Integer Partition - Further Discussion of the Number of the Integer-side Triangles

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Abstract

The concept "partition of integers" refers to representing a positive integer m as a sum of indeterminate integers. Multiple researches concerning integer triangles have been conducted, and satisfying results have been reached. This research utilizes a different method to find the number of integer triangles with a specific perimeter, tackling the problem from an algebraic point of view, and applies this method to polygons. In the following research, given the integers m and n, consider the partition of the integer m into n summands, with the condition that the value of each summand is smaller than the sum of the other $n-1$ summands, so that the summands of the integer can be used to construct the sides of an integral polygon. We discuss the case of triangles first, using inductive reasoning to find a pattern to the number of integral triangles. The numbers of integral quadrilaterals have also been discovered through this method. However, a pattern cannot be determined for polygons that have more than four sides. Thus, we have searched for a recurrence relation between the partition of quadrilaterals and triangles, and generalized the formula to find the number of partitions into n summands for a given integer m.

1 Introduction

"Integer partition" is a significant problem in number theory and combinatorics. It aims to represent a positive integer m as a sum of indeterminate integers. For example, 5 can be partitioned in 7 ways, which are 5, 1+4, 2+3, 1+1+3, 1+2+2, 1+1+1+2, 1+1+1+1+1; thus, the indeterminately partitioning number of 5, $p(5)$, is 7. This problem was first issued by *G. W. Leibniz, 1646~1716*, "I think the question is not easy" asserted by him in the letter sent in *1699* to *John Bernoulli, 1667~1748*. It was mentioned several times in his unpublished manuscripts. Later *L. Wuler, 1707~1783*, developed the topic to a kind of complete partitioning theory with important publishments in 1741 and 1748. German combinatorics scholar *C. F. Hindenburg, 1741~1808,* as well as *J. F. W. Herschel, 1792~1871, A. De Morgan, 1806~1871, J. J. Sylvester, 1814~1897, A. Cayley, 1821~1891, P. A. MacMahon, 1854~1929*, etc., also made great contribution to this field successively. Nowadays, it has become a relatively robust theory, but there's still no simple formula of m for $p(m)$.

It is as hard as the original problem $p(m)$ if we restrict the partition of m in n parts, say $p(m,n)$, unless n is small. In the case of 5, we can easily tell $p(5,1)=p(5,4)=p(5,5)=1$ and $p(5,2)=p(5,3)=2$. Generally speaking, $p(m,1)=1, p(m,2)=\left[\frac{m}{2}\right]$, where $[x]$ is the largest integer smaller or equal to x. In specific, for positive integers m and n, $p(m,n)$ is

equivalent to the number of positive integer sequences $(a_1,a_2,\dots,a_n)$ satisfying the following conditions:

(P1) $a_1+a_2+\cdots+a_n=m$

(P2) $1\le a_1\le a_2\le\cdots\le a_n$

The target of this research is to add the third condition (P3) in addition to (P1) and (P2), regarding it as a conditional partition problem. In the following sections, $f(m,n)$ is used to represent the number of positive integer sequences $(a_1, a_2, \dots, a_n)$ fulfilling (P1), (P2) and (P3).

(P3) $a_1 + a_2 + \cdots + a_{n-1} > a_n$

These sequences are also equivalent to the sides which can construct a n-polygon, without considering the order of the sides. Actually, $f(m,3)$ is the number of distinct integer triangles whose perimeters are m.

In Advanced Combinatorics, published by a French mathematician Louis Comtet in 1974, an exercise clearly stated that there are $\left[\frac{1}{48}(m^2 + 3m + 21 + (-1)^{m-1}3m)\right]$ distinct integer triangles with perimeters equal to m. Four students in East China Normal University (*Yajing Cai, Zhengli Tan, Fugang Chao, Han Ren*) and *Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice* published a paper in 2015, using Ferrers diagram and generating function as well as triangular coordinates to give out two approaches to prove the formula written by Comtet.

The goal of this research is to further find $f(m,n)$. First, different discussing methods will be conducted to show the result of Comtet's formula again. Also, the approach can be extended to four-part partition problem. Finally, recursive relationship between different n values for $p(m,n)$ and $f(m,n)$ will be shown.

2 Three-integer Partition

Table 1: Simple Examples for (a, b, c)

m = 3			m= 4			m= 5			m= 6			m = 7			m = 8		
a	b	c	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c
1	1	1				1	1	3	2	2	2	1	3	3	2	3	3
												2	2	3			
m = 9			m = 10			m = 11			m = 12			m = 13			m = 14		
a	b	c	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c
1	4	4	2	4	4	1	5	5	2	5	5	1	6	6	2	6	6
2	3	4	3	3	4	2	4	5	3	4	5	2	5	6	3	5	6
3	3	3				3	3	5	4	4	4	3	4	6	4	4	6
						3	4	4				3	5	5	4	5	5
												4	4	5			

$f(m,3)$ will be discussed first in this section, with two approaches different from the aforementioned researches, to prove Comtet's formula again.

$$f(m,3) = \left[\frac{1}{48}(m^2 + 3m + 21 + (-1)^{m-1}3m)\right]$$

For simple notation, we denote the sequence as (a, b, c), where a, b, c are three sides of the triangle. Without losing the generality, we can also assume that $c \geq b \geq a$. Then, it's obvious that $b + c > a$ and $a + c > b$, so only $a + b > c$ is required to be discussed.

For instance, a positive-side triangle with perimeter equal to 3 can only be (1, 1, 1). Hence, $f(3,3) = 1$. By brute-force approach, the examples in **Table 1** can be easily found. According to these data, we can deduce the result in **Table 2**, where shows the fact that $f(m,3) < f(m+1,3)$ when m is even and less than 15.

Table 2: $f(m, 3)$ for small m

m	3	4	5	6	7	8	9	10	11	12	13	14	15
f(m, 3)	1	0	1	1	2	1	3	2	4	3	5	4	7

2.1 First approach for $f(m, 3)$

We first show the first approach to get $f(m, 3)$. For a positive integer m, since $1 \leq a \leq b \leq c$, we obtain $3a \leq a + b + c = m$ and $a \leq \frac{m}{3}$; therefore, the valid range for a can be concluded.

$$1 \leq a \leq \frac{m}{3}, \text{which is equivalent to } 1 \leq a \leq \left[\frac{m}{3}\right]$$

After fixing a, as $a \leq b = m - a - c \leq c$, we can deduce that $\frac{m-a}{2} \leq c \leq m - 2a$. On the other hand, $c + 1 \leq a + b = m - c$, so $c \leq \frac{m-1}{2}$, which means $c \leq \left[\frac{m-1}{2}\right]$. We obtain two upper bounds for c, both of which are compulsory. Further discussion about these two bounds can be found that $\left[\frac{m-1}{2}\right] \leq m - 2a$ is equivalent to $a \leq \left[\frac{m+2}{4}\right]$. Hence, the valid range for c satisfies the following conditions:

(2.1) If $a \leq \left[\frac{m+2}{4}\right], \frac{m-a}{2} \leq c \leq \left[\frac{m-1}{2}\right]$, which means $\left[\frac{m+1-a}{2}\right] \leq c \leq \left[\frac{m-1}{2}\right]$.

(2.2) If $a \geq \left[\frac{m+6}{4}\right], \frac{m-a}{2} \leq c \leq m - 2a$, which means $\left[\frac{m+1-a}{2}\right] \leq c \leq m - 2a$.

Thus, we obtain that

$$\begin{aligned}
&\mathrm{f(m, 3)} \\
&= \sum_{a=1}^{\left[\frac{m+2}{4}\right]} \left(\left[\frac{\mathrm{m}-1}{2}\right] - \left[\frac{\mathrm{m}+1-\mathrm{a}}{2}\right] + 1\right) + \sum_{a=\left[\frac{m+6}{4}\right]}^{\left[\frac{\mathrm{m}}{3}\right]} \left(m - 2a - \left[\frac{\mathrm{m}+1-\mathrm{a}}{2}\right] + 1\right) \\
&= \sum_{a=1}^{\left[\frac{m}{3}\right]} \left(\left[\frac{\mathrm{m}+1}{2}\right] - \left[\frac{\mathrm{m}+1-\mathrm{a}}{2}\right]\right) - \sum_{a=\left[\frac{m+6}{4}\right]}^{[m/3]} \left(\left[\frac{\mathrm{m}+1}{2}\right] - m + 2a - 1\right) \\
&= S_1 - S_2.
\end{aligned}$$

When m is even, S_1 is the sum of 0, 1, 1, 2, 2, 3, 3, … (there are $x = \left[\frac{m}{3}\right]$ numbers in total.); it can be further discussed in two cases. If $x = 2y$ for some integer y, then $S_1 = 0 + 1 + 1 + 2 + 2 +$

$\cdots + (y-1) + (y-1) + y = y^2 = \frac{x^2}{4}$. Otherwise, if $x = 2y+1$ for some integer y, $S_1 = 0 + 1 + 1 + 2 + 2 + \cdots + (y-1) + (y-1) + y + y = y(y+1) = \frac{x^2-1}{4}$. When m is odd, however, S_1 is the sum of 1, 1, 2, 2, 3, 3, … (there are also $x = \left[\frac{m}{3}\right]$ numbers in total.); likewise, if $x = 2y$ for some integer y, $S_1 = 1 + 1 + 2 + 2 + \cdots + y + y = y(y+1) = \frac{x(x+2)}{4}$. If $x = 2y+1$ instead, then $S_1 = 1 + 1 + 2 + 2 + \cdots + y + y + (y+1) = (y+1)^2 = \frac{(x+1)^2}{4}$.

S_2 is an arithmetic series with length $z = \left[\frac{m}{3}\right] - \left[\frac{m+2}{4}\right]$, common difference 2, and first number $\left[\frac{\mathrm{m}+1}{2}\right] - m + 2\left[\frac{\mathrm{m}+6}{4}\right] - 1 = \left[\frac{4\left[\frac{\mathrm{m}+6}{4}\right] - \mathrm{m} - 1}{2}\right] = \left[\frac{\mathrm{w}}{2}\right]$, where $w = 3, 2, 5, 4$ respectively when $m \equiv 0, 1, 2, 3 \ (mod\ 4)$, so the first item is 1, 1, 2, 2 respectively. When m $\equiv 0, 1 \ (mod\ 4)$, $S_2 = \frac{(1+2z-1)z}{2} = z^2$; when m $\equiv 2, 3 \ (mod\ 4)$, $S_2 = \frac{(2+2z)z}{2} = z(z+1)$.

As S_1, S_2 are related to $\left[\frac{\mathrm{m}}{3}\right], \left[\frac{m+2}{4}\right]$, we can divide the problem into 12 cases by mod 12 and find the solution for $f(m,3)$ respectively. Details are shown in **Table 3**.

With the 12 results shown in **Table 3**, we can then induce the conclusion that

$$f(m,3) = \left[\frac{1}{48}(m^2 + 3m + (-1)^{m-1}3m) + f'(m,3)\right]$$
$$= \left[\frac{1}{48}(m^2 + 3m + 21 + (-1)^{m-1}3m)\right]$$

Table 3: 12 cases of $f(m,3)$

m	x=[m/3]	S_1	z=[m/3] – [(m+2)/4]	S_2	f(m, 3)= $S_1 - S_2$
12t	4t	$4t^2$	4t-3t=t	t^2	$3t^2 = \frac{m^2}{48}$
12t+1	4t	t(4t+2)	4t-3t=t	t^2	$3t^2 + 2t = (m^2 + 6m - 7)/48$
12t+2	4t	$4t^2$	4t-(3t+1)=t-1	t(t-1)	$3t^2 + t = (m^2 - 4)/48$
12t+3	4t+1	$(2t+1)^2$	(4t+1)-(3t+1)=t	t(t+1)	$3t^2 + 3t + 1 = (m^2 + 6m + 21)/48$
12t+4	4t+1	t(4t+2)	(4t+1)-(3t+1)=t	t^2	$3t^2 + 2t = (m^2 - 16)/48$
12t+5	4t+1	$(2t+1)^2$	(4t+1)-(3t+1)=t	t^2	$3t^2 + 4t + 1 = (m^2 + 6m - 7)/48$
12t+6	4t+2	$(2t+1)^2$	(4t+2)-(3t+2)=t	t(t+1)	$3t^2 + 3t + 1 = (m^2 + 12)/48$
12t+7	4t+2	(t+1)(4t+2)	(4t+2)-(3t+2)=t	t(t+1)	$3t^2 + 5t + 2 = (m^2 + 6m + 5)/48$
12t+8	4t+2	$(2t+1)^2$	(4t+2)-(3t+2)=t	t^2	$3t^2 + 4t + 1 = (m^2 - 16)/48$
12t+9	4t+3	$(2t+2)^2$	(4t+3)-(3t+2)=t+1	$(t+1)^2$	$3t^2 + 6t + 3 = (m^2 + 6m + 9)/48$
12t+10	4t+3	(2t+1)(2t+2)	(4t+3)-(3t+3)=t	t(t+1)	$3t^2 + 5t + 2 = (m^2 - 4)/48$
12t+11	4t+3	$(2t+2)^2$	(4t+3)-(3t+3)=t	t(t+1)	$3t^2 + 7t + 4 = (m^2 + 6m + 5)/48$

, where the value of $f'(m,3)$ can be found in the third row in **Table 4** as well as the sixth column in **Table 3**. In fact, the scalar 21 in the formula can be replaced by any number between 21 and 31.

2.2 Second approach for $f(m,3)$

Then we introduce the second approach to find $f(m,3)$. Recall that $f(m,3)$ is the number of positive integer sequences (a,b,c) satisfying

$$a \le b \le c, \text{and } a+b>c.$$

These sequences can be grouped in two cases: $a+b=c+1$ and $a+b \ge c+2$.

For the first case $a+b=c+1$, as $a+b+c=m$, m must be odd, and $a+b=\frac{m+1}{2}$, $c=\frac{m-1}{2}$. Therefore, since $a \le b$, there are $\left[\frac{m+1}{4}\right]$ solutions.

For the second case $a+b \ge c+2$, as $c \ge b$, $a+b \ge b+2$, so $a \ge 2$. Consider a new sequence (a',b',c'), where $a'=a-1, b'=b-1, c'=c-1$. It satisfies that $a'+b'+c'=m-3, a' \le b' \le c'$, and $a'+b'>c'$. Thus, the case corresponds to the solution of $f(m-3,3)$.

To sum up these two cases, we obtain: if m is even, $f(m,3)=f(m-3,3)$; if m is odd, $f(m,3)=f(m-3,3)+\left[\frac{m+1}{4}\right]$.

One step further, when m is even,

$$\begin{aligned}
&f(m,3)\\
&=f(m-3,3)\\
&=f(m-6,3)+\left[\frac{m-2}{4}\right]\\
&=f(m-12,3)+\left[\frac{m-8}{4}\right]+\left[\frac{m-2}{4}\right]\\
&=f(m-12,3)+\frac{m-6}{2}
\end{aligned}$$

The last equation is true because $\left[\frac{m-2}{4}\right]+\left[\frac{m-8}{4}\right]=\frac{(m-8)+(m-2)-2}{4}$. Similarly, when m is odd,

$$\begin{aligned}
&f(m,3)\\
&=f(m-3,3)+\left[\frac{m+1}{4}\right]\\
&=f(m-6,3)+\left[\frac{m+1}{4}\right]\\
&=f(m-12,3)+\left[\frac{m-5}{4}\right]+\left[\frac{m+1}{4}\right]\\
&=f(m-12,3)+\frac{m-3}{2}
\end{aligned}$$

The last equation is true because $\left\lfloor\frac{m-5}{4}\right\rfloor + \left\lfloor\frac{m+1}{4}\right\rfloor = \frac{(m-5)+(m+1)-2}{4}$. We can induce the two equations as $f(m,3) = f(m-12) + \frac{m-m'}{2}$, where $m' = 6$ if m is even; $m' = 3$ if m is odd. Let $m = 12t + r$ for some $t \in \mathbb{N} \cup \{0\}, r$ is an integer and $0 \le r \le 11$. Then,

$$
\begin{aligned}
& f(m,3) \\
&= f(m-12,3) + \frac{m-m'}{2} \\
&= f(m-24,3) + \frac{(m-12)-m'}{2} + \frac{m-m'}{2} \\
&= f(m-36,3) + \frac{(m-24)-m'}{2} + \frac{(m-12)-m'}{2} + \frac{m-m'}{2} \\
&= \cdots \\
&= \ f(r,3) + \frac{(12+r)-m'}{2} + \cdots + \frac{(m-24)-m'}{2} + \frac{(m-12)-m'}{2} + \frac{m-m'}{2} \\
&= f(r,3) + \frac{m-r}{24} \times \frac{12+r-m'+m-m'}{2} \\
&= \frac{\mathrm{m}^2 + (12-2m')\mathrm{m} - \mathrm{r}(\mathrm{r}+12-2m') + 48f(r,3)}{48}
\end{aligned}
$$

, where $12 - 2m' = 0$ if m is even; $12 - 2m' = 6$ if m is odd. Hence, $(12 - 2m')m = 3m + (-1)^{m-1}3m$. As of the scalar $f'(m,3) = -r(r + 12 - 2m') + 48f(r,3)$ can be calculated from **Table 4**.

Table 4: scalar discussion in 12 cases

$r = m \bmod 12$	0	1	2	3	4	5	6	7	8	9	10	11
$f(r,3)$	0	0	0	1	0	1	1	2	1	3	2	4
$f'(m,3)$	0	−7	−4	21	−16	−7	12	5	−16	9	−4	5

Thus, we can conclude that

$$
\begin{aligned}
& f(m,3) \\
&= \frac{1}{48}\left(m^2 + 3m + (-1)^{m-1}3m + f'(m,3)\right) \\
&= \left[\frac{1}{48}(m^2 + 3m + (-1)^{m-1}3m)\right]
\end{aligned}
$$

3 Four-integer Partition

In this section, two different approaches to deduce $f(m,4)$ will be introduced. For simple notation, we set the four integers to be a,b,c,d, and they fulfill that $a+b+c+d=m, 1\le a\le b\le c\le d, a+b+c>d$, so that they can be the sides of a quadrilateral (not taking the order into account).

3.1 First approach for $f(m,4)$

We introduce the first way to get $f(m,4)$ in this section. We know that $4a\le a+b+c+d=m$, so $a\le\left[\frac{m}{4}\right]$. In addition, to simplify the calculation, assume $b=a-1+x, c=a-1+y, d=a-1+z$. Then the solution of positive integer (a,b,c,d) with $1\le a\le b\le c\le d$ is equivalent to the solution of positive integer (a,x,y,z) with $1\le x\le y\le z$ and $1\le a$. Also, $a+b+c+d=m$ means $x+y+z=m-4a+3$. In the following parts, we discuss $f(m,4)$ in two cases: $x+y>z$ and $x+y\le z$.

When $x+y>z$, $a+b+c=3a-2+x+y>3a-2+z>2a-2+z\ge a-1+z=d$. Thus, there are

$$\sum_{i=1}^{\left[\frac{m}{4}\right]} f(m-4i+3,3) \text{ solutions.}$$

When $x+y\le z$, the condition $a+b+c>d$ means $x+y>z-2a+1$, so the valid range for $x+y$ is $(z-2a+1,z)$. Let $x+y=z-r$. Then $2z-r=m-4a+3$. We know that r must be even if m is odd, and r must be odd if m is even, in order to find the integer solution of z. Therefore, we divide the problem into two cases.

If m is even, set $m=2k$, then $x+y$ is odd. Also, $z-2a+1<x+y\le z$ implies: when $a=i$, (x,y,z) has positive integer solutions only when $x+y\in\{z-1, z-3, \dots, z-(2i-3)\}$. Besides, along with $x+y+z=m-4i+3$, we obtain: when $x+y=z-1, z-3, \dots, z-(2i-3)$, there are

$$\left[\frac{k-(2i-1)}{2}\right], \left[\frac{k-2i}{2}\right], \dots, \left[\frac{k-(3i-3)}{2}\right]$$

integer solutions of (x,y,z) respectively. While if $k-j<0$ for $2i-1\le j\le 3i-3$, $x+y<0$, it leads to a contradiction. For simplicity, let

$$[n]^+=\begin{cases}[n], & n\ge 0\\ 0, & n<0\end{cases}$$

As a result, there are $\sum_{i=2}^{[\frac{m}{4}]}\left(\sum_{j=2i-1}^{3i-3}\left[\frac{k-j}{2}\right]^{+}\right)$ solutions if $x+y\leq z$ and $m=2k$. Based on the discussion, we find: if $m=2k$,

$$f(m,4)=\sum_{i=1}^{[\frac{m}{4}]} f(m-4i+3,3)+\sum_{i=2}^{[\frac{m}{4}]}\left(\sum_{j=2i-1}^{3i-3}\left[\frac{k-j}{2}\right]^{+}\right)$$

Furthermore, assuming $m=12t+r$ for $r=0,2,4,6,8,10$, we obtain

$$\sum_{i=1}^{[\frac{m}{4}]} f(m-4i+3,3)$$

$$=f(m-1,3)+f(m-5,3)+f(m-9,3)+\cdots+f\left(m+3-4\left[\frac{m}{4}\right],3\right)$$

$$=\sum_{i=0}^{t-1}\big(f(12i+a,3)+f(12i+b,3)+f(12i+c,3)\big)+m'$$

, where the information of a,b,c, and m' can be found in **Table 5**.

Table 5: 6 cases of even numbers

$r=$	0	2	4	6	8	10
$[m/4]$	$3t$	$3t$	$3t+1$	$3t+1$	$3t+2$	$3t+2$
$m+3-4[m/4]$	3	5	3	5	3	5
(a,b,c)	$(3,7,11)$	$(5,9,13)$	$(7,11,15)$	$(9,13,17)$	$(11,15,19)$	$(13,17,21)$
m'	0	0	$f(3,3)$	$f(5,3)$	$f(3,3)+f(7,3)$	$f(5,3)+f(9,3)$

Along with the content of the section **2**, we deduce the result in **Table 6**.

Table 6: first part of $f(m,4)$

$r=$	0	2	4	6	8	10
$\sum_{i=1}^{[m/4]} f(m-4i+3,3)$	$\left(\sum_{i=0}^{r-1} 9i^2+15i+7\right)$	$\left(\sum_{i=0}^{r-1} 9i^2+12i+4\right)+3r^2+2r$	$\left(\sum_{i=0}^{r-1} 9i^2+15i+7\right)+3r^2+3r+1$	$\left(\sum_{i=0}^{r-1} 9i^2+12i+4\right)+6r^2+6r+1$	$\left(\sum_{i=0}^{r-1} 9i^2+15i+7\right)+6r^2+8r+3$	$\left(\sum_{i=0}^{r} 9i^2+12i+4\right)$

Then consider $\sum_{i=2}^{\left[\frac{m}{4}\right]}\left(\sum_{j=2i-1}^{3i-3}\left[\frac{k-j}{2}\right]^{+}\right)$. For all natural numbers n with $2n-1 \leq k \leq 3n-3$, set the minimum value of n be α_1 and maximum value be β_1. For $2 \leq i \leq \alpha_1 - 1$ $(i \in \mathbb{N} - \{1\})$, $\left[\frac{k-j}{2}\right]^{+} = \left[\frac{k-j}{2}\right]$ when $j = 2i-1, 2i, \ldots, 3i-3$. Also, set $\left[\frac{k-3}{2}\right] = x$ when $k \geq 3$. Let $g(i) = \sum_{j=2i-1}^{3i-3}\left[\frac{k-j}{2}\right]$. We have

$$\sum_{i=2}^{\left[\frac{m}{4}\right]}\left(\sum_{j=2i-1}^{3i-3}\left[\frac{k-j}{2}\right]^{+}\right) = \sum_{i=2}^{\alpha_1-1} g(i) + \sum_{i=\alpha_1}^{\beta_1}\left(\sum_{j=2i-1}^{3i-3}\left[\frac{k-j}{2}\right]^{+}\right),$$

For all $i \geq \beta_1 + 1$, $\sum_{j=2i-1}^{3i-3}\left[\frac{k-j}{2}\right]^{+} = 0$. To make a deeper discussion about $\sum_{i=2}^{\alpha_1-1} g(i)$, we have two cases: k is odd, or k is even.

For k is even, $\left[\frac{k-(2n+1)}{2}\right] = \left[\frac{k-(2n+2)}{2}\right]$ $(2n+2 \leq k)$, so we have the following pattern of $g(i)$ in **Table 7**.

Table 7: $g(i)$ pattern when k is even

$g(2)$	x
$g(3)$	$(x-1)+(x-1)$
$g(4)$	$(x-2)+(x-2)+(x-3)$
$g(5)$	$(x-3)+(x-3)+(x-4)+(x-4)$
$g(6)$	$(x-4)+(x-4)+(x-5)+(x-5)+(x-6)$

Similarly, we obtain $g(i) = \begin{cases}(i-1)x - \left(\frac{5}{4}i^2 - 4i + 3\right), & i \text{ is even} \\ (i-1)x - \left(\frac{5}{4}i^2 - 4i + \frac{11}{4}\right), & i \text{ is odd}\end{cases}$. For simple computation, we discuss the problem in two cases again:

When α_1 is even:

$$\begin{aligned}\sum_{i=2}^{\alpha_1-1} g(i) &= \sum_{\gamma=1}^{\frac{\alpha_1-2}{2}}\left((2\gamma-1)x - (5\gamma^2 - 8\gamma + 3)\right) + \sum_{\delta=1}^{\frac{\alpha_1-2}{2}}\left(2\delta x - (5\delta^2 - 3\delta)\right) \\ &= \sum_{\varepsilon=1}^{\frac{\alpha_1-2}{2}}(4x\varepsilon - x - 10\varepsilon^2 + 11\varepsilon - 3) \\ &= \sum_{\varepsilon=1}^{\frac{\alpha_1-2}{2}}\left((-10)\varepsilon^2 + (4x+11)\varepsilon - (x+3)\right)\end{aligned}$$

When α_1 is odd:

$$\begin{aligned}\sum_{i=2}^{\alpha_1-1} g(i) &= \sum_{\gamma=1}^{\frac{\alpha_1-1}{2}}\left((2\gamma-1)x - (5\gamma^2 - 8\gamma + 3)\right) + \sum_{\delta=1}^{\frac{\alpha_1-3}{2}}\left(2\delta x - (5\delta^2 - 3\delta)\right) \\ &= \sum_{\varepsilon=1}^{\frac{\alpha_1-2}{2}}[(-10)\varepsilon^2 + (4x+11)\varepsilon - (x+3)] + (\alpha_1 - 2)x - \frac{5}{4}(\alpha_1 - 1)^2 + 4(\alpha_1 - 1) - 3\end{aligned}$$

The second situation is that k is odd. With the same method as the case k is even, we have:

$$\sum_{i=2}^{\alpha_1-1} g(i)=$$
$$\begin{cases}\sum_{\varepsilon=1}^{\frac{\alpha_1-2}{2}}[(-10)\varepsilon^2+(4x+9)\varepsilon-(x+2)], & \text{if } \alpha_1 \text{ is even}\\ \sum_{\varepsilon=1}^{\frac{\alpha_1-3}{2}}[(-10)\varepsilon^2+(4x+9)\varepsilon-(x+2)]+\left[(\alpha_1-2)x-\frac{5}{4}(\alpha_1-1)^2+\frac{7}{2}(\alpha_1-1)-2\right], & \text{if } \alpha_1 \text{ is odd}\end{cases}$$

Lastly, we can discuss $\sum_{i=\alpha_1}^{\beta_1}(\sum_{j=2i-1}^{3i-3}\left[\frac{k-j}{2}\right]^+)$ in two cases.

The first case happens when k is even. Let $a_{p,q}$ be the q-th j-value for $i=p$, which is $(2p-1)+q-1$. Some examples are shown in **Table 8**:

Table 8: $a_{p,q}$ examples

i	j	$a_{p,q}$
2	3	$a_{2,1}$
3	5, 6	$a_{3,1}, a_{3,2}$
4	7, 8, 9	$a_{4,1}, a_{4,2}, a_{4,3}$
5	9, 10, 11, 12	$a_{5,1}, a_{5,2}, a_{5,3}, a_{5,4}$

Assume that $k=a_{\alpha_1,\omega_1}=a_{\alpha_1+1,\omega_2}=\cdots=a_{\beta_1,\omega_\sigma}$. For $a_{\alpha_1,\tau}(\tau>\omega_1)$, $\left[\frac{k-a_{\alpha_1,\tau}}{2}\right]^+=0$. Similarly, we can induce the result in **Table 9** by discussing the $a_{p,q}$ value with $p=\alpha_t$ and $q\le\omega_t$, i.e. the $a_{p,q}$ in the same row as k but smaller column.

Table 9: $\left[\frac{k-j}{2}\right]^+$ pattern

i	j	$\left[\frac{k-j}{2}\right]^+$
β_1	$2\beta_1-1, a_{\beta_1,\omega_\sigma}(k)$	0, 0
β_1-1	$2\beta_1-3, 2\beta_1-2, 2\beta_1-1, a_{\beta_1-1,\omega_{\sigma-1}}(k)$	1, 1, 0, 0
$\vdots$	$\vdots$	$\vdots$
α_1	$2\alpha_1-1, 2\alpha_1, \dots, a_{\alpha_1,\omega_1}(k)$	$\frac{k-2\alpha_1}{2}, \frac{k-2\alpha_1}{2}, \dots, 0$

From **Table 9**, we know that when k is even,

$$\sum_{i=\alpha_1}^{\beta_1}\left(\sum_{j=2i-1}^{3i-3}\left[\frac{k-j}{2}\right]^+\right)$$

$$= \sum_{i=\alpha_1}^{\beta_1} 2\left(0+1+2+\cdots+\frac{k-2i}{2}\right)$$

$$= \sum_{i=\alpha_1}^{\beta_1} (\frac{k-2i+2}{2})(\frac{k-2i}{2})$$

The second case is that k is odd; similarly, we have $\sum_{i=\alpha_1}^{\beta_1}\left(\sum_{j=2i-1}^{3i-3}\left[\frac{k-j}{2}\right]^+\right) = \sum_{i=\alpha_1}^{\beta_1}(\frac{k-2i+1}{2})^2$.

To sum up, we successfully deduce the value of $\sum_{i=2}^{[\frac{m}{4}]}(\sum_{j=2i-1}^{3i-3}\left[\frac{k-j}{2}\right]^+)$. In detail, we have:

1. When k is even,

   $\sum_{i=2}^{t}\left(\sum_{j=2i-1}^{3i-3}\left[\frac{2t-j}{2}\right]^+\right) = \sum_{i=2}^{\alpha_1-1} g(i) + \sum_{i=\alpha_1}^{\beta_1}(\frac{k-2i+2}{2})(\frac{k-2i}{2})$

   , where $\sum_{i=2}^{\alpha_1-1} g(i)$

   $$=\begin{cases}\sum_{\varepsilon=1}^{\frac{\alpha_1-2}{2}}((-10)\varepsilon^2+(4x+11)\varepsilon-(x+3)), & \text{if } \alpha_1 \text{ is even}\\ \sum_{\varepsilon=1}^{\frac{\alpha_1-2}{2}}[(-10)\varepsilon^2+(4x+11)\varepsilon-(x+3)]+(\alpha_1-2)x-\frac{5}{4}(\alpha_1-1)^2+4(\alpha_1-1)-3, & \text{if } \alpha_1 \text{ is odd}\end{cases}.$$

2. When k is odd,

   $\sum_{i=2}^{t}\left(\sum_{j=2i-1}^{3i-3}\left[\frac{2t+1-j}{2}\right]^+\right) = \sum_{i=2}^{\alpha_1-1} g(i) + \sum_{i=\alpha_1}^{\beta_1}(\frac{k-2i+1}{2})^2$

   , where $\sum_{i=2}^{\alpha_1-1} g(i)$

   $$=\begin{cases}\sum_{\varepsilon=1}^{\frac{\alpha_1-2}{2}}[(-10)\varepsilon^2+(4x+9)\varepsilon-(x+2)], & \text{if } \alpha_1 \text{ is even}\\ \sum_{\varepsilon=1}^{\frac{\alpha_1-3}{2}}[(-10)\varepsilon^2+(4x+9)\varepsilon-(x+2)]+\left[(\alpha_1-2)x-\frac{5}{4}(\alpha_1-1)^2+\frac{7}{2}(\alpha_1-1)-2\right], & \text{if } \alpha_1 \text{ is odd}\end{cases}.$$

Replace m with $12t+r$. For example, for $r=0$, $\alpha_1=2t+1$, $\beta_1=3t$, $x=3t-2$. We obtain:

$$f(12t,4)$$

$$= (\sum_{i=0}^{r-1} 9i^2+15i+7)+\sum_{\varepsilon=1}^{\frac{\alpha_1-2}{2}}[(-10)\varepsilon^2+(4x+11)\varepsilon-(x+3)]+(\alpha_1-2)x-\frac{5}{4}(\alpha_1-1)^2+4(\alpha_1-1)-3+\sum_{i=\alpha_1}^{\beta_1}(\frac{k-2i+2}{2})(\frac{k-2i}{2})$$

$$= \frac{9(t-1)t(2t-1)}{6}+15\frac{t(t-1)}{2}+7t+\frac{(-10)(t-1)t(2t-1)}{6}+(12t+3)\frac{(t-1)t}{2}-(3t+1)(t-1)+(2t-1)(3t-2)-5t^2+8t-3+\frac{3t(3t+1)(6t+1)}{6}-\frac{2t(2t+1)(4t+1)}{6}-(6t+1)\frac{(5t+1)t}{2}+3t(3t+1)t$$

$$= 6t^3+\frac{3}{2}t^2+\frac{1}{2}t$$

Similarly, for different r-value, we have the result in **Table 10**:

Table 10: $f(m,4)$ results when m is even

r	0	2	4	6	8	10
$f(12t+\mathrm{r},4)$	$6t^3+\frac{3}{2}t^2+\frac{1}{2}t$	$6t^3+\frac{9}{2}t^2+\frac{3}{2}t$	$6t^3+\frac{15}{2}t^2+\frac{7}{2}t+1$	$6t^3+\frac{21}{2}t^2+\frac{13}{2}t+1$	$6t^3+\frac{27}{2}t^2+\frac{21}{2}t+3$	$6t^3+\frac{33}{2}t^2+\frac{31}{2}t+5$
$f(m,4)$	$\frac{1}{288}m^3+\frac{1}{96}m^2+\frac{1}{24}m$	$\frac{1}{288}m^3+\frac{1}{96}m^2+\frac{1}{24}m-\frac{11}{72}$	$\frac{1}{288}m^3+\frac{1}{96}m^2+\frac{1}{24}m+\frac{4}{9}$	$\frac{1}{288}m^3+\frac{1}{96}m^2+\frac{1}{24}m-\frac{3}{8}$	$\frac{1}{288}m^3+\frac{1}{96}m^2+\frac{1}{24}m+\frac{2}{9}$	$\frac{1}{288}m^3+\frac{1}{96}m^2+\frac{1}{24}m+\frac{5}{72}$

When m is odd, we use the same approach to deduce the equations in **Table 11**:

Table 11: $f(m,4)$ results when m is odd

r	0	2	4	6	8	10
$f(12t+\mathrm{r},4)$	$6t^3+\frac{3}{2}t^2+\frac{1}{2}t$	$6t^3+\frac{9}{2}t^2+\frac{3}{2}t$	$6t^3+\frac{15}{2}t^2+\frac{7}{2}t+1$	$6t^3+\frac{21}{2}t^2+\frac{13}{2}t+1$	$6t^3+\frac{27}{2}t^2+\frac{21}{2}t+3$	$6t^3+\frac{33}{2}t^2+\frac{31}{2}t+5$
$f(m,4)$	$\frac{1}{288}m^3+\frac{1}{96}m^2+\frac{1}{24}m$	$\frac{1}{288}m^3+\frac{1}{96}m^2+\frac{1}{24}m-\frac{11}{72}$	$\frac{1}{288}m^3+\frac{1}{96}m^2+\frac{1}{24}m+\frac{4}{9}$	$\frac{1}{288}m^3+\frac{1}{96}m^2+\frac{1}{24}m-\frac{3}{8}$	$\frac{1}{288}m^3+\frac{1}{96}m^2+\frac{1}{24}m+\frac{2}{9}$	$\frac{1}{288}m^3+\frac{1}{96}m^2+\frac{1}{24}m+\frac{5}{72}$

Hence, by induction, we have:

$$f(m,4)=\left[\frac{1}{288}\left(m^3+\frac{9}{2}m^2+\frac{9}{2}m+128+(-1)^{m+1}\left(\frac{3}{2}m^2-\frac{15}{2}m\right)\right)\right]$$

3.2 Second approach for $f(m,4)$

In this section, we introduce the second approach to deduce $f(m,4)$. Similar to the method in **2.2**, for natural number m, four positive integers a,b,c,d with $a+b+c+d=m$ and $a+b+c\geq d+1$ can be discussed in three situations:

(3.1) $a+b+c=d+1=\frac{m+1}{2}$. It's obvious that a,b,c,d has integer solution if and only if m is odd; in specific, there are $p\left(\frac{m+1}{2},3\right)=p\left([\frac{m+2}{2}],3\right)$ solutions.

(3.2) $a+b+c=d+2=\frac{m+2}{2}$. It's obvious that a,b,c,d has integer solution if and only if m is even; in specific, there are $p\left(\frac{m+2}{2},3\right)=p\left([\frac{m+2}{2}],3\right)$ solutions.

(3.3) $a+b+c\geq d+3$. If $a=1$ then $b+c\geq d+2\geq c+2$, which implies $b\geq 2$ and $(b-1)+(c-1)\geq(d-1)+1$, so there are $f(m-4,3)$ solutions; if $a\geq 2$, it can be represented as $(a-1)+(b-1)+(c-1) \geq (d-1)+1$, which has $f(m-4,4)$ solutions. Hence, when $a+b+c \geq d+3$, there are $f(m-4,3)+f(m-4,4)$ solutions.

To sum up, we obtain:

$$f(m,4) = f(m-4,4) + f(m-4,3) + p\left(\left[\frac{m+2}{2}\right],3\right)$$

To deduce the closed-form solution of $f(m,4)$, we refer to the approach we use to find $f(m,3)$ in **2.2** to calculate $p(m,3)$. Recall that $p(m,3)$ is the number of natural number sequences (a,b,c) with $a \leq b \leq c$ and $a+b+c=m$. These sequences can grouped into two categories: one satisfies $a=1$, the others fulfills $a \geq 2$.

Consider the first situation $a=1$. The equation $a+b+c=m$ implies that $b+c=m-1$. Along with $1 \leq b \leq c$, we can conclude that there are $p(m-1,2)=\left[\frac{m-1}{2}\right]$ solutions. For the second case $a \geq 2$, consider a positive integer sequence (a',b',c') with $a'=a-1$, $b'=b-1$, $c'=c-1$, which satisfies $a'+b'+c'=m-3$ and $a' \leq b' \leq c'$. This maps to a solution of $p(m-3,3)$. Combining the two cases, we obtain $p(m,3)=p(m-3,3)+\left[\frac{m-1}{2}\right]$. One step further,

$$\begin{aligned} p(m,3) &= p(m-3,3)+\left[\tfrac{m-1}{2}\right] \\ &= p(m-6,3)+\left[\tfrac{m-1}{2}\right]+\left[\tfrac{m-4}{2}\right] \\ &= p(m-6,3)+\tfrac{(m-1)+(m-4)-1}{2} \\ &= p(m-6,3)+(m-3) \end{aligned}$$

Let $m=6t+r$, where $r \in \{0,1,2,3,4,5\}$. Then

$$\begin{aligned} p(m,3) &= p(m-6,3)+(m-3) \\ &= p(m-12,3)+(m-3)+(m-9) \\ &= \cdots \\ &= p(r,3)+(m-3)+(m-9)+\cdots+(m-6t+3) \\ &= p(r,3)+\tfrac{m-r}{6}\,\tfrac{(m-3)+(m-6t+3)}{2} \\ &= \frac{m^2-r^2+12p(r,3)}{12} \end{aligned}$$

,where $p'(m,3) \coloneqq -r^2+12p(r,3)$ can be calculated as shown in **Table 12**.

Table 12: $p'(m,3)$ discussion

r	0	1	2	3	4	5
$p(r,3)$	0	0	0	1	1	2
$p'(m,3) = -r^2 + 12p(r,3)$	0	-1	-4	3	-4	-1

Therefore, we obtain that $p(m,3) = \frac{1}{12}(m^2 + p'(m,3)) = \left[\frac{1}{12}(m^2+3)\right]$. In fact, the scalar 3 can be replaced by any number between 3 and 7.

With the formula of $p(m,3)$, along with the result of $f(m,3)$ in section **2**. We obtain that

$$f(m,4) = f(m-4,4) + f(m-4,3) + p\left(\left[\frac{m+2}{2}\right],3\right)$$
$$= f(m-8,4) + f(m-8,3) + p\left(\left[\frac{m-2}{2}\right],3\right) + f(m-4,3) + p\left(\left[\frac{m+2}{2}\right],3\right)$$
$$= f(m-12,4) + f(m-12,3) + p\left(\left[\frac{m-6}{2}\right],3\right) + f(m-8,3) + p\left(\left[\frac{m-2}{2}\right],3\right) + f(m-4,3) + p\left(\left[\frac{m+2}{2}\right],3\right)$$
$$= f(m-12,4) + f^*(m,4)$$

, where $f^*(m,4) = A + B$, and

$$A = \frac{1}{48}\left((m-12)^2 + 3(m-12) + (-1)^{m-12-1}3(m-12) + 4[\tfrac{m-6}{2}]^2\right) + \frac{1}{48}\left((m-8)^2 + 3(m-8) + (-1)^{m-8-1}3(m-8) + 4[\tfrac{m-2}{2}]^2\right) + \frac{1}{48}\left((m-4)^2 + 3(m-4) + (-1)^{m-4-1}3(m-4) + 4[\tfrac{m+2}{2}]^2\right)$$
$$= \begin{cases} \frac{1}{48}(6m^2 - 60m + 268), & \text{if } m \text{ is even,} \\ \frac{1}{48}(6m^2 - 48m + 139), & \text{if } m \text{ if odd,} \end{cases}$$

$$B = \frac{1}{48}\left(f'(m-12,3) + 4p'\left(\left[\frac{m-6}{2}\right],3\right) + f'(m-8,3) + 4p'\left(\left[\frac{m-2}{2}\right],3\right) + f'(m-4,3) + 4p'([\frac{m+2}{2}],3)\right)$$

Use **Table 13** to compute B. We know that $B = \frac{1}{48}(-28)$ if m is even, $B = \frac{1}{48}(-1)$ if m is odd.

Hence, $f^*(m,4) = \begin{cases} \frac{1}{48}(6m^2 - 60m + 240), & \text{if } m \text{ is even,} \\ \frac{1}{48}(6m^2 - 48m + 138), & \text{if } m \text{ if odd,} \end{cases}$

Table 13: B's discussion for $f^*(m,4)$

(m-12) mod 12	0	1	2	3	4	5	6	7	8	9	10	11
$f'(m-12,3)$	0	-7	-4	21	-16	-7	12	5	-16	9	-4	5
(m-8) mod 12	4	5	6	7	8	9	10	11	0	1	2	3
$f'(m-8,3)$	-16	-7	12	5	-16	9	-4	5	0	-7	-4	21
(m-4) mod 12	8	9	10	11	0	1	2	3	4	5	6	7
$f'(m-4,3)$	-16	9	-4	5	0	-7	-4	21	-16	-7	12	5

$\left[\frac{m-6}{2}\right]$ mod 6	3	3	4	4	5	5	0	0	1	1	2	2
$4\,p'(\left[\frac{m-6}{2}\right],3)$	12	12	-16	-16	-4	-4	0	0	-4	-4	-16	-16
$\left[\frac{m-2}{2}\right]$ mod 6	5	5	0	0	1	1	2	2	3	3	4	4
$4\,p'(\left[\frac{m-2}{2}\right],3)$	-4	-4	0	0	-4	-4	-16	-16	12	12	-16	-16
$\left[\frac{m+2}{2}\right]$ mod 6	1	1	2	2	3	3	4	4	5	5	0	0
$4\,p'(\left[\frac{m+2}{2}\right],3)$	-4	-4	-16	-16	12	12	-16	-16	-4	-4	0	0
$48B$	-28	-1	-28	-1	28	-1	-28	-1	-28	-1	-28	-1

Then set $m = 12t + r$ with $0 \le r \le 12$. Then

$$\begin{aligned}
f(m,4) &= f(m-12,4) + f^*(m,4) \\
&= f(m-24,4) + f^*(m-12,4) + f^*(m,4) \\
&= \cdots \\
&= f(r,4) + \textstyle\sum_{i=0}^{t-1} f^*(m-12i,4) \\
&= f(r,4) + \begin{cases} \sum_{i=0}^{t-1} \frac{1}{48}\left(6(m-12i)^2 - 60(m-12i) + 240\right), \text{if m is even} \\ \sum_{i=0}^{t-1} \frac{1}{48}\left(6(m-12i)^2 - 48(m-12i) + 138\right), \text{if m is odd} \end{cases} \\
&= \begin{cases} \frac{1}{288}\left(m^3 + 3m^2 + 12m - r^3 - 3r^2 - 12r + 288f(r,4)\right), \text{if m is even} \\ \frac{1}{288}\left(m^3 + 6m^2 - 3m - r^3 - 6r^2 + 3r + 288f(r,4)\right), \text{if m is odd} \end{cases} \\
&= \begin{cases} \frac{1}{288}\left(m^3 + 3m^2 + 12m + f'(m,4)\right), \text{if m is even} \\ \frac{1}{288}\left(m^3 + 6m^2 - 3m + f'(m,4)\right), \text{if m is odd} \end{cases}
\end{aligned}$$

, where $f'(m,4)$ can be found in **Table 14**.

Table 14: scalar $f'(m,4)$ discussion

m mod 12	0	1	2	3	4	5	6	7	8	9	10	11
$-r^3 - 3r^2 - 12r$	0		-44		-160		-396		-800		-1420	
$-r^3 - 6r^2 + 3r$		-4		-72		-260		-616		-1188		-2024
$f(r,4)$	0	0	0	0	1	1	1	2	3	4	5	7
$f'(m,4)$	0	-4	-68	-72	128	28	-108	-40	64	-36	20	-8

In conclusion, we can induce that

$$f(m,4) = \frac{1}{288}\left(m^3 + \frac{9}{2}m^2 + \frac{9}{2}m + f'(m,4) + (-1)^{m+1}\left(\frac{3}{2}m^2 - \frac{15}{2}m\right)\right)$$

$$= \left[\frac{1}{288}\left(m^3 + \frac{9}{2}m^2 + \frac{9}{2}m + 128 + (-1)^{m+1}\left(\frac{3}{2}m^2 - \frac{15}{2}m\right)\right)\right]$$

4 Multi-integer Partition

In this section, we extend the approach in **2.2** and **3.2**, and apply it to find the multi-integer partition for more than four numbers. We first consider $p(m,n)$, which will be used in the process of deducing $f(m,n)$

4.1 Approach for $p(m,4)$

Recall that $p(m,n)$ is defined as the number of natural number sequences $(a_1, a_2, \dots, a_n)$ with

$$1 \le a_1 \le a_2 \le \cdots \le a_n, \qquad a_1 + a_2 + \cdots + a_n = m$$

for positive integer m, n. These sequences can be grouped in two situations: one is $a_1 = 1$, and the others is $a_1 \ge 2$.

For the first case $a_1 = 1$, as $a_1 + a_2 + \cdots + a_n = m$ is equivalent to $a_2 + \cdots + a_n = m - 1$, along with $1 \le a_2 \le \cdots \le a_n$, there are $p(m-1, n-1)$ solutions.

For the second case $a_1 \ge 2$, consider another natural number sequence $(a_1', a_2', \dots, a_n')$, where $a_i' = a_i - 1$ for $i = 1, 2, \dots, n$. It satisfies the conditions $a_1' + a_2' + \cdots + a_n' = m - n$ and $1 \le a_1' \le a_2' \le \cdots \le a_n'$, which maps to the solution of $p(m-n, n)$

Combining these two cases, we obtain $p(m,n) = p(m-n,n) + p(m-1,n-1)$. One step further,

$$\begin{aligned} p(m,n) &= p(m-n,n) + p(m-1,n-1) \\ &= p(m-2n,n) + p(m-n-1,n-1) + p(m-1,n-1) \\ &= p\left(m - \left[\frac{m}{n}\right]n, n\right) + \sum_{i=1}^{\left[\frac{m}{n}\right]} p(m - in + n - 1, n-1) \\ &= \sum_{i=1}^{\left[\frac{m}{n}\right]} p(m - in + n - 1, n-1) \end{aligned}$$

The last equation is true as $\frac{m}{n} - 1 < \left[\frac{m}{n}\right] \le \frac{m}{n}$ is equivalent to $0 \le m - \left[\frac{m}{n}\right]n < n$.

The answer can be found by applying $p(m,2) = \left[\frac{m}{2}\right]$ to the recurrence relation.

4.2 Generating Function of $p(m,4)$

In this sub-section, we discuss $p(m,n)$ in the view of generating function. Recall that $p(m,n)$ is defined as the number of natural number sequences $(a_1, a_2, \dots, a_n)$ with

$$1 \le a_1 \le a_2 \le \cdots \le a_n, \qquad a_1 + a_2 + \cdots + a_n = m$$

for positive integer m, n. Set $b_i = a_i - a_{i-1}$ for $i = 1, 2, \dots, n$ with $a_0 = 1$. The number of natural number sequences $(a_1, a_2, \dots, a_n)$ is equivalent to the number of non-negative integer sequences $(b_1, b_2, \dots, b_n)$. $a_1 \le a_2 \le \cdots \le a_n$ is equivalent to $b_1, b_2, \dots, b_n \ge 0$. $a_1 + a_2 + \cdots + a_n = m$ is equivalent to $nb_1 + (n-1)b_2 + \cdots + 1b_n = m - n$.

The number of non-negative integer sequences $(b_1, b_2, \dots, b_n)$ with $nb_1 + (n-1)b_2 + \cdots + 1b_n = m - n$ is equal to the coefficient of x^{m-n} in $(1 + x + x^2 + \cdots)\ (1 + x^2 + x^4 + \cdots) \dots (1 + x^n + x^{2n} + \cdots)$. This is because $nb_1 = 0, n, 2n, \dots$; $(n-1)b_2 = 0, (n-1), 2(n-1), \dots$; $1b_n = 0, 1, 2, \dots$, and so on.

We can simplify the polynomial. The coefficient of x^{m-n} in $(1 + x + x^2 + \cdots)\ (1 + x^2 + x^4 + \cdots) \dots (1 + x^n + x^{2n} + \cdots)$ is equal to the coefficient of x^m in $x^n(1 + x + x^2 + \cdots)\ (1 + x^2 + x^4 + \cdots) \dots (1 + x^n + x^{2n} + \cdots)$. By Taylor series, we obtain that $x^n(1 + x + x^2 + \cdots)\ (1 + x^2 + x^4 + \cdots) \dots (1 + x^n + x^{2n} + \cdots) = \prod_{i=1}^{n} \frac{x}{(1-x^i)}$. Hence, the coefficient of x^m in $\prod_{i=1}^{n} \frac{x}{(1-x^i)}$ is the number of non-negative integer sequences $(b_1, b_2, \dots, b_n)$, and that's the number of natural number sequences $(a_1, a_2, \dots, a_n)$.

In conclusion, the generating function of $p(m, n)$ is

$$\prod_{i=1}^{n} \frac{x}{(1-x^i)}$$

4.3 Approach for $f(m, n)$

We can apply the approach we used for $f(m, 3)$ and $f(m, 4)$ in **2.2** and **3.2** to find $f(m, n)$. Recall that $f(m, n)$ is the number of natural number sequences $(a_1, a_2, \dots, a_n)$ satisfying

$$a_1 + a_2 + \cdots + a_{n-1} \ge a_n + 1, \qquad a_1 + a_2 + \cdots + a_n = m, \qquad a_1 \le a_2 \le \cdots \le a_n.$$

These sequences can be divided in $n - 1$ groups: $a_1 + a_2 + \cdots + a_{n-1} = a_n + 1, a_1 + a_2 + \cdots + a_{n-1} = a_n + 2, \dots, a_1 + a_2 + \cdots + a_{n-1} = a_n + n - 2$, and $a_1 + a_2 + \cdots + a_{n-1} \ge a_n + (n-1)$. These groups can be categorized in three types.

For the first type, $k = 1, 2, \dots, \left[\frac{n-1}{2}\right]$ and $a_1 + a_2 + \cdots + a_{n-1} = a_n + 2k - 1 = \frac{m+2k-1}{2}$, it's obvious that $a_1, a_2, \dots, a_n$ are positive integers if and only if m is positive odd number, and there are $p(\frac{m+2k-1}{2}, n-1)$ solutions.

For the second type, $k = 1, 2, \dots, \left[\frac{n-2}{2}\right]$ and $a_1 + a_2 + \cdots + a_{n-1} = a_n + 2k = \frac{m+2k}{2}$, it's obvious that $a_1, a_2, \dots, a_n$ are positive integers if and only if m is positive even number, and there are $p(\frac{m+2k}{2}, n-1)$ solutions.

For the third type, $a_1 + a_2 + \cdots + a_{n-1} \geq a_n + (n-1)$, two sub-types are considered: $a_1 = 1$ or $a_1 \geq 2$. If $a_1 = 1$, then $a_2 + a_3 + \cdots + a_{n-1} \geq a_n + (n-2)$, which can be shown to have $\sum_{i=3}^{n-1} f(m-n\,,i)$.

Proof: Consider $n = 4$. $a_2 + a_3 \geq a_4 + 2$. There is no solution when $a_2 = 1$, as it implies $a_3 \geq a_4 + 1$. Hence $a_2 \geq 2$ and $(a_2 - 1) + (a_3 - 1) \geq (a_4 - 1) + 1$, which has $f(m-4,3)$ solution. Suppose that there are $\sum_{i=3}^{n-1} f(m-k\,,i)$ solutions for $a_2 + a_3 + \cdots + a_{k-1} \geq a_k + (k-2)$. Consider $n = k+1$ with $a_2 + a_3 + \cdots + a_k \geq a_{k+1} + (k-1)$. If $a_2 = 1$, then $a_3 + \cdots + a_k \geq a_{k+1} + (k-2)$, which has $\sum_{i=3}^{k-1} f(m-1-k\,,i)$ solutions. If $a_2 \geq 2$, then $(a_2 - 1) + (a_3 - 1) + \cdots + (a_k - 1) \geq (a_{k+1} - 1) + 1$, which has $f(m-1-k, k)$ solutions. Thus, there are $\sum_{i=3}^{k} f(m-1-k\,,i)$ solutions when $n = k+1$. By mathematical induction, we prove that there are $\sum_{i=3}^{n-1} f(m-n\,,i)$ solutions when $a_1 = 1$.

If $a_1 \geq 2$, then it can be represented as $(a_1 - 1) + (a_2 - 1) + (a_3 - 1) + \ldots + (a_{n-1} - 1) \geq (a_n - 1) + 1$, which has $f(m-n, n)$ solutions. Hence, there are $\sum_{i=3}^{n} f(m-n\,,i)$ solutions for the third type.

To sum up, we obtain the following result:

$$\begin{cases} \text{When } m \text{ is even, } f(m,n) = \sum_{i=3}^{n} f(m-n\,,i) + \sum_{k=1}^{\lceil \frac{n-2}{2} \rceil} p\left(\frac{m+2k}{2}, n-1\right) \\ \text{When } m \text{ is odd, } f(m,n) = \sum_{i=3}^{n} f(m-n\,,i) + \sum_{k=1}^{\lceil \frac{n-1}{2} \rceil} p\left(\frac{m+2k-1}{2}, n-1\right) \end{cases}$$

5 Reference

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