two triangles of $\mathcal{F}_- \cup \mathcal{F}$, which have only one common edge $e \subset \xi$ and $u \in e$ (see Fig. 6(i)). Here "χ passes through two triangles" means that points of the intersection of χ and each triangle are at least two points,

Case 2 (Non-common edge case for u): χ passes through two triangles of $\mathcal{F}_- \cup \mathcal{F}$, which have only one common point that is u (see Fig. 6(ii)).

Consider the triple $(u_-, u, u_+) \in \xi_- \times \xi \times \xi_+$, we take $x \in \mathrm{PSGP}(u, u_+)$, $x_- \in \mathrm{PSGP}(u_-, u)$ such that x, x_- do not coincide with u, u_-, u_+. We need to construct a sequence of adjacent triangles from $\mathcal{F}_-$ and $\mathcal{F}$ joining x_- and x, denoted by $\mathcal{F}_- \bowtie \mathcal{F}$ as follows:

(A1) If Common Edge Case for u happens, $\mathcal{F}_- \bowtie \mathcal{F} = \mathcal{F}_- \cup \mathcal{F}$,

(A2) If Non-common Edge Case for u happens, $\mathcal{F}_- \bowtie \mathcal{F}$ consists of triangles of $\mathcal{F}_- \cup \mathcal{F}$ and the minimum number of adjacent triangles having the common vertex u such that $\mathcal{F}_- \bowtie \mathcal{F}$ contains the sub-path of χ joining x_- and x.
Since χ passes through u, some triangles sharing u in the counter-clockwise or clockwise direction are added to $\mathcal{F}_- \bowtie \mathcal{F}$ to ensure the adjacency of triangles. There are two such sequences, denote by $\mathcal{F}_- \bowtie \mathcal{F}^{ccw}$ and $\mathcal{F}_- \bowtie \mathcal{F}^{cw}$.

We now state the factor (f2): Straightness Condition For each shooting point u, we check a so-called *Straightness Condition* to decide whether the algorithm stops or continues. We say that the straightness condition holds at u $(u \neq p, q)$ if $\mathrm{PSGP}_{\mathcal{F}_-}(x_-, u) \cup \mathrm{PSGP}_{\mathcal{F}}(u, x)$ is a pseudo path of shortest θ-gentle paths joining x_- and x along $\mathcal{F}_- \bowtie \mathcal{F}$. In particular,

(B1) If Common Edge Case for u happens, then the straightness condition is

$$lp\left(\mathrm{PSGP}_{\mathcal{F}_- \bowtie \mathcal{F}}(x_-, x)\right) = lp\left(\mathrm{PSGP}_{\mathcal{F}_-}(x_-, u)\right) + lp\left(\mathrm{PSGP}_{\mathcal{F}}(u, x)\right),$$

(B2) If Non-common Edge Case for u happens, then the straightness condition is

$$\begin{aligned} lp\left(\mathrm{PSGP}_{\mathcal{F}_- \bowtie \mathcal{F}^{ccw}}(x_-, x)\right) &= lp\left(\mathrm{PSGP}_{\mathcal{F}_- \bowtie \mathcal{F}^{cw}}(x_-, x)\right) \\ &= lp\left(\mathrm{PSGP}_{\mathcal{F}_-}(x_-, u)\right) + lp\left(\mathrm{PSGP}_{\mathcal{F}}(u, x)\right). \end{aligned}$$

4.3. The factor (f3): Update of shooting points

If Straightness Condition (B1–B2) holds at all shooting points, the algorithm stops, otherwise, when Straightness Condition (B1–B2) does not hold at least one shooting point, we update u to u^{next} $(u \neq p, q)$. By Proposition 5, which will be presented in Section 4.4, the length of the path formed by the set of new shooting points is less than that of the previous path.

Then we update u to u^{next} where u^{next} is selected as an intersection point between a pseudo path of shortest θ-gentle paths joining x_- and x along $\mathcal{F}_- \bowtie \mathcal{F}$ and edges of ξ. Particularly,

(C1) If Common Edge Case for u happens and u does not satisfy Straightness Condition (B1), we then set u^{next} as the intersection point between $\mathrm{PSGP}_{\mathcal{F}_- \bowtie \mathcal{F}}(x_-, x)$ and e, where e is the common edge of two triangles which is passed by χ (see Fig. 7(i)),

(C2) If Non-common Edge Case for u happens and u does not satisfy Straightness Condition (B2), then there are two edges e, e' sharing u such that they are edges of ξ. We then set u^{next} as the intersection point between $\mathrm{PSGP}_{\mathcal{F}_- \bowtie \mathcal{F}}(x_-, x)$ and e or e' (see Fig. 7(ii)). We denote

$$lp\left(\mathrm{PSGP}_{\mathcal{F}_- \bowtie \mathcal{F}}(x_-, x)\right) = \min\{lp\left(\mathrm{PSGP}_{\mathcal{F}_- \bowtie \mathcal{F}^{ccw}}(x_-, x)\right), lp\left(\mathrm{PSGP}_{\mathcal{F}_- \bowtie \mathcal{F}^{cw}}(x_-, x)\right)\}.$$

Since Straightness Condition (B1–B2) does not hold at u, we can easily claim the following for (C1–C2)

$$lp\left(\mathrm{PSGP}_{\mathcal{F}_- \bowtie \mathcal{F}}(x_-, x)\right) < lp\left(\mathrm{PSGP}_{\mathcal{F}_-}(x_-, u)\right) + lp\left(\mathrm{PSGP}_{\mathcal{F}}(u, x)\right). \quad (6)$$

We can update shooting points in which the straightness condition does not hold and keep the remaining shooting points. But the proposed algorithm will update all shooting points. Because if u satisfies

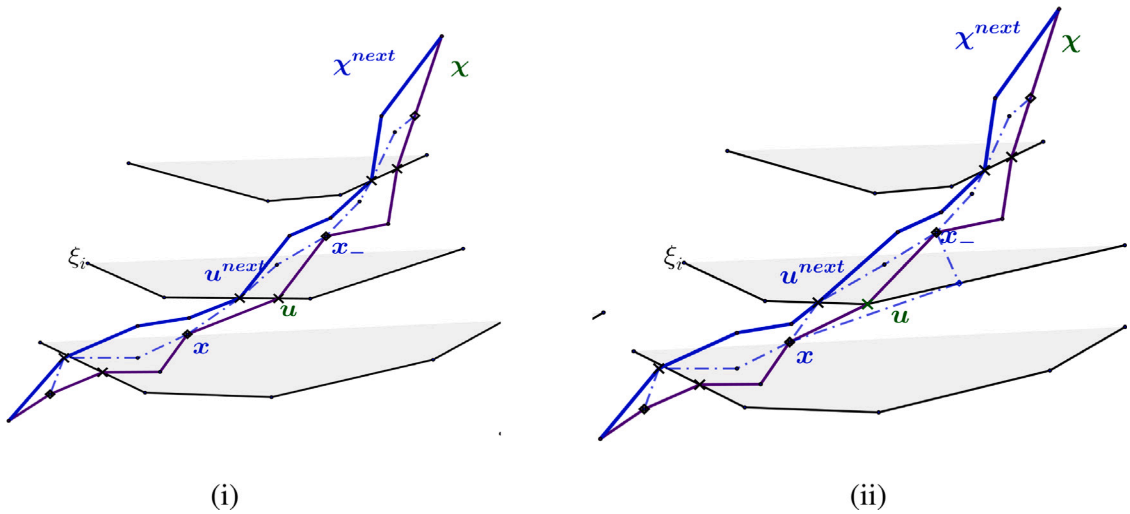


Fig. 7. After updating shooting points u to u^{next}, we get $\chi^{next} = \cup\text{PSGP}_{\mathcal{F}^{next}}(u^{next}, u_+^{next})$ from $\chi = \cup\text{PSGP}_{\mathcal{F}}(u, u_+)$, where (i) illustrates Common Edge Case for u and (ii) illustrates Non-common Edge Case for u.

Straightness Condition (B1–B2), then the update (C1–C2) gives $u = u^{next}$.

4.4. Proposed algorithm

The algorithm for finding an approximate shortest θ-gentle path joining two given points on terrain is as follows:

Input: A terrain $\mathcal{T}$, $\theta \in (0, \pi/2)$ and $p, q \in \mathcal{T}$
Output: An approximate shortest θ-gentle path joining p and q on $\mathcal{T}$

1: Divide $\mathcal{T}$ into sub-terrains by a set of cutting slices satisfying (5) ▷ *partition*
2: Take a set of ordered initial shooting points. Let χ be the pseudo path of the path formed by the set of initial shooting points, $flag = true$
3: Call Procedure Straightness_Update $(\chi, \chi^{next}, flag)$
4: If Straightness Condition (B1-B2) holds at all shooting points, i.e., $flag = true$, **go to** step 6
5: Otherwise, i.e, $flag = false$, then $\chi \leftarrow \chi^{next}$ and **go to** step 3
6: Adjust the pseudo path χ^{next} to get a θ-gentle path γ ▷ γ *satisfies slope requirement and its length is computed by* (4)
7: **return** γ.

In the rest of the paper, we turn to use subscripts $i-1$, i, $i+1$ for shooting points, sequences of adjacent triangles, etc corresponding to each cutting slice.

For each iterative step, if Straightness Condition (B1–B2) does not hold, we update shooting points to get a better path. Then Procedure Straightness_Update $(\chi, \chi^{next}, flag)$ performs checking Straightness Condition and updating shooting points u_i to u_i^{next} $(u_i \neq p, q)$. Note that at the next iterative step, after updating shooting points, we use the same way indicated in Section 4.1 to construct the sequence $\mathcal{F}_i^{next}$ joining u_i^{next} and u_{i+1}^{next}. Proposition 5 shows the decrease in the sequence of lengths of obtained paths.

Proposition 4. *Assume that γ_0 is a shortest θ-gentle path joining p and q on $\mathcal{T}$. Let u_i be the intersection point between γ_0 and each cutting slice ξ_i which is not through p and q. Then Straightness Condition (B1–B2) holds at u_i.*

The proof of Proposition 4 is given in Appendix.

Proposition 5. *Procedure* Straightness_Update $(\chi, \chi^{next}, flag)$ *gives $lp(\chi^{next}) \leq lp(\chi)$, i.e.,*

$$l\left(\bigcup_{i=0}^{k} \text{SGP}_{\mathcal{F}_i^{next}}(u_i^{next}, u_{i+1}^{next})\right) \leq l\left(\bigcup_{i=0}^{k} \text{SGP}_{\mathcal{F}_i}(u_i, u_{i+1})\right).$$

procedure Straightness_Update$(\chi, \chi^{next}, flag)$
Input: χ is a pseudo path joining shooting points u_i, i.e., $\chi = \cup_{i=0}^{k}\text{PSGP}(u_i, u_{i+1})$.
Output: Determined if $flag = true$ or $flag = false$ and the set of new shooting points such that $lp(\chi^{next}) \leq lp(\chi)$, where χ^{next} is a pseudo path joining new shooting points u_i^{next}.
  $flag \leftarrow true$
  $x_0 \in \text{PSGP}(u_0, u_1)$ such that $x_0 \neq u_0, x_0 \neq u_1$
  for $i = 1, 2, \dots, k$ **do**
    Take $x_i \in \text{PSGP}_{\mathcal{F}_i}(u_i, u_{i+1})$ such that x_i does not coincide with u_i, u_{i+1}
    if $\text{PSGP}_{\mathcal{F}_{i-1}}(x_{i-1}, u_1) \cup \text{PSGP}_{\mathcal{F}_i}(u_i, x_i)$ is a pseudo path of shortest θ-gentle paths joining x_{i-1} and x_i along $\mathcal{F}_{i-1} \bowtie \mathcal{F}_i$ **then** ▷ *Straightness Condition* (B1-B2) *holds at* u_i
      Set $u_i^{next} \leftarrow u_i$
    else Set u_i^{next} to be the intersection of $\text{PSGP}_{\mathcal{F}_{i-1} \bowtie \mathcal{F}_i}(x_{i-1}, x_i)$ and edges of ξ_i ▷ *due to (C1-C2)*
      $flag \leftarrow false$
  $u_0^{next} = p, u_{k+1}^{next} = q$
  $\chi^{next} = \cup\text{PSGP}_{\mathcal{F}_i^{next}}(u_i^{next}, u_{i+1}^{next})$

If Straightness Condition (B1–B2) does not hold at some shooting point u_i, then the inequality above is strict. Therefore the sequence of lengths of paths obtained by our algorithm is convergent.

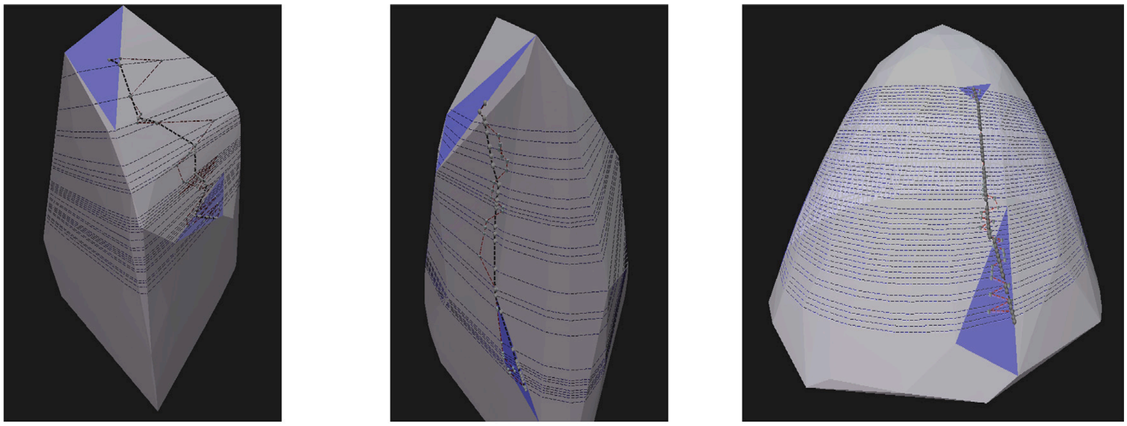
The proof of Proposition 5 is given in Appendix.

Theorem 1. *The path obtained at jth-iterative step of the proposed algorithm is an approximate shortest θ-gentle path joining p and q on $\mathcal{T}$, where j is a large enough natural number.*

The proof of Theorem 1 is given in Appendix.

5. Numerical examples

In the section, we implement the proposed algorithm in C++ code using CGAL then compile and run the code on Ubuntu Linux platform Intel Core i5-7200U, CPU 568 2.50 GHz with 4 GB RAM. Our experiments are executed on convex polyhedral terrains to ensure the assumption of the connectedness of cutting slices (see Section 4.1).



(i) (ii) (iii)

Fig. 8. A visualization of the proposed algorithm on convex polyhedral terrains: the boundaries of cutting slices are represented by parallel dashed curves; the light brown curves are the initial pseudo paths of the initial θ-gentle paths formed by the set of shooting points; the dark gray curves show the final pseudo paths which are pseudo paths of the final θ-gentle paths formed by the set of final shooting points obtained by MMS. Therein, the lengths of the initial and final pseudo paths under $\|.\|_s$ norm are the lengths of the initial and final θ-gentle paths under the Euclidean norm.

We run the algorithms several times to compute the lengths of obtained paths and corresponding running times. Average results are then reported. The results are visualized by OpenGL.

Fig. 8 gives a visualization of the proposed algorithm in which the boundaries of cutting slices are represented by parallel dashed curves, the initial pseudo paths are shown in light brown and the final pseudo paths obtained by MMS are shown in dark gray.

For computing shortest θ-gentle paths joining two shooting points on consecutive cutting slices, we use an algorithm for finding classical shortest paths on convex polyhedral surfaces introduced by Algawal et al. [13]. Polyhedral terrains is discretized in the way of Algawal et al. to obtain a weighted graph, where the weight of each edge in the graph is the length under $\|.\|_s$ calculated by the formula (3). We then find the weighted shortest path between two shooting points, which plays a role as a shortest θ-gentle path joining two shooting points.

For finding a shortest θ-gentle path joining two points p and q on a polyhedral terrain, we test the proposed algorithm on the datasets given in Table 1 which consists of convex polyhedral terrains of a few tens to several tens of thousands of vertices to show the correctness of our algorithm. Each terrain was generated randomly and the pair of p and q also is randomly selected on each of the terrains for the first test. Table 2 reports results obtained by the proposed algorithm for the input data set corresponding to Table 1, where the slope parameter θ was fixed as 0.3. In order to evaluate the MMS-based algorithm, we use the shortest path joining p and q without the slope constraint on the terrains which can be obtained by any algorithm for finding shortest paths. We then perform the step of adjusting the path to get a θ-gentle path, denoted by $\bar{\gamma}(p,q)$.

Table 2 also gives a comparison between the lengths of final θ-gentle paths obtained by MMS and that of $\bar{\gamma}(p,q)$. In general, the final θ-gentle paths obtained by MMS are shorter than $\bar{\gamma}(p,q)$. The last row of the table presents the test of the algorithm for a complicated terrain consisting of 17,115 vertices and 34,226 triangular faces. The running time is long due to the current specification of the PC platform. Herein, it shows the feasibility of the algorithm. For more complicated terrains of huge size, we can apply an approximate scheme in which an original terrain can be approximated by a simpler surface. A shortest θ-gentle path is computed on the simplified surface. It is then used to approximately obtain the corresponding shortest θ-gentle path on the original one. Such a method was given in [8].

Clearly, the running time of the algorithm based on MMS is influenced by factors: the slope parameter θ, the size of the data sets (the number of vertices of terrains), the initial path, the number of iterations, and the number of cutting slices. We use the term the average running time which is the quotient of the total running time and the number of iterative steps to evaluate the quality of the proposed algorithm. The values of running time and average time of each iteration given in Table 2 also indicate that the computing times increase as the size of input terrains raises.

We also test the algorithm for different values of θ. Fig. 9 shows the testing results. The experiment is executed on terrains of 118, 290, and 414 vertices with p and q given in Table 1. We can see that the running time has no clear pattern as a function of θ, while the length of obtained θ-gentle paths is decreasing when θ increases. This is because the length of a shortest θ-gentle path tends to that of shortest paths without slope constraints as θ varies from 0 to $\frac{\pi}{2}$.

A comparison with Liu and Wong's algorithm

In Liu and Wong's algorithm [8], a surface simplification is performed to simplify a terrain. Shortest θ-gentle paths are then computed based on the simplified surface. We here aim to compare the performance of finding shortest θ-gentle paths. Therefore, we do not use simplification in this comparison.

For finding a shortest θ-gentle path joining two points p and q on a terrain $\mathcal{T}$, Liu and Wong used a naive search algorithm to search all non-self-cutting sequences of adjacent triangles joining p and q on the entire $\mathcal{T}$. They implemented the brute-force approach to find all possible candidates for the solution and then constructed a tree of sequences of adjacent triangles joining p and q to find a shortest θ-gentle path along each sequence. Then sub-problems of finding shortest θ-gentle paths joining those two points along a sequence of adjacent triangles are solved to obtain an exact shortest θ-gentle path.

In particular, for each sequence of m adjacent triangles, the length of the shortest θ-gentle paths can be represented by a function in m-variables of the coordinates of p, q and all the vertices of the triangles in this sequence. Slope constraints can be specified by using the coordinates of these vertices and θ. They referred to minimizing the function of the total length under $\|.\|_s$ norm of a polyline whose endpoints belong to common edges of adjacent triangles. There are two drawbacks to this approach: its dependence on optimization tools for solving non-smooth convex optimization problems and the time consume to execute the brute-force search.

Table 3 gives the comparison result between two algorithms. As shown in the table, the lengths of the final θ-gentle paths obtained by our algorithm are similar to that of the shortest θ-gentle paths by Liu and Wong's one. It however takes too long time to complete Liu and Wong's algorithm. For terrains of more than or equal to 20 vertices, the consuming time of Liu and Wong's algorithm is greater than 24 h (= 86,400 s).

Table 1
The coordinates of p to q corresponding to the polyhedral terrains for the implementation.

Num. of vertices	Num. of faces	$p = (x(p), y(p), z(p))$	$q = (x(q), y(q), z(q))$
30	56	$(38.7153, -17.2593, 81.5168)$	$(20.2045, -17.2593, 62.1578)$
69	134	$(39.291, 23.4091, 157.183)$	$(22.4181, 23.4091, 61.4292)$
110	216	$(94.4766, 29.6616, 166.05)$	$(124.73, 29.6616, 87.4252)$
118	232	$(75.5987, 25.7441, 184.372)$	$(100.763, 25.7441, 105.191)$
290	576	$(18.6394, -31.885, -14.3982)$	$(69.8596, -31.885, -62.0098)$
414	824	$(57.1076, -39.3121, 153.314)$	$(88.2201, -39.3121, 77.9623)$
735	1,446	$(56.7353, -30.2738, 157.961)$	$(73.0712, -30.2738, 69.1942)$
1,065	2,126	$(44.7474, -63.5466, 155.602)$	$(69.907, -63.5466, 74.4962)$
1,280	2,556	$(56.9648, -41.0489, 153.936)$	$(92.906, -41.0489, 66.261)$
1,516	3,028	$(73.486, -5.976, 133.893)$	$(88.1203, -5.976, 93.4566)$
7,607	15,210	$(259.728, -586.539, -411.306)$	$(308.102, -586.539, -554.833)$
17,115	34,226	$(2526.01, 247.731, -645.779)$	$(3436.61, 247.731, -1183.03)$

Table 2
Experiments of finding θ-gentle paths on convex polyhedral terrains by MMS, where p, q, and terrains are given in Table 1, and $\theta = 0.3$.

Num. of vertices	Num. of cutting slices[a]	Num. of iterations	Length of $\bar{\gamma}(p,q)$[b]	Length of the initial θ-gentle paths	Length of the final θ-gentle paths	Running time (s)	Average time for each iteration (s)
30	7	21	85.41195	189.97232	68.1006	8.713	0.415
69	27	13	324.14001	409.69764	324.01891	8.661	0.666
110[c]	28	14	272.33611	527.21885	267.08084	18.555	1.325
118[d]	26	46	269.07095	493.82249	267.93636	59.790	1.299
290[e]	45	62	164.04168	198.72688	162.37734	101.997	1.645
414	62	56	264.91539	371.28881	257.30488	197.710	3.530
735	85	14	308.29389	336.17432	300.78357	32.748	2.339
1,065	80	11	278.76214	350.64900	274.44965	68.633	6.239
1,280	87	18	302.70465	374.82565	296.68171	234.84	13.047
1,516	40	25	137.73332	146.88255	136.83118	293.155	11.726
7,607	143	14	486.25417	991.86302	485.67421	671.947	47.996
17,115	453	26	1,917.98817	8,427.00653	1,821.00829	13,715.133	527.505

[a]The number of cutting slices except for two slices passing through p and q.
[b]$\bar{\gamma}(p,q)$ is a θ-gentle path adjusted from the shortest path p to q on these terrains.
[c]Example showed in Fig. 8 (i).
[d]Example showed in Fig. 8 (ii).
[e]Example showed in Fig. 8 (iii).

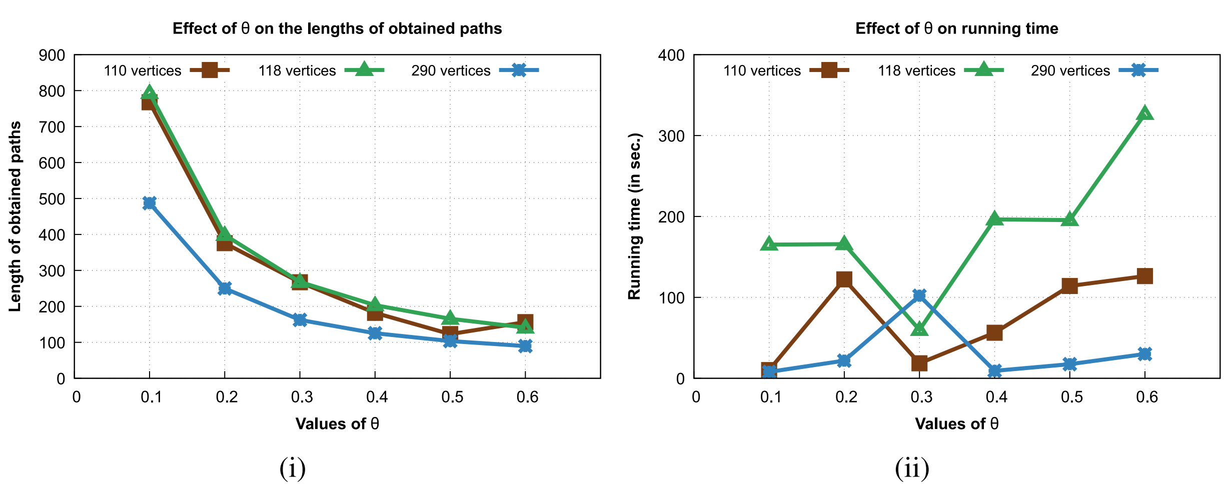


Fig. 9. Effect of the slope parameter θ on the length of the obtained θ-gentle paths and the running time of MMS, where experiments are processed on terrains of 110, 118, and 290 vertices with the corresponding coordinates of p and q are given in Table 1.

Table 3
Comparison between Liu and Wong's algorithm and MMS with $\theta = 0.2$.

Num. of	Liu and Wong's algorithm				MMS	
vertices	Num. of sub-problems	Max. num. of variables	Length	Running time (s)	Length	Running time(s)
10	158	14	210.773	1,106.136	211.37189	10.895
15	3,176	24	456.156	59,294.118	457.10281	18.781
20	69,809	34	303.173	> 86,400.000	309.06892	4.435
25	2,002,522	44	368.0512	> 86,400.000	368.72831	6.708
30	32,320,556	54	97.11615	> 86,400.000	100.06106	3.243

Due to the use of the brute-force approach, the calculation speed and the number of sub-problems and their number of variables are increasing if the number of vertices of terrains raises. The running time of Liu and Wong's algorithm is thousands of times higher than our algorithm. However, the input data of MMS requires terrains that satisfy the connectedness assumption of cutting slices, while Liu and Wong's algorithm can be applied for arbitrary terrains. It should also note that Liu and Wong's algorithm focuses on simplifying terrains to reduce the number of their triangles rather than solving the SGP problem.

6. Concluding remarks

Generally, the problem of finding shortest θ-gentle path is quite hard to solve, and no polynomial time algorithm is known. Moreover, not many properties of shortest θ-gentle paths have been shown. In this paper, we give some geometric and analysis properties of shortest θ-gentle paths: characterization of a θ-unreachable vertex, the existence of shortest θ gentle paths along a sequence of adjacent triangles, and an example of shortest θ-gentle paths which can visit a convex vertex of a polyhedral terrain. After that, we present an algorithm for computing an approximate shortest θ-gentle path joining two given points on a polyhedral terrain using a new approach based on the method of multiple shooting, but under the assumption of the connectedness given in Section 4.1 for terrains.

Ideas of MMS may be used for finding energy-minimizing paths on terrains [14]. It is also suitable for the problem of finding shortest paths on polyhedral terrains where its faces are weighted polygons. In the next paper, we will deal with such problems.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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Appendix

The proof of Proposition 3:

Proof. If x is an intersection point of χ and some edge of a triangle of $\hat{\mathcal{F}}$, then x clearly belongs to shortest θ-gentle paths whose pseudo is χ. Otherwise, x is then a point in some line segment, say $[a,b]$, such that a and b are intersection points between χ and edges of a triangle in $\hat{\mathcal{F}}$ ($x \neq a, b$). Let us denote by f the triangle. We consider two cases:

(a) If $b \notin \mathrm{SR}_f(a)$, where $\mathrm{SR}_f(a)$ is the steep region of a in f, then $[a,b]$ belongs to all shortest θ-gentle paths whose pseudo is χ and so is x.

(b) If $b \in \mathrm{SR}_f(a)$, then $[a,b]$ is steep. By constructing the adjusted paths, $\mathrm{adj}[a,x] \cup \mathrm{adj}[x,b]$ is a θ-gentle path that lies completely in f. Since $x \in [a,b]$,

$$l(\mathrm{adj}[a,x] \cup \mathrm{adj}[x,b]) = \frac{|z(x)-z(a)|}{\sin\theta} + \frac{|z(b)-z(x)|}{\sin\theta} = \frac{|z(b)-z(a)|}{\sin\theta} = l(\mathrm{adj}[a,b]).$$

Consequently, $\mathrm{adj}[a,x] \cup \mathrm{adj}[x,b]$ is also a shortest θ-gentle path joining a and b in f. Let τ be a shortest θ-gentle path whose pseudo path is χ. Then $a, b \in \tau$. We can replace the sub-path of τ joining a and b with $\mathrm{adj}[a,x] \cup \mathrm{adj}[x,b]$ to obtain a θ-gentle path, denoted by γ in which its length do not exceed that of τ, which completes the proof. □

The proof of Proposition 4:

Proof. Let $u_0 = p, u_{k+1} = q$. In what follows, for $a, b \in \gamma_0$, $\gamma_0(a,b)$ stands for the sub-path joining a and b of γ_0. We can assume that $\mathcal{F}_i$ is a sequence of adjacent triangles joining u_i and u_{i+1}, for $i = 0, 1, \ldots, k$ such that $\hat{\mathcal{F}} = \bigcup_{i=0}^{k} \mathcal{F}_i$ is a sequence of adjacent triangles along γ_0. It is easy to check that any sub-path of a shortest θ-gentle path is also a shortest θ-gentle path. Then we can write $\gamma_0 = \cup_{i=0}^{k} \gamma_0(u_i, u_{i+1})$, where $\gamma_0(u_i, u_{i+1})$ is a shortest θ-gentle path joining u_i and u_{i+1} along $\mathcal{F}_i$. Let us denote by $\mathrm{PSGP}_{\mathcal{F}_i}(u_i, u_{i+1})$ the pseudo path of $\gamma_0(u_i, u_{i+1})$. For $i = 0, 1, \ldots, k$, take x_i on $\mathrm{PSGP}_{\mathcal{F}_i}(u_i, u_{i+1})$ such that x_i does not coincide with u_i and u_{i+1}. According to Remark 4, there exists at least a shortest θ-gentle path joining u_i and u_{i+1} along $\mathcal{F}_i$ such that x_i belongs to it. Therefore without loss of generality, we can assume that $x_i \in \gamma_0(u_i, u_{i+1})$, for $i = 0, 1, \ldots, k$. Then $\gamma_0(x_{i-1}, x_i)$ is a shortest θ-gentle path joining x_{i-1} and x_i on the entire $\mathcal{T}$. If Common Edge Case for u_i happens, $\gamma_0(x_{i-1}, x_i)$ is a shortest θ-gentle path along $\mathcal{F}_{i-1} \bowtie \mathcal{F}_i$. Therefore (B1) of Straightness Condition must happen at u_i.

If Non-common Edge Case for u_i happens, $\gamma_0(x_{i-1}, x_i)$ is a shortest θ-gentle path joining x_{i-1} and x_i along both $\mathcal{F}_{i-1} \bowtie \mathcal{F}_i^{ccw}$ and $\mathcal{F}_{i-1} \bowtie \mathcal{F}_i^{cw}$. Therefore (B2) of Straightness Condition must happen at u_i, which completes the proof. □

Remark 5. By constructing sequences of adjacent triangles $\mathcal{F}_i$, $\mathcal{F}_i^{next}$, and $\mathcal{F}_{i-1} \bowtie \mathcal{F}_i$, we deduce that

$$l\left(\mathrm{SGP}_{\mathcal{F}_{i-1}\bowtie\mathcal{F}_i}(x_{i-1}, x_i)\right) \le l\left(\mathrm{SGP}_{\mathcal{F}_{i-1}}(x_{i-1}, u_i^{next})\right) + l\left(\mathrm{SGP}_{\mathcal{F}_i}(u_i^{next}, x_i)\right);$$

$$l\left(\mathrm{SGP}_{\mathcal{F}_i^{next}}(u_i^{next}, u_{i+1}^{next})\right) \le l\left(\mathrm{SGP}_{\mathcal{F}_{i-1}\bowtie\mathcal{F}_i}(u_i^{next}, x_i)\right) + l\left(\mathrm{SGP}_{\mathcal{F}_i\bowtie\mathcal{F}_{i+1}}(x_i, u_{i+1}^{next})\right).$$

The proof of Proposition 5:

Proof. Assume that Straightness Condition (B1–B2) does not hold at some shooting point u_i, $\gamma = \cup_{i=0}^{k} \mathrm{SGP}_{\mathcal{F}_i}(u_i, u_{i+1})$. According to Remark 4 and $x_i \in \mathrm{PSGP}_{\mathcal{F}_i}(u_i, u_{i+1})$, x_i belongs to $\mathrm{SGP}_{\mathcal{F}_i}(u_i, u_{i+1})$, for $j = 0, 1, \ldots, k$ (see Fig. 7). Hence instead of proving the results on the length of shortest θ-gentle paths, we can move to prove the corresponding results about the sum of $\|.\|_s$ norms of their pseudo paths.

On the sub-terrain $\mathcal{T}_{i-1}$, we have:

$$lp\left(\mathrm{PSGP}_{\mathcal{F}_{i-1}}(u_{i-1}, u_i)\right) = lp\left(\mathrm{PSGP}_{\mathcal{F}_{i-1}}(u_{i-1}, x_{i-1})\right) + lp\left(\mathrm{PSGP}_{\mathcal{F}_{i-1}}(x_{i-1}, u_i)\right). \tag{7}$$

On the sub-terrain $\mathcal{T}_i$, we get:

$$lp\left(\mathrm{PSGP}_{\mathcal{F}_i}(u_i, u_{i+1})\right) = lp\left(\mathrm{PSGP}_{\mathcal{F}_i}(u_i, x_i)\right) + lp\left(\mathrm{PSGP}_{\mathcal{F}_i}(x_i, u_{i+1})\right). \tag{8}$$

Combining Remark 5, with formulas (7) and (8), yields:

$$l\left(\bigcup_{i=0}^{k} \mathrm{SGP}_{\mathcal{F}_i}(u_i, u_{i+1})\right) = \sum_{i=0}^{k} l\left(\mathrm{SGP}_{\mathcal{F}_i}(u_i, u_{i+1})\right) = \sum_{i=0}^{k} lp\left(\mathrm{PSGP}_{\mathcal{F}_i}(u_i, u_{i+1})\right)$$

$$= lp\left(\mathrm{PSGP}_{\mathcal{F}_0}(u_0,x_0)\right) + \sum_{i=1}^{k-1} lp\left(\mathrm{PSGP}_{\mathcal{F}_{i-1}}(x_{i-1},u_i)\cup \mathrm{PSGP}_{\mathcal{F}_i}(u_i,x_i)\right) + lp\left(\mathrm{PSGP}_{\mathcal{F}_k}(x_k,u_{k+1})\right)$$

Since the construction of the sequence of adjacent triangles $\mathcal{F}_{i-1}\bowtie\mathcal{F}_i$ from $\mathcal{F}_i$ by adding triangles to ensure the adjacency of triangles in $\mathcal{F}_{i-1}\bowtie\mathcal{F}_i$, we see that:

$$lp\left(\mathrm{PSGP}_{\mathcal{F}_{i-1}}(x_{i-1},u_i)\cup \mathrm{PSGP}_{\mathcal{F}_i}(u_i,x_i)\right) \ge lp\left(\mathrm{PSGP}_{\mathcal{F}_{i-1}\bowtie\mathcal{F}_i}(x_{i-1},x_i)\right). \quad (9)$$

Updating shooting points u_i^* yields:

$$lp\left(\mathrm{PSGP}_{\mathcal{F}_{i-1}\bowtie\mathcal{F}_i}(x_{i-1},x_i)\right) = lp\left(\mathrm{PSGP}_{\mathcal{F}_{i-1}\bowtie\mathcal{F}_i}(x_{i-1},u_i^*)\cup \mathrm{PSGP}_{\mathcal{F}_{i-1}\bowtie\mathcal{F}_i}(u_i^*,x_i)\right). \quad (10)$$

Combining (9) with (10) gives:

$$\begin{aligned} l\left(\bigcup_{i=0}^{k}\mathrm{SGP}_{\mathcal{F}_i}(u_i,u_{i+1})\right) &\ge lp\left(\mathrm{PSGP}_{\mathcal{F}_0}(u_0,x_0)\right) + \sum_{i=1}^{k-1} lp\left(\mathrm{PSGP}_{\mathcal{F}_{i-1}\bowtie\mathcal{F}_i}(x_{i-1},u_i^*)\cup \mathrm{PSGP}_{\mathcal{F}_{i-1}\bowtie\mathcal{F}_i}(u_i^*,x_i)\right) + lp\left(\mathrm{PSGP}_{\mathcal{F}_{k+1}}(x_k,u_{k+1})\right)\\ &= lp\left(\mathrm{PSGP}_{\mathcal{F}_0}(u_0^{next},x_0)\right) + \sum_{i=1}^{k-1} lp\left(\mathrm{PSGP}_{\mathcal{F}_{i-1}\bowtie\mathcal{F}_i}(x_{i-1},u_i^{next})\cup \mathrm{PSGP}_{\mathcal{F}_{i-1}\bowtie\mathcal{F}_i}(u_i^{next},x_i)\right) + lp\left(\mathrm{PSGP}_{\mathcal{F}_{k+1}}(x_k,u_{k+1}^{next})\right)\\ &= lp\left(\mathrm{PSGP}_{\mathcal{F}_0}(u_0^{next},x_0)\right) + lp\left(\mathrm{PSGP}_{\mathcal{F}_0\bowtie\mathcal{F}_1}(x_0,u_1^{next})\right) + lp\left(\mathrm{PSGP}_{\mathcal{F}_0\bowtie\mathcal{F}_1}(u_1^{next},x_1)\right)\\ &\quad + lp\left(\mathrm{PSGP}_{\mathcal{F}_1\bowtie\mathcal{F}_2}(x_1,u_2^{next})\right) + \cdots + lp\left(\mathrm{PSGP}_{\mathcal{F}_{k-1}\bowtie\mathcal{F}_k}(u_k^{next},x_k)\right) + lp\left(\mathrm{PSGP}_{\mathcal{F}_{k+1}}(x_k,u_{k+1}^{next})\right). \end{aligned}$$

For $i = 0,1,\ldots,k$,

$$lp\left(\mathrm{PSGP}_{\mathcal{F}_{i-1}\bowtie\mathcal{F}_i}(u_i^{next},x_i)\right) + lp\left(\mathrm{PSGP}_{\mathcal{F}_i\bowtie\mathcal{F}_{i+1}}(x_i,u_{i+1}^{next})\right) \ge lp\left(\mathrm{SGP}_{\mathcal{F}_i}(u_i,u_{i+1})\right),$$

we obtain:

$$\begin{aligned} l\left(\bigcup_{i=0}^{k}\mathrm{SGP}_{\mathcal{F}_i}(u_i,u_{i+1})\right) &\ge lp\left(\mathrm{PSGP}_{\mathcal{F}_0^{next}}(u_0^{next},u_1^{next})\right) + lp\left(\mathrm{PSGP}_{\mathcal{F}_1^{next}}(u_1^{next},u_2^{next})\right) + \cdots + lp\left(\mathrm{PSGP}_{\mathcal{F}_k^{next}}(u_k^{next},u_{k+1}^{next})\right)\\ &= l\left(\bigcup_{i=0}^{k}\mathrm{SGP}_{\mathcal{F}_i^{next}}(u_i^{next},u_{i+1}^{next})\right). \end{aligned}$$

If Straightness Condition (B1–B2) does not hold at some shooting point u_i then the inequality in the formula (9) is strict. This completes the proof. □

The proof of Theorem 1:

Proof. At jth-iterative step, suppose that Main Algorithm defines a path $\gamma^j = \bigcup_{i=0}^{k}\mathrm{SGP}_{\mathcal{F}_i}(u_i,u_{i+1})$. For $i = 0,1,\ldots,k$, take x_i on $\mathrm{PSGP}_{\mathcal{F}_i}(u_i,u_{i+1})$ such that x_i does not coincide with u_i and u_{i+1}. We get a set of points $\{x_i, i = 0,1,\ldots,k\}$ on the pseudo path $\chi = \cup_{i=0}^{k}\mathrm{PSGP}_{\mathcal{F}_i}(u_i,u_{i+1})$.

Firstly, suppose that Straightness Condition (B1–B2) in Section 4.2 does not hold at some shooting point u_i. If Common Edge Case for u_i happens, then u_i does not satisfy Straightness Condition (B1). It follows

$$lp\left(\mathrm{PSGP}_{\mathcal{F}_{i-1}\bowtie\mathcal{F}_i}(x_{i-1},x_i)\right) < lp\left(\mathrm{PSGP}_{\mathcal{F}_{i-1}}(x_{i-1},u_i)\right) + lp\left(\mathrm{PSGP}_{\mathcal{F}_i}(u_i,x_{i+1})\right)$$

In such case, u_i is updated to u_i^{next} which is the intersection point of $\mathrm{PSGP}_{\mathcal{F}_{i-1}\bowtie\mathcal{F}_i}(x_{i-1},x_i)$ and e_i.

If Non-common Edge Case for u_i happens, then u_i does not satisfy Straightness Condition (B2). It follows

$$\min\{lp\left(\mathrm{PSGP}_{\mathcal{F}_{i-1}\bowtie\mathcal{F}_i^{ccw}}(x_{i-1},x_i)\right), lp\left(\mathrm{PSGP}_{\mathcal{F}_{i-1}\bowtie\mathcal{F}_i^{cw}}(x_{i-1},x_i)\right)\} < lp\left(\mathrm{PSGP}_{\mathcal{F}_{i-1}}(x_{i-1},u_i)\right) + lp\left(\mathrm{PSGP}_{\mathcal{F}_i}(u_i,x_{i+1})\right)$$

Let $\mathcal{F}_{i-1}\bowtie\mathcal{F}_i \in \{\mathcal{F}_{i-1}\bowtie{\mathcal{F}_i}^{ccw}, \mathcal{F}_{i-1}\bowtie{\mathcal{F}_i}^{cw}\}$ such that

$$lp\left(\mathrm{PSGP}_{\mathcal{F}_{i-1}\bowtie\mathcal{F}_i}(x_{i-1},x_i)\right) = \min\{lp\left(\mathrm{PSGP}_{\mathcal{F}_{i-1}\bowtie\mathcal{F}_i^{ccw}}(x_{i-1},x_i)\right), lp\left(\mathrm{PSGP}_{\mathcal{F}_{i-1}\bowtie\mathcal{F}_i^{cw}}(x_{i-1},x_i)\right)\}.$$

In such case, u_i is updated to u_i^{next} which is the intersection point of $\mathrm{PSGP}_{\mathcal{F}_{i-1}\bowtie\mathcal{F}_i}(x_{i-1},x_i)$ and e_i or ${e'}_i$ is u_i^*.

Applying Proposition 5 gives $l(\gamma^{j\ next}) < l(\gamma^j)$, for $j\in\mathbb{N}$. Let $\{\gamma^j\}_{j\in\mathbb{N}}$ be the sequence of θ-gentle paths joining p and q which are obtained by Main Algorithm. Then the sequence $\{l(\gamma^j)\}_{j\in\mathbb{N}}$ is strictly reduced and it is then convergent. Denote $\sigma = \inf\{l(\gamma^j), j\in\mathbb{N}\}$. Since two norms $\|.\|_2$ and $\|.\|_s$ are equivalent in $\mathbb{R}^3$, analysis similar to that in the proof of Proposition 1.4.11 in [12] shows that the sequence $\{\chi^j\}_{j\in\mathbb{N}}$ has a subsequence $\{\chi^{j_n}\}_{n\in\mathbb{N}}$ that converges uniformly to a path, denoted by χ. Let us denote by γ a θ-gentle path which is an adjusted path of χ. Since $lp(\chi^{j_n}) = l(\gamma^{j_n})$, we get $\lim_{n\to\infty} l(\gamma^{j_n}) = l(\gamma)$. Due to the formula defining σ and $\{\gamma^{j_n}\}_{n\in\mathbb{N}} \subset \{\gamma^j\}_{j\in\mathbb{N}}$, we get $\sigma \le \lim_{n\to\infty} l(\gamma^{j_n}) = l(\gamma)$, i.e., $\sigma\le l(\gamma)$. Since $\chi^{j_n} \rightrightarrows \chi$ as $n\to\infty$, we get $lp(\chi) \le \liminf_{n\to\infty} lp(\chi^{j_n})$. Otherwise, $lp(\chi^{j_n}) = l(\gamma^{j_n})$ and $lp(\chi) = l(\gamma)$. Then $l(\gamma)\le\sigma$. In conclusion, $l(\gamma) = \sigma$. By Definition 5, γ is an approximate shortest θ-gentle path.

The proof is complete. □

References

[1] P.K. Agarwal, B. Aronov, J. O'Rourke, C.A. Schevon, Star unfolding of a polytope with applications, SIAM J. Comput. 26 (6) (1997) 1689–1713.

[2] J. Chen, Y. Han, Shortest paths on a polyhedron, in: Proceedings of the Sixth Annual Symposium on Computational Geometry, 1990, pp. 360–369.

[3] M. Sharir, A. Schorr, On shortest paths in polyhedral spaces, SIAM J. Comput. 15 (1) (1986) 193–215.

[4] S.-W. Cheng, J. Jin, Approximate shortest descending paths, SIAM J. Comput. 43 (2) (2014) 410–428.

[5] M. Ahmed, A. Lubiw, Shortest anisotropic path with few bends is NP-complete, in: Proceedings of the 18th Fall Workshop on Computational Geometry, 2008, pp. 28–29.

[6] M. Ahmed, A. Lubiw, A. Maheshwari, Shortest gently descending paths, in: Proceedings of the 3rd International Workshop on Algorithms and Computation, 2009, pp. 59–70.

[7] L. Sautter, Approximate Shortest Gentle Paths on Terrains (Diplome thesis), Karlsruhle Institute of Technology, 2014, (under the guidance of Nöllenburg, M.).

[8] L. Liu, R.C.-W. Wong, Finding shortest path on land surface, in: Proceedings of the 2011 ACM SIGMOD International Conference on Management of Data, 2011, pp. 433–444.

[9] P.T. An, N.N. Hai, T.V. Hoai, Direct multiple shooting method for solving approximate shortest path problems, J. Comput. Appl. Math. 244 (2013) 67–76.

[10] P.T. An, L.H. Trang, Multiple shooting approach for computing shortest descending paths on convex terrains, Comput. Appl. Math. 37 (4) (2018) 1–31.

[11] T.V. Hoai, P.T. An, N.N. Hai, Multiple shooting approach for computing approximately shortest paths on convex polytopes, J. Comput. Appl. Math. 317 (2017) 235–246.

[12] A. Papadopoulos, Metric Spaces, Convexity and Nonpositive Curvature, Vol. 6, European Mathematical Society, 2005.
[13] P.K. Agarwal, S. Har-Peled, M. Karia, Computing approximate shortest paths on convex polytopes, Algorithmica 33 (2) (2002) 227–242.
[14] Z. Sun, J.H. Reif, On finding energy-minimizing paths on terrains, IEEE Trans. Robot. 21 (1) (2005) 102–114.

Dr. Phan Thanh An is now an associate professor in Applied Mathematics at Faculty of Applied Science of the Ho Chi Minh City University of Technology (HCMUT), Vietnam. He also serves as director of the Institute of Mathematical and Computational Sciences – IMACS of HCMUT. He completed his Ph.D. in the field of optimization at the Vinh University in Vietnam in 1999 under the supervision of Professors Hoang Xuan Phu and Nguyen Nhuy. Then he had worked as a researcher at the Hanoi Institute of Mathematics, Vietnam (as an associate professor of the institute since 2009), and has specialized in the use of optimization methods (the method of orienting curves and the method of multiple shooting) for solving computational geometry problems. During the years 2009–2014 he worked at the Center for Computational and Stochastic Mathematics (CEMAT) of the Instituto Superior Técnico, the University of Lisbon, Portugal and supervised successfully two Ph.D. theses for this university in the research trend “Optimization approaches for computational geometry”. After staying in the Institute of Mathematics and Computer Sciences (ICMC), University of São Paulo, Brasil in 3 years as a visiting professor in Computer Science from 2017 to 2019, he has moved to HCMUT. He regularly visits the Interdisciplinary Center for Scientific Computing (IWR), Heidelberg University, Germany to collaborate with colleagues from the group “Simulation and Optimization” of the IWR on numerical optimization. He has also visited the Freie Universität Berlin, Germany since 2014 to collaborate with the Mathematical Geometry Processing working group on discrete geodesics. To date, Dr. An has supervised successfully 5 Ph.D. theses. He has authored or co-authored more than 40 scientific publications and two books. His research works and their applications can be found at his research website: https://sites.google.com/a/math.ac.vn/phanthanh-an-s-homepage/.